NONLINEARITY AND CHAOS IN ECONOMIC MODELS: IMPLICATIONS FOR POLICY DECISIONS

James Bullard
Federal Reserve Bank of St. Louis

Alison Butler
Federal Reserve Bank of St. Louis

August 1991
Revised July 1992

Abstract. This survey paper discusses the policy implications that can be expected from the recent research on nonlinearity and chaos in economic models. Expected policy implications are interpreted as a driving force behind the recent proliferation of research in this area. In general, it appears that no *new* justification for policy intervention is developed in models of endogenous fluctuations, although this conclusion depends in part on the definition of equilibrium. When justified, however, policy tends to be very effective in these models.

The authors thank Robert Becker, William Brock, Mark French and four anonymous referees for their constructive comments. All errors are the responsibility of the authors. Butler gratefully acknowledges the hospitality of the Swiss National Bank, where she was visiting when this paper was written. The views expressed in this paper are those of the authors and do not necessarily represent the views of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Swiss National Bank.

Please send correspondence to the authors at: Research Department, Federal Reserve Bank of St. Louis, P.O. Box 442, St. Louis, MO 63166. FAX: 314—444—8731.

I. INTRODUCTION

A substantial amount of recent research has sought to elucidate the role of nonlinearity and chaos in macroeconomic models. Some of the work has been theoretical, attempting to ascertain whether simple nonlinear deterministic models can exhibit the kind of fluctuations typically found in economic data. Other work has been empirical, and discusses the possibility that actual economic time series are characterized by chaotic dynamics.[1] Both lines of research are regarded as being in the early stages. According to Scheinkman (1990, p. 46), "... none of these developments is far enough along to bring about a change in the way economic practitioners proceed." This point is generally conceded. Nevertheless, the fact that so much research is being done suggests that some economists expect new and important policy implications from these models.[2] This paper surveys the recent research and attempts to present a picture of these potential policy implications. In doing so, focus is placed first on whether stabilization policy is in itself desirable in recent examples of nonlinear models, secondly on the efficacy of policy in these examples in the cases where intervention is justified, and thirdly on the types of policy errors that can occur if policy advice is based on linear models when the economy is actually characterized by significant nonlinearities.

Recognizing that theorists are often critical of pragmatists, and vice–versa, the point of view taken in this paper is that the researchers working on nonlinear methodologies are offering a critique of present day linear methodologies, that is, the linear stochastic difference equation approach popular in macroeconomics since the mid–1950s. Such a critique is only important to the extent that linear models are fundamentally wrong

---

[1]See Grandmont (1987a) for some of the theoretical work, and LeBaron (1991) for a survey of the empirical work. For an introduction to the subject, see Baumol and Behabib (1989), Boldrin and Woodford (1990), Brock (1990), Butler (1990), Kelsey (1988), or Scheinkman (1990).

[2]Woodford (1990, pp. 24–25) suggests that "... the fact that [nonlinear] models could well have consequences for policy analysis that are different from those associated with more conventional models [is] an important reason for being interested in the question of [these] models' logical coherence ...."

or misleading, skewing our understanding of the economy and perhaps corrupting the associated policy advice. It is possible to hold belief in a "wrong" theory that generates "incorrect" policy advice, in the sense that if the advice is executed, the actual net effects will be fundamentally different from those predicted by the theory. This survey attempts to illuminate the extent to which the work in nonlinear economic dynamics has generated a successful critique.[3]

Several broad conclusions are espoused. Based on the survey presented here, it is not clear that nonlinear models inducing endogenous fluctuations provide, *per se*, a satisfying *new* justification for stabilization policy, that is, government intervention to eliminate or mitigate the fluctuations. Of paramount importance is that, under the equilibrium modeling strategy typically followed in this literature, the endogenous fluctuations generated tend to be Pareto optimal. Models of this type that do generate a role for policy rely not on nonlinearities, but on some other assumption such as incomplete markets, deviations from rational expectations, or important externalities. While such justifications of policy may be quite reasonable, they are not innovative. However, such a conclusion depends in part on how equilibrium is defined in these models, and this point is emphasized throughout the discussion.

The above conclusion is tempered somewhat by a concern about the treatment of expectations in these models. When equilibrium paths characterized by perfect foresight become extremely complicated, the perfect foresight assumption itself becomes less tenable. Maintaining the perfect foresight assumption can imply that the agents in the model never

[3]Most present day policy advice is linked to linear theories, and while few would claim this approach is exactly correct, many believe that linear specifications provide an approximation to the true law of motion for the macroeconomy. Both measurement and misspecification introduce errors into these equations. This paper is not concerned, then, with arguing that the linear frameworks in use today are theoretically inaccurate; the fact that these models are merely approximations is well accepted. Instead, an attempt is made to illustrate *fundamental* ways in which the nonlinear approach yields alternative policy advice.

make a forecasting mistake even though an observer of the economy might conclude that the state vector follows a white noise process. This seems to imply that alternative expectational assumptions will play an important role in future research on these models.

When stabilization is justified, simple and effective policy rules often exist in the nonlinear framework which can achieve the desired objective, in contrast to some linear stochastic models where the efficacy of policy is in question. The mathematics that produce the cyclical and chaotic dynamics, the hallmark of this literature, are based on variations in parameters which lead to changes in the dynamic properties of the model. When some of the parameters of the model can be set by the policy authorities, the authorities have considerable control over the dynamic outcome.

Finally, and not surprisingly, it appears that fundamentally mistaken policy inferences can be made when linear frameworks are used to approximate nonlinear relationships. This idea is illustrated via an extended example (with optimizing agents and rational expectations) where answers to policy questions change dramatically as certain assumptions concerning linearity are relaxed.

The remainder of the paper includes a discussion of the impact of nonlinear modeling strategies on economic theory, followed by a consideration of policy advice in models characterized by nonlinear dynamics. The section on policy includes a brief look at prominent examples in the literature. Sections five and six briefly discuss stochastic considerations and assess the empirical implications and evidence, respectively, and the final section provides summary comments. The upcoming section consists of mathematical preliminaries. It develops a simple description of the operational differences between linear and nonlinear models and provides some definitions that serve as a basis for the subsequent discussion.

II. PRELIMINARIES: LINEAR AND NONLINEAR DYNAMIC MODELS

Consider a discrete time dynamic model given by

$$z_t = G_\alpha(z_{t-1}). \tag{1}$$

where z_t is a $p\times 1$ vector of state variables, $\alpha \in [0,1]$ parameterizes a family of differentiable maps $G_\alpha(\cdot)$, and $G_\alpha(0) = 0$ for every α, that is, the null vector is a fixed point of $G_\alpha(\cdot)$.[4] The current state vector z_t is given by $G_\alpha^t(z_0)$, where G_α^t represents t iterations of $G_\alpha(\cdot)$, and z_0 is the initial state vector. The map $G_\alpha(\cdot)$ displays continuous dependence on the parameter α. If the initial vector z_0 is not the null vector, or if the system is perturbed slightly from the steady state, the system described by equation (1) may or may not evolve in such a way that the steady state is asymptotically attained. If it does, the steady state is said to be stable; otherwise it is unstable.[5]

Suppose first that $G_\alpha(\cdot)$ is linear. Then (1) can be written as $z_t = Az_{t-1}$ where A is a fixed non–null $p\times p$ matrix which depends continuously on the parameter α. Analysis of stability properties involves computing eigenvalue conditions for the A matrix, and the intuition is not difficult. Suppose the system is univariate, A is a real number, and $z_0 \neq 0$. Then $|A| > 1$ implies an ever–diverging z_t, while $|A| < 1$ implies that the steady state is stable. If $A = \pm 1$, a special case is obtained; in particular, the system remains in a two period cycle if $A = -1$, and it remains at $z_t = z_0$ if $A = 1$. Analogously, for the vector system with no zero elements in z_0, the stability condition is that all the eigenvalues of A are inside the unit circle in the complex plane.

---

[4]For an exposition of the material in the remainder of this section, see Grandmont (1988) or Wiggins (1990). While the distinction between differential and difference equations can be important in this field, many economic theories attempt to explain dynamic relationships in a discrete time format. Stochastic considerations are also important but will be deferred until section five.

[5]In economics, a distinction is often drawn between historical variables, such as the capital stock, initial values of which might be thought of as being determined by history, and future–oriented variables, such as prices, initial values of which might be deduced by forward–looking agents who understand the model in which they operate. The discussion here relates best to the historical variables interpretation, as the initial vector is taken as given.

Economists often simply impose stability conditions in linear dynamic models. This is because, in linear models, dynamic paths other than those converging to a steady state often do not make economic sense.[6] Cycles are a special case of parameter values, and explosive dynamics are rarely observed in economic time series. Therefore, when working with linear models, it is natural for economists to equate the notion of economic equilibrium with the mathematical concept of the steady state.

When $G_\alpha(\cdot)$ is nonlinear, one simple way to analyze stability is to linearize the system at the steady state by evaluating the Jacobian matrix DG at the null vector. This approximation will be valid for a sufficiently small neighborhood of the fixed point. The condition for (now local) stability is that the eigenvalues of $DG(0)$ lie inside the unit circle. Most of the new and controversial analysis of nonlinear economic dynamics focuses on models where this condition is not met, that is, where the system does not tend toward a steady state. With linear dynamic models, this case was essentially uninteresting because it implied either very special parameter values in order to get cyclic outcomes, or explosive dynamics. In a nonlinear system, by contrast, cycles are possible for a wide range of parameter values, and, perhaps more importantly, even more complicated dynamic paths are possible for still other ranges of parameter values.[7] *The main benefit of nonlinear dynamic modeling is that it is possible to consider reasonable and simple economic models that never converge to a steady state; even deterministic versions can display endogenous fluctuations.*

By using an appropriate parameter index, one can consider how the dynamics of the

---

[6]For instance, a linear model with dynamics diverging from the steady state might imply negative prices at some time $t \geq 1$.

[7]It is important to stress that nonlinearity alone is insufficient for the map to display these dynamic properties—certain conditions must be met. This is easiest to see in the univariate case, where a graph of $z_{t+1} = G(z_t)$ in (z_{t+1}, z_t) space demonstrates that the map must be nonmonotonic before one can observe dynamics qualitatively different from those associated with linear maps. See Grandmont (1988).

system change as the condition for the stability of the steady state is approached and violated. Typically, α is chosen so that greater values imply that the eigenvalue condition for the local stability of the steady state no longer holds and hence more complicated trajectories become stable. Suppose that when α is near zero, $G_{\alpha}^{t}(z_0)$ tends toward the steady state for z_0 near the null vector, and that linearization at the steady state reveals this fact. Now imagine examining the stability condition with successively larger values of the α parameter. At some value for α, say α_0, the stability condition for the steady state no longer holds. This is known as a bifurcation point.

There are three ways that the condition can be violated; one of the eigenvalues must be either greater than 1, less than negative one, or else a pair of complex conjugate roots must lie outside the unit circle in the complex plane. Each of these cases is associated with a different type of bifurcation, known as fold, flip, and Hopf bifurcations respectively. However, even though $\alpha > \alpha_0$ and the steady state is unstable, the system need not explode. Instead, the nonlinearity of $G_{\alpha}(\cdot)$ can contain the dynamics to some neighborhood of the steady state. For instance, the observed time series from such a system can be periodic.

One might imagine further increases in the bifurcation parameter α leading eventually to a violation of the stability condition for the cycle, but where the dynamics of the system are again contained by the nonlinearity of $G_{\alpha}(\cdot)$. The observed dynamic path in such a case might be more complicated still. One relatively well–known example is the Feigenbaum cascade, a period–doubling bifurcation process where the system at first converges to the steady state, and then to cycles of period 2, 4, 8, ..., for successively higher values of the bifurcation parameter. At still higher values of the bifurcation parameter, the character of the observed time series undergoes a qualitative change: it becomes completely aperiodic. This is known as a chaotic trajectory, although aperiodicity alone is

insufficient to define chaos.[8]

The set of points on the dynamic path as time tends to infinity is known as the *attractor* of the system; this is simply a way to describe limiting behavior. An attractor might simply consist of k points, where k is finite. In particular, k might be unity. In this case, according to the local analysis described above, arbitrary initial conditions near the steady state converge to the steady state; hence, the "attractor" is a single point. But the attractor might instead consist of an infinity of points and may have special topological properties. In particular, simple attractors are topological manifolds, and strange attractors are fractals. Simple attractors are associated with relatively simple dynamics, and strange attractors are associated with chaos. Conceptualizing long–run equilibria as attractors offers an intuitive way to understand the heart of models using nonlinear dynamics.

In summary, the possible asymptotic outcomes in nonlinear models are roughly characterized steady states, cycles, and chaos. For some sets of parameter values, a steady state may be the stable outcome, while for other sets, the stable outcome may be a cycle, and the cycle may be of arbitrary length. A qualitatively different third possibility, deterministic chaos, may exist for a third set of parameter values. Relative to linear models, nonlinear approaches exhibit a rich set of possible observed outcomes. In the next sections an assessment of these facts will be made as they apply to policy considerations.

---

[8]A rigorous definition of chaos is beyond the scope of this paper; see Wiggins (1990). For the purposes of this survey, it is sufficient to note that chaotic sequences are aperiodic and may be indistinguishable from white noise (Brock and Malliaris, 1989). In addition, chaotic sequences are sensitive to initial conditions, in the sense that for initial conditions very close in the domain, the implied dynamics diverge and quickly become dissimilar. This is widely interpreted as implying that chaotic systems are difficult or even impossible to predict, since errors in measuring the initial state eventually imply a completely different time sequence.

III. THE THEORETICAL IMPLICATIONS OF NONLINEAR MODELS

In actual economic data, cyclical and noisy sequences are often observed. Nonlinear dynamic models therefore offer the possibility of explaining economic phenomena in a purely endogenous manner, without resorting to *ad hoc* stochastic specifications. Such a development is important for economic theory and seems quite promising. Furthermore, the mathematical assumptions needed to obtain cycles and chaos do not translate in any obvious way to special economic assumptions—there is nothing in the mathematics that is inherently contradictory with economic theory. Thus, while these outcomes of nonlinear models are not equilibria in the mathematical sense of the word (i.e., the variable is never at rest), from an economic perspective, all equilibrium conditions may be met.

Terminology, then, presents a problem. The concept of equilibrium is used both to mean a steady state and to mean that a set of conditions imposed by economists are satisfied. Therefore, the definition of equilibrium becomes particularly important when discussing the behavior induced by nonlinear systems. For the purposes of the remainder of this paper, the term equilibrium will be used in the economic context, and the generic mathematical concepts of steady states, cycles, and chaos will be used to describe dynamic behavior. Thus, logical discussion of "chaotic equilibrium" or "periodic equilibrium" is possible, meaning that economic equilibrium conditions such as market clearing are met, but that the limiting dynamic behavior of the system is chaotic or periodic. Similarly, reference can be made to steady states and cycles generically, without reference to aspects like market clearing, or to economic equilibrium without reference to its dynamic properties. In short, the terminology draws a distinction between economic equilibrium and the dynamic properties of an equilibrium sequence, because in nonlinear models it is no longer sensible to presume economic equilibrium is described by a steady state.

Of the non–steady state outcomes in these models, the possibility of chaotic trajectories are generally considered much more interesting. In particular, for business

cycle modeling, one of the criticisms of the early post–war research in nonlinear dynamics was that actual business cycles are not exactly repetitive as the deterministic model would imply. (Of course, one could always add noise, interpreted as measurement error, to these models; see section five.) However, the discovery that complicated, random–looking sequences might prevail in deterministic models has dampened that criticism substantially. Therefore, it is really the possibility of chaos that has revived interest in nonlinear dynamic models.

Because these models require certain conditions in order to produce chaos, it is important that examples be developed of *standard* optimizing models with *plausible* parameter values that generate chaotic sequences from a large set of initial conditions. The assumptions should be standard, so that the emergence of chaotic equilibrium paths cannot be attributed to unusual features of the model. The parameter values must be plausible, and not simply possible, because otherwise chaos can be dismissed as empirically unlikely. While many authors have attempted such a feat, they have so far fallen short of a completely convincing demonstration.

It should be emphasized, however, that much of the problem in writing down coherent examples may stem from the relatively rudimentary mathematical techniques available to study chaotic systems. The phenomenon is simply not yet well understood. Economists attempting to apply the mathematics have been forced to mold simple dynamic economic models into a framework for studying chaos that is relatively well known, such as the period–doubling bifurcation process mentioned earlier. The research on chaos in economic models is therefore at best suggestive and at worst uninformative about the likelihood that chaos is a relevant possibility in more realistic dynamic economic models where the mathematics to study the system have not been fully worked out.

For researchers who equate economic equilibrium with dynamic steady states, nonlinear systems provide a rigorous way to think about disequilibrium modeling, as

deviations from steady states can arise endogenously and persist indefinitely. The discussion in this section has highlighted, however, how this conceptualization can be misleading. There is no sense in which non–steady state outcomes of nonlinear models are *necessarily* associated with "disequilibrium" in the economic sense. It may well be that a nonlinear model which never converges to a steady state nevertheless has equilibrium paths characterized by perfect competition, perfect foresight, and continuous market clearing. The fact that most examples of cycles and chaos in well–specified models are developed in exactly this type of framework drives most of the conclusions about policy surveyed in the next section.

IV. THE POLICY IMPLICATIONS OF NONLINEAR METHODOLOGY

There is no clear agreement in the economics profession concerning the policy interpretation of nonlinear methodologies. For instance, Brock and Malliaris (1989, pp. 305–6) have stated that the policy implications of deterministic models with endogenous fluctuations are obvious, because these theories typically suggest "... strong government stabilization policies," in contrast to theories where fluctuations are caused by exogenous shocks, where stabilization is, "... at best, an exercise in futility ...." However, the following review of three prominent examples of endogenous fluctuations does not suggest such a blanket conclusion.

In particular, it appears to be difficult to obtain a *new* justification for stabilization policy in these models, that is, a justification for removing or mitigating the fluctuations that does not rely on weakening some aspect of either perfect foresight, perfect competition, or continuous market clearing. The cyclic and chaotic equilibria that are generated in these examples tend to be Pareto optimal outcomes unless one or more of these assumptions is abandoned. Thus while the efficacy of policy actions may be greater in these models, in the sense that the authorities may have great ability to influence the

dynamics of the economy relative to some other models where they are essentially impotent, the desirability of interventionist policy remains in doubt.

Justifying Policy in Nonlinear Models

The three examples reviewed in this section involve two frameworks, or versions of them, that are commonly employed in theoretical discussions. The first is the Ramsey problem of optimal growth and the other is the overlapping generations model. The third example is a variation on the first.[9]

A sketch of a one–sector optimal growth model consists of a representative consumer who maximizes discounted utility over an infinite horizon; typically $\sum_{t=0}^{\infty}\beta^t U(c_t)$, where $U(\cdot)$ is a utility function with standard properties, c_t is consumption, and $\beta \in (0,1)$ is the discount factor. A single good is produced using both labor, ℓ_t, and capital, k_t. The good can either be consumed or used as an input into production, given by $F(\ell_t,k_t)$. Capital depreciates at a constant rate $\delta \in (0,1)$.

An important result in the literature on nonlinear economic dynamics, developed in detail by Dechert (1984), is that in this baseline one–sector model the optimal capital accumulation path is described by $k_{t+1} = H(k_t)$, where $H(\cdot)$ is monotonically increasing and independent of β, provided the utility function is concave, and regardless of whether or not the production function is concave. Cycles and chaos are ruled out in this case. This result does not hold for a two–sector optimal growth model, however.

The two–sector model also has a representative consumer maximizing utility over an infinite horizon, say $\sum_{t=0}^{\infty}\beta^t U(c_t)$, except that there are now two goods, a consumption good and a capital good, each of which is produced using capital and labor as inputs. The production functions are given by $F^1(\ell_t^1,k_t^1)$ and $F^2(\ell_t^2,k_t^2)$ for the consumption and

[9]There are many other examples, but they are not all in optimizing frameworks, and Boldrin and Woodford (1990) have suggested that they can be inconsistent with utility maximization.

capital good, respectively, and the constraints are $c_t \leq F^1(\ell_t^1, k_t^1)$, $k_{t+1} \leq F^2(\ell_t^2, k_t^2)$, $\ell_t^1 + \ell_t^2 \leq L$ (the labor endowment), and $k_t^1 + k_t^2 \leq k_t$, with k_0 given. Boldrin and Montrucchio (1986) proved that for this case, and in fact for the n–sector case, the optimal capital accumulation path could be chaotic. Specifically, they show how to construct an economy with any given dynamics obeying the assumptions of the model, including the choice of a discount parameter. For some of the original examples of chaos in this framework, the necessary values of the discount parameters were implausibly low; in fact, they were near zero.[10]

Boldrin and Montrucchio (1986) derive their results in an optimizing framework with perfect competition and perfect foresight, and the cycles and chaos they produce are equilibrium outcomes, in the economic sense defined above. The complicated capital accumulation paths that result are Pareto optimal—no alternative path exists that makes someone in the economy better off without making someone else worse off. Under the assumptions laid out, there is no role for stabilization policy, as government intervention cannot lead to a Pareto superior outcome.

The *laissez faire* policy advice implied is hard to reconcile with the concept of equilibrium as a steady state. However, by restricting the notion of equilibrium to mean only continuous market clearing under perfect foresight, the Pareto optimality of a complicated dynamic path becomes feasible. Grandmont (1985) noted the Pareto optimality of the cycles and chaos generated in a standard overlapping generations model and suggested reasons why government intervention might nevertheless be justified. Grandmont's (1985) example serves both as an illustration of how another of the commonly used dynamic models in economics can generate chaotic time paths and how stabilization

[10]Boldrin and Montrucchio (1986) calculate a two–sector example with $\beta = .01072$. However, in more recent research, Boldrin and Montrucchio (1992) show that any dynamic path can be justified as optimal by the choice of the utility function and the technology, even if given an arbitrary value for the discount parameter. See also the discussion in Boldrin and Woodford (1990).

policy might be considered appropriate.

The overlapping generations model consists of agents that live for two periods and maximize time separable utility $U_1(c_1) + U_2(c_2)$ where c_τ, $\tau = 1, 2$, is consumption in the first and second period of an agent's life. There is a single perishable good and a single asset, a fixed stock of money M. Endowments are given by $w_\tau \geq 0$, $\tau = 1, 2$. The optimum consumption choice for the consumer depends only on the ratio of current price to expected price, $p_t/p^e_{t+1} \equiv \theta_t$. The dynamics of the model can be analyzed via a one dimensional difference equation $\theta_t = G(\theta_{t+1})$ where time is reversed.

Grandmont (1985) derives conditions on the function $G(\cdot)$ necessary for a period–doubling bifurcation process, the same process described in section two, to occur. The condition is that, roughly, the degree of risk aversion of the old agents must be "large" relative to that of the young agents. If the condition is met, cycles and chaos exist as perfect foresight equilibrium outcomes. As stressed by Boldrin and Woodford (1990), Grandmont's (1985) framework produces many potential equilibrium outcomes, among which a steady state is one possibility. Without placing more restrictions on the model, no prediction can be made about which among these many equilibria will be realized; in this sense the model is *indeterminate*. The complicated trajectories generated in this model are, in a sense, of a qualitatively different type than those generated in the optimal growth framework.

The price paths generated, whether constant, cyclical, or chaotic, are once again Pareto optimal outcomes. However, Grandmont (1985) asserts that while complicated dynamics and perfect foresight are compatible mathematically, they may not be compatible economically. That is, if the economic variables of interest to agents are following chaotic trajectories, and therefore appear to be evolving as white noise from the point of view of the outside observer, how much sense does it make to continue to endow the agents in the model with perfect foresight? That kind of accuracy would be, at best, uncanny, and

Grandmont (1985) proceeds with policy analysis by assuming the economy is initially away from the equilibrium path. Grandmont (1985, p. 1034) explains,

> Periodic equilibria with perfect foresight are Pareto optimal, so it would seem that there is nothing to do about them .... [But] the argument is misleading. Perfect foresight is *very unlikely* to obtain out of long run (here periodic) equilibr[ium]. This is especially true ... when there are many cyclical equilibria. Traders will therefore "learn" ... until the ... environment becomes repetitive enough.[11]

Grandmont's (1985) point about the incompatibility of perfect foresight and chaos should perhaps be emphasized. When economic equilibrium is equated with the concept of a steady state, it may be quite reasonable to presume that agents learn to avoid expectational errors and thus assess their future economic circumstances accurately. However, when the concept of economic equilibrium is viewed as compatible with any attractor, including those that describe chaotic trajectories, the wisdom of assuming expectational errors will naturally be eliminated is called into question.[12] The idea receives some support from Sargent (1987a), who notes that a central tenet of rational expectations is that agents in the model can predict as well as the economist manipulating the model. In Grandmont's economy, however, agents have perfect foresight *even along a chaotic trajectory.* Thus, even though the dynamics appear to be random to the observer, agents in the model never make a forecasting mistake.

If this rationale for government intervention in the model economy is accepted, it is not difficult to show that the government can indeed effectively intervene. Grandmont (1987b) examines a version of the model with a government sector that has three traditional instruments at its disposal, interpreted as forms of budgetary, fiscal, and

---

[11]Italics in original.

[12]The "learning" implied refers to alternative models of expectations formation. Agents might be viewed as updating their beliefs by assessing past data on the variables they wish to forecast, perhaps using standard statistical techniques. See Bullard (1992) for an introduction. Analyzing the effects of introducing learning would lead us too far astray—the point for the purposes of this paper is simply that the treatment of expectations is a concern for these models.

monetary policy. It is shown that simple policy rules exist that can, if they are credible, force the economy onto an arbitrary perfect foresight equilibrium path. According to Grandmont (1987b), then, the government has at its disposal the necessary tools to stabilize the economy, in the sense of moving to the steady state path, should it so desire. In this model the efficacy of policy is clear. On the other hand, this discussion of countercyclical policy relies on the economic incompatibility of complicated dynamics and perfect foresight—the "learning" implied is neither commonly modeled nor well understood in economics.

In fact, deviations from the baseline assumptions of perfect foresight, perfect competition, and continuous market clearing, perhaps quite reasonable and very slight, tend to lead to new classes of models where *laissez faire* policy advice is suboptimal. On this basis, one might be led to question the robustness of the results on the optimality of deterministic cycles reviewed so far. For instance, a version of the one–sector optimal growth model analyzed earlier in this section can produce endogenous cycles when a loan market is missing. In particular, Woodford (1989) demonstrates how a model with two types of infinitely–lived agents, one with only a capital endowment and the other endowed solely with labor, can generate complicated dynamics when the capital owners (firms) can only finance investment projects out of internal funds.[13] By assuming that the entrepreneurs maximize a (special) utility function given by $\sum_{t=0}^{\infty}\beta^t log\ c_t$, Woodford (1989) deduces that the dynamics of the economy can be described by a first order nonlinear difference equation. The existence of chaotic time paths, at some parameter values (but regardless of the discount rate β), is proven using standard techniques.

The policy implications in Woodford's (1989) model are unambiguously interventionist. Because the framework is that of one–sector optimal growth, the

[13]See also the related model of Becker and Foias (1992).

endogenous fluctuations are Pareto inferior in this model: the previously cited research of Dechert (1984) indicates that the Pareto optimal outcome for the same one–sector technology is a steady state. Therefore, the complicated dynamics should be eliminated if possible. Furthermore, the steady state of the model is not the Pareto optimal steady state. Woodford (1989, p. 331) concludes that "... achiev[ing] an efficient use of resources ... would involve continuing intervention, even in the steady state."

The activist advice enters into this model from the incomplete markets assumption, and the incompleteness takes a specific form. Such an approach may or may not be reasonable; there may be empirical methods available to test the assumption. However, it remains that there is no published example of a well–specified, optimizing model, *obeying baseline assumptions*, where *Pareto inferior* endogenous fluctuations exist. The reason for this seems clear—one must allow for some type of market incompleteness to justify government intervention under a criterion of Pareto optimality. In the examples developed so far, simply being more explicit about dynamics and conceptualizing long–run equilibria as attractors has been insufficient to break the fundamental welfare results. Unless one is willing to accept variations on the baseline assumptions, which may of course be quite reasonable, the preliminary conclusion seems to be that when endogenous fluctuations exist in optimizing models, the associated policy advice is *laissez–faire*. Subject to the caveats mentioned earlier about the mathematics, and except to the extent that complicated attractors may be incompatible with the assumption of perfect foresight, the generation of endogenous cycles through nonlinear modeling does not appear to provide any *new* ammunition for the debate between those who believe in activist policy and those who do not. This is not to say that *laissez faire* is the best policy; it is merely to say that the debate on that topic so far has not been altered by contributions from the literature on endogenous fluctuations.

As for the efficacy of policy in nonlinear models, the effects of employing nonlinear

methodologies seem clear. As has been noted, these models are based on bifurcation theory, which involves the analysis of indexed families of maps. Generally speaking, the long–run outcomes of such systems, that is, the nature of their attracting sets, display continuous dependence on parameters. In practice, equilibrium maximizing models contain a handful of parameters, some of which may represent policy rules such as the rate of taxation or the rate of money growth. In the vernacular of section two, the bifurcation parameter in a particular application might be taken to be one of the policy parameters, and the long–run outcomes of the system will then display continuous dependence on the policy rule. Often, therefore, one will conclude from these models that the authorities have considerable control over the asymptotic outcomes of the system, and thus that the efficacy of policy is great.

Misleading Policy Inference from Linear Models

Policy conclusions, including conclusions about the efficacy of policy, can change as assumptions about linearity are relaxed. Thus, even if nonlinear dynamic models offer no new justification for policy, it still remains that linear models may be misleading in a fundamental way, skewing our understanding of the economy and corrupting the associated policy advice, as alluded to in the introduction. This section emphasizes how such a corruption can occur by consideration of a specific economic environment and a specific policy question. There is no pedagogically clear, noncontroversial example to use, and there currently exist only a few economic models with sufficient nonlinearity to generate cycles and chaos. Nevertheless the following history of the Cagan theory of hyperinflation will serve to illustrate the key points.

Three versions of the following basic model are considered. Money demand depends on expected inflation, and the government raises revenue to fund a fixed real deficit via the inflation tax. The idea is to analyze monetary policy taking explicit account of the

government budget constraint. Fiscal policy is held constant—the real deficit is exogenous, fixed, and nonnegative. Particular emphasis will be placed on the characterization of the stationary inflation paths that may be observed in this type of model. In addition, when multiple economies are considered with different fiscal policies (different values for the fixed real deficit), an interesting question is whether higher deficits are associated with higher stationary inflation rates.

One version of this model was presented by Friedman (1971) using a continuous time framework. Money demand is given by

(2a) $$H(t)/P(t) = f(\pi(t))$$

where $f'(\cdot) < 0$, $\pi(t)$ is expected inflation, $H(t)$ is money, and $P(t)$ is the price level. Friedman (1971) assumed perfect foresight by setting $\pi(t) = \dot{P}/P$, a dot indicating a time derivative. The fixed real deficit ξ is given by

(2b) $$\xi = \dot{H}(t)/P(t).$$

Friedman was interested in finding an equilibrium stationary inflation rate in this model. By differentiating equation (2a) with respect to time and setting $\dot{\pi} = 0$, equation (2b) implies $\xi = \pi f(\pi)$. The first derivative of the real deficit with respect to inflation, the "revenue maximizing rate of inflation," satisfies $\pi \frac{d \ ln f(\pi)}{d\pi} = -1$. Friedman (1971) noted, based on empirical work, that many countries seemed to experience inflation rates much greater than their revenue maximizing rates.

Friedman (1971) imposed substantial linearity on the model before concluding that the equilibrium was a unique steady state. Explicit microfoundations were lacking, and stability properties of the equilibrium were not analyzed. However, the model described by (2ab) was shown in subsequent research to be closely related to models of money based on the overlapping generations framework, where explicit microfoundations were available and dynamic analysis is possible. Also, later authors dropped the assumption that the fiscal authority maximizes revenue, which, while it may be a reasonable assumption to make,

removes the possibility of comparing otherwise identical economies with different real deficits.

A two–period overlapping generations framework that produces discrete time versions of equations (2ab) was studied by Sargent and Wallace (1981,1987). Each generation t consists of many agents $n = 1 \ldots N$, each with perfect foresight. There are no bequests, and there is no storage or population growth. The agents born at time t are endowed with $w_t^n(t)$, $w_t^n(t+1)$, taxed at $\tau_t^n(t)$, $\tau_t^n(t+1)$, and consume $c_t^n(t)$, $c_t^n(t+1)$. They maximize a utility function $U_t^n\left[c_t^n(t), c_t^n(t+1)\right]$ with standard properties, where each period's consumption is a normal good. First period saving, defined as $w_t^n(t) - \tau_t^n(t) - c_t^n(t)$, is a function of the rate of return to saving, denoted $f[R(t)]$, where $R(t)$ is the gross rate of interest. Sargent and Wallace (1981,1987) assume $f'(\cdot) > 0$.

The government consumes $G(t)$, issues currency $H(t)$, and must meet a budget constraint which equates government spending with seignorage revenue plus tax revenue. Arbitrage requires that the return to saving equals the return to holding currency, $R(t) = P(t)/P(t+1)$. The real value of the government deficit, government purchases less tax revenue, is assumed to be constant and is denoted by ξ. These assumptions imply a version of Friedman's model as

$$H(t)/P(t) = f[P(t)/P(t+1)] \tag{3a}$$

$$\xi = \frac{H(t) - H(t-1)}{P(t)}. \tag{3b}$$

This system can be written as a simple difference equation in real balances $h(t) = H(t)/P(t)$,

$$h(t) = \xi + \psi(h(t-1)) \tag{4}$$

for $t \geq 2$, given $h(1) = \xi$, where $\psi'(\cdot), \psi''(\cdot) > 0$ based on the assumption that $f'(\cdot) > 0$. When the deficit is positive but less than some maximum amount, ξ_{max}, this difference equation has two stationary points, one where inflation is high and another where inflation is low. If $\xi = \xi_{max}$, there is a unique stationary rate of inflation; this is

the "revenue maximizing rate" that Friedman found. No equilibrium exists when $\xi > \xi_{max}$.

The analysis of versions of this model by Sargent and Wallace (1981,1987) can be interpreted as taking fuller account of the inherent nonlinearity in this framework. By explicitly specifying the microfoundations of a general equilibrium model, the policy interpretation was altered. In particular, when there are two stationary states, the one where inflation is high is an attractor for almost all feasible initial conditions, even though the low inflation steady state finances the fixed real deficit equally well. Sargent and Wallace (1987) interpreted this result as suggestive of a theory of hyperinflation. The Friedman (1971) case where the stationary rate of inflation is unique corresponds to a unique but unstable steady state in the Sargent and Wallace (1981,1987) analysis. Comparison of economies with different values of the fixed real deficit ξ indicates that higher deficits are associated with higher stationary inflation rates if the economy is at the low inflation steady state, but that higher deficits are associated with lower inflation if the economy is at the high inflation steady state.

The work of Grandmont (1985), previously discussed, can be interpreted as taking still fuller account of the inherent nonlinearity in this model. The results of Sargent and Wallace (1981,1987) are based on a (perhaps quite reasonable) assumption that the aggregate savings function depends positively on the rate of interest, that is, $f'(\cdot) > 0$. This amounts to an auxiliary assumption that $c_t^n(t)$ and $c_t^n(t+1)$ are gross substitutes. The standard assumptions on utility assumed above are not sufficient for the gross substitutes hypothesis to hold. Grandmont (1985) noted that $f(\cdot)$ is an aggregate excess demand function and that, in general, it could be any continuous function and still satisfy the assumptions on the utility function.[14] The difference equation given by equation (4)

[14]See Sonnenschein (1973).

could therefore be any continuous mapping and still describe an economy with optimizing individuals possessing perfect foresight. In particular, Grandmont (1985) isolated some conditions under which the mapping would be sufficiently nonlinear to generate cyclic and chaotic equilibrium paths. Furthermore, as has been noted, these equilibria are Pareto optimal.

When the full implications of nonlinearity are taken into account, the relatively sharp policy conclusions of earlier analyses of this model tend to be clouded. Since equation (4) can embody any continuous mapping, the stationary inflation rates and the dynamic behavior of the system remain in doubt. Based on economic theory alone, few policy inferences are available. In particular, it is no longer clear what the revenue maximizing rate of inflation would be, which if any steady state is likely to be attained, what the effects of higher values of the deficit would be, or whether the system is likely to display complicated dynamics.

V. STOCHASTIC CONSIDERATIONS

The arguments presented so far are based exclusively on deterministic models, whereas the trend in economics for some time has been to consider stochastic models. One simple criticism of the stabilization policy results surveyed so far might be that the results are peculiar to simple deterministic models. In this section, two methods introducing stochastic elements in nonlinear dynamic models will be considered.

The first method is direct, where the stochastic element of the model is viewed perhaps as measurement error, and is added to an otherwise deterministic nonlinear dynamic model. Crutchfield, Farmer, and Huberman (1982) consider the consequences of introducing noise directly in a particular deterministic equation known to display a period–doubling bifurcation process culminating in chaotic trajectories. They analyze both the case where the bifurcation parameter is a random variable and where a white noise

term is simply added to the entire equation. They show that the two stochastic systems display an equivalence. More importantly, they show that the addition of noise does not alter qualitatively the period–doubling bifurcation process or the chaotic dynamics. Furthermore, the distribution of the stochastic term is not of qualitative importance. In fact, there is a sense in which chaos is "more likely" when the system is stochastic.[15] These results, though far from definitive, seem to suggest that the deterministic cycles themselves are likely to survive the introduction of noise via direct methods. It appears that the endogenous fluctuations come not from the determinism, but from the nonlinearity.

The second method of adding stochastic elements places more emphasis on an integral role for the uncertainty in the model. In recent years, a great deal of work has been done on a class of models which are otherwise deterministic but where agents' expectations are assumed to be conditioned on a frivolous variable which follows a random process. Guesnerie and Woodford (1991, p. 28) characterize these models, often called models of sunspot equilibria or extrinsic uncertainty, as ones where expectations remain rational but are "garbled by 'extraneous' noise." These models are closely related to the nonlinear deterministic models that were reviewed earlier in the paper in the sense that deterministic cycles and chaos can be viewed as degenerate cases of sunspot equilibria, where the sunspot variable follows a particular deterministic process. Normally, however, sunspot processes are modeled as being stochastic. Similarities are also apparent in that the conditions for the existence of deterministic cycles to exist are closely related to the conditions for the existence of sunspot equilibria. For instance, the conditions for sunspot equilibria to exist in the overlapping generations model reviewed earlier are the same as the conditions for deterministic cycles to exist.

One of the first questions asked of models with extrinsic uncertainty was whether

[15]See also the discussion in Kelsey (1988).

there a sense in which the extraneous noise can play an important role. A result on this question was developed by Cass and Shell (1983). Roughly, their key theorem states that in an Arrow–Debreu economy with extrinsic uncertainty, the existence of a full set of claims contingent on sunspot realizations is sufficient to imply that any sunspot equilibrium is a standard competitive equilibrium, and that therefore the fundamental welfare theorems apply. In other words, the sunspots do not matter if full insurance is available. While this result is of some importance, one should perhaps be careful to note its limited applicability, as stressed by Guesnerie and Woodford (1991). First, full insurability itself might be viewed as a dubious hypothesis. Second, and more broadly, the theorem as stated does not necessarily hold for the infinite horizon economies discussed earlier, and it does not hold for similar economies with, say, incomplete markets, monopolistic competition, distortionary taxation, or other deviations from standard assumptions. One might therefore conclude that sunspot equilibria are more likely than not to be Pareto inferior. As in the models discussed earlier in the survey, however, such a conclusion relies on rather traditional arguments, again probably quite reasonable, which justify the stabilization policy, instead of on the introduction of extrinsic noise *per se.*[16] The parallel with the conclusions concerning the deterministic models also holds with regard to the efficacy of policy, as in cases where sunspot equilibria *are* Pareto inferior, there is little doubt that it is at least possible to design policy rules to eliminate the fluctuations.[17]

Of course, the stochastic elements may enter a nonlinear dynamic model in a more conventional way, perhaps through preferences or technology, instead of through agents' beliefs. In such cases, the model is one that incorporates more familiar *intrinsic* noise.

---

[16]Woodford (1990, p. 46) argues, "The most plausible conditions under which ... sunspot equilibria ... can occur would seem to be conditions under which such equilibria are possible only because of market imperfections. As a result, endogenous fluctuations ... will indicate [inefficiency] ...."

[17]See Guesnerie and Woodford (1991) for a discussion.

Less research concerning complicated dynamics in models of this type has been completed to date, partly because a continuity argument has sometimes been employed to argue that the results concerning endogenous fluctuations from models with extrinsic uncertainty carry over to models with a "small" amount of intrinsic uncertainty.[18] Therefore, models with both intrinsic and extrinsic uncertainty are sometimes viewed as more convoluted versions of models with only extrinsic uncertainty, and therefore theorists prefer to work with the extreme case of no intrinsic uncertainty. Since models with sunspot equilibria can also represent the complicated dynamics of a deterministic model when the sunspot variable is assumed to follow a deterministic process, a similar argument applies for the case of no extrinsic uncertainty but a small amount of intrinsic uncertainty. Generally, however, the question of the relationship between models with either or both types of uncertainty and the deterministic models remains a topic of current research.

VI. THE EMPIRICAL EVIDENCE

Many of the examples of economies with chaotic dynamics surveyed earlier in the paper suggest the conclusion that any dynamics are possible, even when restricting attention to models based on utility maximization and continuous market clearing. However, all the theorizing might be viewed as empty if there are no observed instances of chaotic dynamics in measurable economic variables. Therefore, some researchers have focused on testing for the presence of nonlinear dependence in general and chaos in particular in macroeconomic and financial time series.[19] Whether or not chaos has actually been observed is a key source of contention in the literature, but one that is unfortunately beyond the scope of this paper, and only a few comments can be offered.

---

[18]See, for instance, Woodford (1986).

[19]For a discussion of the tests mentioned here, see, for example, Brock and Malliaris (1989) or LeBaron (1991).

Trying to determine if a variable has followed a chaotic path is problematic. One difficulty is that present tests are designed for deterministic data from controlled experiments, and measurement error in the data can invalidate the procedure or lead to misleading results. In addition to highly accurate data, these tests require very large samples; a sample size of 1500 would be considered "small."

The most common approach tests for the correlation dimension or the Liapunov exponent. One significant problem with estimating the correlation dimension is that, particularly with small samples, distinguishing between chaos, nonlinear stochastic processes and autocorrelated processes is nearly impossible with current techniques.[20] In addition, the statistical properties of these tests are not well understood. As a result, most authors claim to find evidence "suggestive" of chaos, and any claims of clear evidence, either affirmative or negative, should probably be viewed with the skepticism normally accorded an open issue.[21]

The study of nonlinear dynamic models and deterministic chaos offers several lessons to econometricians. If forecasting is a goal of economic modeling, the possible existence of significant nonlinearity in the data suggests standard linear models may provide misleading results. In the case of chaos, the precision of a forecast when there is only a very small error in the initial conditions worsens exponentially over time. Nevertheless, strange attractors often have well–defined overall structure that might possibly be exploited by economists in some situations. Generally, it seems clear that more work needs to be done in understanding nonlinear estimation so that econometric models

---

[20]This issue was raised by Ramsey, Sayers, and Rothman (1991). Barnett and Hinich (1991) pointed out some problems with the methodology used in that paper. Unfortunately, as Barnett and Hinich readily admit, their methodology, while perhaps more consistent with conventional statistical methodology, also cannot distinguish between chaos and other types of complicated dynamics. A promising new approach, however, has been developed by McCaffrey, *et al.*, (1991). In their work, the problems of measurement error are accomodated and the sample size requirements are considerably reduced.

[21]Evidence consistent with chaos is presented by, for example, Barnett and Chen (1988).

can describe a greater variety of behavior and be more precise as well.

In addition, this research suggests that economists might want to try nonlinear specifications of economic relationships before resorting to statistical modeling. Of course, the nonlinearity must be significant enough—nonmonotonic in the univariate case—to generate complicated dynamics. The hope is that the study of such systems in turn will help to improve the quality of economic forecasts in the presence of nonlinear relationships.

VII. CONCLUSIONS

Present linear modeling techniques face a critique from economists working on nonlinear dynamic models. While it is perhaps unrealistic to try to infer the results of a line of inquiry still in an early stage, this survey has attempted to offer the uninitiated a feel for the potential of this research agenda to offer alternative policy advice. One justification for this approach is that potential policy implications seem to be a driving force behind this research.

For some authors, nonlinear models provide a rigorous concept for the notion of an inherently unstable economy. However, as a review of the literature has shown, most of the recent work begins with optimizing models, and the resulting endogenous fluctuations tend to be equilibrium phenomena, in which markets clear and expectations are correct. Therefore, this line of work is changing the way economists think about equilibrium, and the definition of economic versus mathematical equilibrium is an important theme of this survey. While endogenous fluctuations are clearly shown to be a possibility in the examples cited, they tend to be Pareto optimal outcomes. It is not clear that nonlinear dynamic methodologies imply, *per se*, any *new* justification for stabilization policy. Instead, the rationale for policy intervention tends to rely on the same types of arguments that have been used to justify policy in other economic literature. This conclusion seems also to hold for models with extrinsic uncertainty. In short, the work on nonlinear

economic dynamics has so far not provided any new ammunition for the debate between those who believe in activist policy and those who do not.

However, the deviations from baseline assumptions required to obtain a justification for policy in these models need not be in any sense extreme or unreasonable. Grandmont (1985), for example, justified policy intervention to eliminate endogenous fluctuations in his model by assuming that agents learn over time. Of course, if one is willing to depart from perfect foresight (or from rational expectations in stochastic models), then generally speaking it is not difficult to find models where stabilization policy is fruitful. On the other hand, the notion of learning is that agents use the information in available data to eliminate systematic forecast errors—learning therefore seems to imply perfect foresight *asymptotically*, and there may be some question as to how much of a deviation the learning concept actually entails. More directly, it is not clear that perfect foresight is even a feasible assumption when equilibrium dynamics become complicated or chaotic, because then agents in the model predict perfectly even when observers outside the model see only white noise. These issues remain open, but learning has been a theme of this survey because the treatment of expectations is a concern in these models.

An important consideration for policy is that linear approximations in a nonlinear world lead to distorted policy conclusions. In the course of a detailed example, this review illustrated how policy conclusions can be altered substantially by taking increasing account of inherent nonlinearity.

The efficacy of policy in nonlinear models has also been emphasized. In a deterministic nonlinear framework the stability conditions for equilibria play an important role and are dependent on the parameter values of the model. If one or more of the parameters is controlled by the policy authorities, there is often considerable ability to intervene to eliminate any endogenous fluctuations, should the authorities desire to pursue an activist approach.

REFERENCES

Barnett, W. and P. Chen. (1988) "The Aggregation–Theoretic Monetary Aggregates are Chaotic and have Strange Attractors: An Econometric Application of Mathematical Chaos." In W. Barnett, E. Berndt, and H. White, eds., *Dynamic Econometric Modeling*, Proceedings of the Third International Symposium in Economic Theory and Econometrics (Cambridge University Press).

Barnett, W. and M. Hinich. (1991) "Has Chaos Been Discovered with Economic Data?" In Ping Chen and Richard Day, eds., *Evolutionary Dynamics and Nonlinear Economics*. Oxford University Press, forthcoming.

Baumol, W. and J. Benhabib. (1989) "Chaos: Significance, Mechanism, and Economic Applications." *Journal of Economic Perspectives* 3(1): 77–105.

Becker, R., and C. Foias. (1992) "The Local Bifurcation of Ramsey Equilibrium." Working paper, Indiana University.

Boldrin, M., and L. Montrucchio. (1986) "On the Indeterminacy of Capital Accumulation Paths." *Journal of Economic Theory* 40: 26–39.

Boldrin, M., and L. Montrucchio. (1992) *Dynamic Complexities of Intertemporal Competitive Equilibria*. Oxford University Press, forthcoming.

Boldrin, M., and M. Woodford. (1990) "Equilibrium Models Displaying Endogenous Fluctuations and Chaos: A Survey." *Journal of Monetary Economics* 25: 189–222.

Brock, W. (1990) "Chaos and Complexity in Economic and Financial Science." In G. von Furstenberg, ed., *Acting Under Uncertainty: Multidisciplinary Conceptions*. Kluwer.

Brock, W. and A. Malliaris. (1989) *Differential Equations, Stability and Chaos in Dynamic Economics*. North–Holland.

Butler, A. (1990) "A Methodological Approach to Chaos: Are Economists Missing the Point?" *Federal Reserve Bank of St. Louis Review* 72: 36–48.

Bullard, J. (1992) "Learning." In D. Glasner, ed., *The Encyclopedia of Business Cycles, Panics, Crises, and Depressions.* Garland. Forthcoming.

Cass, D., and K. Shell. (1983) "Do Sunspots Matter?" *Journal of Political Economy* 91: 193–227.

Crutchfield, J., J. Farmer, and B. Huberman. (1982) "Fluctuations and Simple Chaotic Dynamics." *Physics Reports* 92: 45–82.

Dechert, W. (1984) "Does Optimal Growth Preclude Chaos? A Theorem on Monotonicity." *Zeitschrift für Nationalökonomie* 44: 57–61.

Friedman, M. (1971) "Government Revenue from Inflation." *Journal of Political Economy*, LXXIX: 846–855. July–August.

Grandmont, J–M. (1985) "Endogenous Competitive Business Cycles." *Econometrica* 53: 995–1045.

Grandmont, J–M, ed. (1987a) *Nonlinear Economic Dynamics.* Academic Press.

Grandmont, J–M. (1987b) "Stabilizing Competitive Business Cycles." In J–M. Grandmont, ed., *Nonlinear Economic Dynamics.* Academic Press.

Grandmont, J–M. (1988) *Nonlinear Difference Equations, Bifurcations, and Chaos: An Introduction.* IMSSS Economics Lecture Notes Series, Stanford.

Guesnerie, R., and M. Woodford. (1991) "Endogenous Fluctuations." Working paper, DELTA, Paris, and University of Chicago.

Kelsey, D. (1988) "The Economics of Chaos or the Chaos of Economics." *Oxford Economic Papers* 40: 1–31.

LeBaron, B. (1991) "Econometria no Lineal del Caos: Resultados Empiricos." (Empirical Evidence for Nonlinearities and Chaos in Economic Time Series: A Summary of Recent Results.) *Cuadernos Economicos*, Volume 47, Bank of Spain. Also available in English as SSRI Working Paper 9117, University of Wisconsin.

McCaffrey, D., S. Ellner, A. Gallant, and D. Nychka. (1991) "Estimating the Lyapunov

Exponent of a Chaotic System with Nonparametric Regression." Working paper, Department of Statistics, North Carolina State University, January.

Ramsey, J., C. Sayers, and P. Rothman. (1990) "The Statistical Properties of Dimension Calculations Using Small Data Sets: Some Economic Applications." *International Economic Review* 31(4): 991–1020.

Sargent, T. (1987a) "Rational Expectations." In J. Eatwell, M. Milgate, and P. Newman, eds., *The New Palgrave: A Dictionary of Economics.* Stockdon Press. Volume 4.

Sargent, T., and N. Wallace. (1981) "Some Unpleasant Monetarist Arithmetic." *Federal Reserve Bank of Minneapolis Quarterly Review* 5(3): 1–17.

Sargent, T., and N. Wallace. (1987) "Inflation and the Government Budget Constraint." In A. Razin and E. Sadka, eds., *Economic Policy in Theory and Practice.*

Scheinkman, J. (1990) "Nonlinearities in Economic Dynamics." *Economic Journal* 100: 33–48.

Sonnenschein, H. (1973) "Do Walras' Identity and Continuity Characterize the Class of Community Excess Demand Functions?" *Journal of Economic Theory* 6(4): 345–354.

Wiggins, S. (1990) *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Springer–Verlag.

Woodford, M. (1986) "Stationary Sunspot Equilibria in a Finance Constrained Economy." *Journal of Economic Theory* 40: 128–137.

Woodford, M. (1989) "Imperfect Financial Intermediation and Complex Dynamics." In W. Barnett, J. Geweke, and K. Shell, eds., *Economic Complexity: Chaos, Sunspots, Bubbles, and Nonlinearity.* Cambridge University Press.

Woodford, M. (1990) "Equilibrium Models of Endogenous Fluctuations: An Introduction." NBER Working Paper No. 3360, May.



ECONOMIC RESEARCH
FEDERAL RESERVE BANK OF ST. LOUIS

WORKING PAPER SERIES

Nonlinearity and Chaos in Economic Models:Implications for Policy Decisions

Authors	James Bullard, and Alison Butler
Working Paper Number	1991-002B
Citable Link	https://doi.org/10.20955/wp.1991.002
Suggested Citation	Bullard, J., Butler, A.; Nonlinearity and Chaos in Economic Models:Implications for Policy Decisions, Federal Reserve Bank of St. Louis Working Paper 1991-002. URL https://doi.org/10.20955/wp.1991.002

Published In	Economic Journal
Publisher Link	https://doi.org/10.2307/2234705

Federal Reserve Bank of St. Louis, Research Division, P.O. Box 442, St. Louis, MO 63166

The views expressed in this paper are those of the author(s) and do not necessarily reflect the views of the Federal Reserve System, the Board of Governors, or the regional Federal Reserve Banks. Federal Reserve Bank of St. Louis Working Papers are preliminary materials circulated to stimulate discussion and critical comment.