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THE INTEREST-ELASTICITY OF TRANSACTIONS DEMAND FOR CASH

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ONE traditionally recognized source of demand for cash holdings is the need for transactions balances, to bridge the gaps in time between the receipts and the expenditures of economic units. By virtually common consent, this transactions demand for cash has been taken to be independent of the rate of interest. The relationship, if any, between the demand for cash holdings and the rate of interest has been sought elsewhere — in inelasticities or uncertainties of expectations of future interest rates. An exception is Professor Hansen, who has argued that even transactions balances will become interest-elastic at high enough interest rates.[1] Above some minimum, he conjectures, the higher the interest rate the more economical of cash balances transactors will be.

The purpose of this paper is to support and to elaborate Professor Hansen's argument. Even if there were unanimity and certainty that prevailing interest rates would continue unchanged indefinitely, so that no motive for holding cash other than transactions requirements existed, the demand for cash would depend inversely on the rate of interest. The reason is simply the cost of transactions between cash and interest-bearing assets.[2]

In traditional explanations of the velocity of active money, the amount of cash holdings needed for a given volume of transactions is taken as determined by the institutions and conventions governing the degree of synchronization of receipts and expenditures. To take a simple example, suppose that an individual receives \$100 the first of each month, but distributes a monthly total outlay of \$100 evenly through the month. His cash balance would vary between \$100 on the first of each month and zero at the end of the month. On the average his cash holdings would equal \$50, or 1/24 of his annual receipts and expenditures. If he were paid once a year this ratio would be ½ instead of 1/24; and if he were paid once a week it would be 1/104.

The failure of receipts and expenditures to be perfectly synchronized certainly creates the need for transactions balances. But it is not obvious that these balances must be cash. By cash I mean generally acceptable media of payment, in which receipts are received and payments must be made. Why not hold transactions balances in assets with higher yields than cash, shifting into cash only at the time an outlay must be made? The individual in the preceding example could buy \$100 of higher-yielding assets at the beginning of the month, and gradually sell these for cash as he needs to make disbursements. On the average his cash holdings would be zero, and his holdings of other assets \$50. The advantage of this procedure is of course the yield. The disadvantage is the cost, pecuniary and non-pecuniary, of such frequent and small transactions between cash and other assets. There are intermediate possibilities, dividing the \$50 average transactions balances between cash and other assets. The greater the individual sets his average cash holding, the lower will be both the yield of his

[1] Alvin H. Hansen, *Monetary Theory and Fiscal Policy* (New York, 1949), 66–67.

[2] The importance of these costs in explaining the demand for cash has been explicitly analyzed by W. J. Baumol, "The Transactions Demand for Cash: An Inventory Theoretic Approach," *Quarterly Journal of Economics*, LVI (November 1952), 545–56, a paper which I should have read before writing this one but did not. Baumol is mainly interested in the implications of his analysis for the theory of the transactions velocity of money at a given rate of interest, while the focus of this paper is on the interest-elasticity of the demand for cash at a given volume of transactions. Other differences between Baumol's model and mine are discussed in the Appendix.

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transactions balances and the cost of his transactions. When the yield disadavantage of cash is slight, the costs of frequent transactions will deter the holding of other assets, and average cash holdings will be large. However, when the yield disadvantage of cash is great, it is worth while to incur large transactions costs and keep average cash holdings low. Thus, it seems plausible that the share of cash in transactions balances will be related inversely to the interest rate on other assets. The remainder of the paper is a more rigorous proof of this possibility.

Let *bonds* represent the alternative asset in which transactions balances might be held. Bonds and cash are the same except in two respects. One difference is that bonds are not a medium of payment. The other is that bonds bear an interest rate.[3] There is no risk of default on bonds, nor any risk of a change in the rate of interest.

A transaction of $\$x$, either way, between bonds and cash is assumed to cost $\$(a + bx)$, where a and b are both positive. Part of the cost of a transaction is independent of the size of the transaction, and part is proportional to that amount.

At the first of each time period ($t = 0$), the individual receives $\$Y$. He disburses this at a uniform rate throughout the period, and at the end of the period ($t = 1$) he has disbursed it all.[4] Thus his total transactions balance, whatever its composition, $T(t)$ is:

$$T(t) = Y(1 - t). \qquad (0 \leq t \leq 1) \quad (1)$$

His average transactions balance:

$$\overline{T} = \int_0^1 Y(1 - t)dt = Y/2. \qquad (2)$$

$T(t)$ is divided between cash $C(t)$ and bonds $B(t)$:

$$T(t) = B(t) + C(t) \qquad 0 \leq B(t),\ C(t). \quad (3)$$

Let $\overline{B}$ and $\overline{C}$ be average bond holding and cash holding respectively:

$$\overline{B} = \int_0^1 B(t)dt$$

$$\overline{C} = \int_0^1 C(t)dt \qquad \overline{B} + \overline{C} = \overline{T} = Y/2. \quad (4)$$

The interest rate per time period is r. Bonds earn interest in proportion to the length of time they are held, no matter how short.

The problem is to find the relationship between $\overline{B}$ (and hence $\overline{C}$) and the interest rate r, on the assumption that the individual chooses $B(t)$ and $C(t)$ so as to maximize his interest earnings, net of transactions costs. The relationship may be found in three steps:

1. Suppose that the number of transactions during the period were fixed at n. Given r, what would be the optimal times ($t_1, t_2, \ldots t_n$) and amounts of these n transactions? What would be the revenue R_n from this optimal plan? What are the corresponding values of $\overline{B}$ and $\overline{C}$?

2. Given r, but now considering n variable, what would be the value of n — call it n^* — for which R_n is a maximum?

3. How does n^*, the optimal number of transactions, depend on r? As n^* varies with r, so will $\overline{B}$ and $\overline{C}$. Also, incidentally, how do n^*, $\overline{B}$, and $\overline{C}$ depend on Y, the volume of transactions?

1. The first problem is the optimal timing and amounts of a given number of transactions. Consider this problem first for the case in which transaction costs are independent of the size of transactions ($b = 0$). In this case transactions costs are fixed by the number of transactions; and, for a given number, the optimal

[3] I shall assume for convenience that the own-rate of interest on cash is zero; if cash bore interest, the argument would be essentially the same. By "the interest rate" is really meant the difference between the yield on bonds and the yield on cash.

[4] The argument would be changed only inessentially by considering instead an individual who receives cash at a uniform rate and must make a single periodic disbursement. It may not be too far-fetched to claim that, at a given season, almost every transactor in the economy can be approximated by one of these two models. Either the transactor is accumulating a series of small receipts toward the day when large disbursements must be made, or he is gradually disbursing in small payments a prior large receipt. At different seasons of the year, or month, or week, the same transactor may sometimes be of one type and sometimes of the other. Of course actual transactions balances $T(t)$ need not decline, or grow, linearly, as assumed in this paper.

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scheduling will be the one which gives the greatest interest earnings.

If there is one transaction, from cash into bonds, there must be at least a second transaction, from bonds back into cash. Bonds cannot be used for payments, and the entire initial transactions balance must be paid out by the end of the period.

In Chart 1, the total transactions balance T is plotted against time, as in equation (1). Chart 1 presents two possible ways of scheduling two transactions. The first way, shown in Chart 1a, is to hold all cash, no bonds, until time t_1; to buy B_1 bonds at that time; to hold these, and earn interest on them, until time t_2; and then to convert them into cash. Total interest earnings are proportional to the shaded area. The second way, shown in Chart 1b, is to buy the same amount of bonds B_1 immediately on receipt of periodic income Y, and to

CHART 1. — SCHEDULING OF TWO TRANSACTIONS

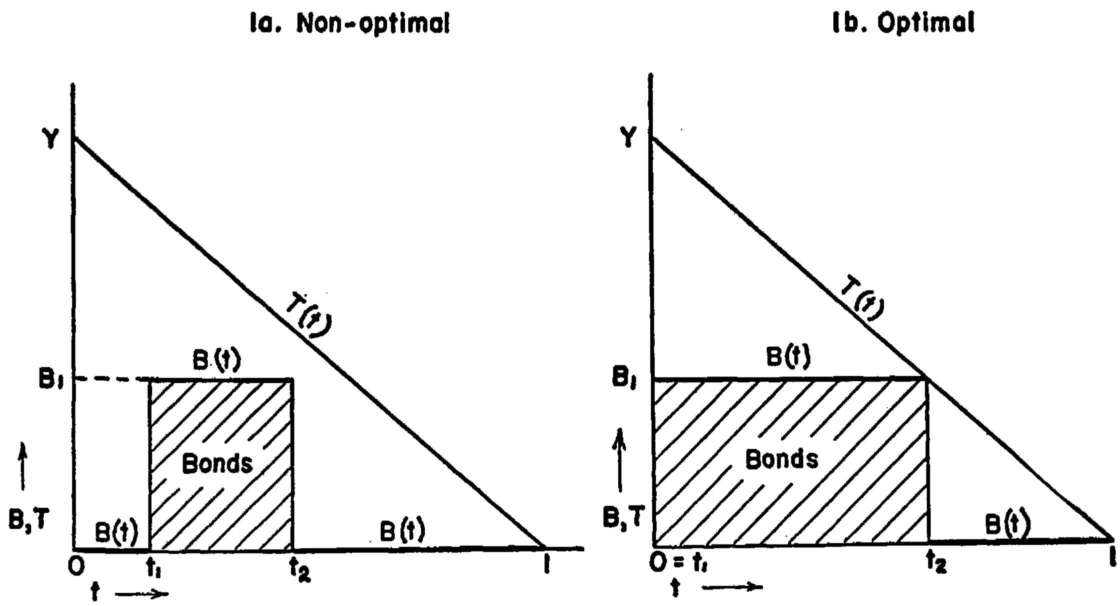
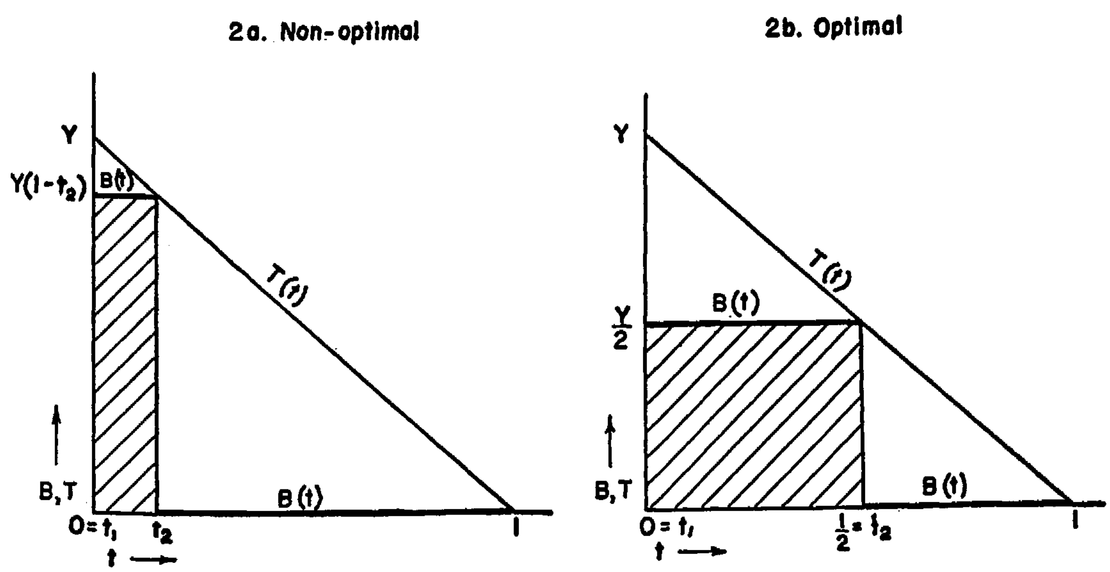


CHART 2. — SCHEDULING OF TWO TRANSACTIONS



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hold them until they absolutely must be sold in order to get the cash necessary for subsequent payments. The revenue from this schedule is proportional to the shaded area in Chart 1b, and is obviously greater than the revenue in Chart 1a. The two general principles exemplified in the superiority of the second schedule to the first are as follows:

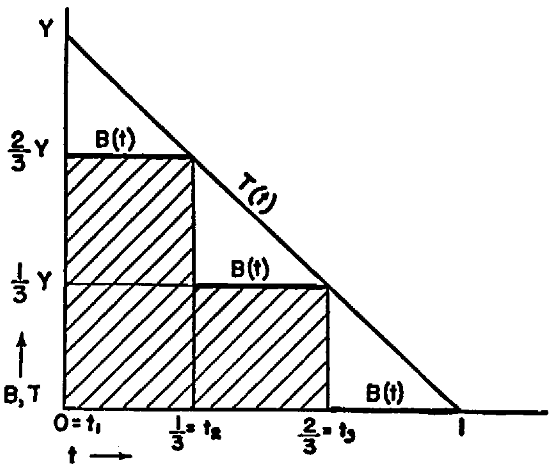
(a) All conversion from cash into bonds should occur at time 0. Whatever the size of a transaction in this direction, to postpone it is only to lose interest.

(b) A transaction from bonds into cash should not occur until the cash balance is zero. To make this transaction before it is necessary only loses interest that would be earned by holding bonds a longer time.

There are many schedules of two transactions that conform to these principles. Two possibilities are shown in Chart 2. In Chart 2a the initial transaction is obviously too great, and the second transaction must therefore be too early. The optimal schedule is to convert half of Y into bonds at time 0 and to sell them for cash at time $\frac{1}{2}$.[5]

If three transactions are allowed, it is not necessary to sell all the bonds at one time. Some may be sold at time t_2 and the remainder at time t_3. This makes it possible to buy more bonds initially. Chart 3 shows the optimal

Chart 3. — Scheduling of Three Transactions



[5] For $\frac{1}{2}$ is the value of t_2 which maximizes the expression representing interest revenue: $rY(1 - t_2)t_2$.

schedule. In general, for n transactions, the optimal schedule is to buy at time zero $\frac{n-1}{n}Y$ bonds, and to sell them in equal installments of Y/n at times $t_2 = 1/n$, $t_3 = 2/n$, . . . $t_i = \frac{i-1}{n}$, . . . , $t_n = \frac{n-1}{n}$. The average bond holding, following this schedule, is half of the initial holding:

$$\bar{B}_n = \frac{n-1}{2n}Y. \qquad (n \geq 2) \qquad (5)$$

Revenue is $r\bar{B}_n$, or:

$$R_n = \frac{n-1}{2n}Yr. \qquad (n \geq 2) \qquad (6)$$

Transaction costs are equal to na, so that net revenue is:

$$\pi_n = \frac{n-1}{2n}Yr - na \quad (n \geq 2) \qquad (7)$$

where a is the cost of a transaction. These results are all proved in the Appendix.

Some modification in the argument is needed to take account of transaction costs proportional to the size of the transaction. If this cost is b per dollar, then every dollar of cash-bonds-cash round trip costs $2b$, no matter how quickly it is made. The interest revenue from such a circuit depends, on the other hand, on how long the dollar stays in bonds. This means that it is worth while to buy bonds only if they can be held long enough so that the interest earnings exceed $2b$. This will be possible at all only if r exceeds $2b$, since the maximum time available is 1. The minimum time for which all bonds purchased at time zero must be held, in order to break even, is $2b/r$. Holding bonds beyond that time, so far as transactions needs permit, will result in interest earnings which are clear gain. The problem is the same as in the simpler case without size-of-transaction costs, except that the effective beginning time is not $t_1 = 0$ but $t_1 = 2b/r$, and consequently the effective beginning total transactions balance is not Y but $Y[1 - (2b/r)]$. With these modifications, the solution for the optimal scheduling of n transactions is the same: Put $(n - 1)/n$ of the beginning balance into bonds, and sell these bonds

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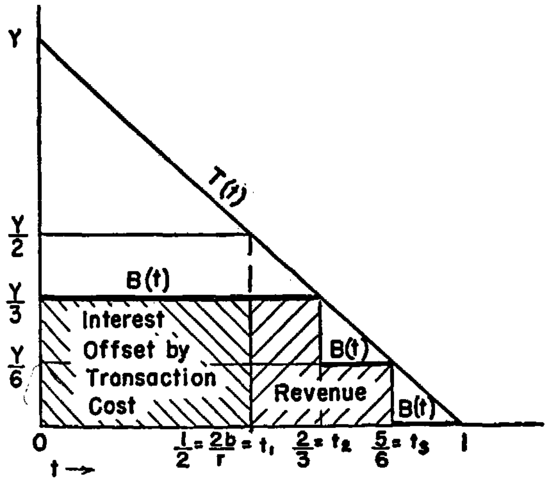
for cash in equal installment at $n - 1$ equally spaced dates.[6]

For $n = 3$, the solution is illustrated in Chart 4, which may be compared with Chart 3. In Chart 4 it is assumed that $2b/r = 1/2$, i.e., that the size of transaction cost per dollar is $1/4$ of the interest rate. The effective beginning time is thus $t_1 = 1/2$, and the effective beginning balance is $T(1/2)$ or $Y/2$.

CHART 4. — SCHEDULING OF TRANSACTIONS

(Transaction cost per dollar equal to one-fourth interest rate)



The initial purchase of bonds amounts to $2/3$ of $Y/2$; half of this purchase is converted back into cash at $t_2 = 2/3$, and the remainder at $t_3 = 5/6$.

For the general case, the following results are proved in the Appendix.

$$\overline{B}_n = \frac{n-1}{2n} Y \left(1 - \frac{4b^2}{r^2}\right) \quad (n \geq 2), \quad (r > 2b) \tag{8}$$

$$R_n = \frac{n-1}{2n} Yr \left(1 - \frac{2b}{r}\right)^2 (n \geq 2), (r \geq 2b) \tag{9}$$

$$\pi_n = \frac{n-1}{2n} Yr \left(1 - \frac{2b}{r}\right)^2 - na. \quad (n \geq 2), \ (r \geq 2b) \tag{10}$$

2. The next step is to determine the optimal number of transactions, i.e., the value of n which maximizes π_n in (10). As shown in Chart 5, revenue R_n is a positive increasing function of n, which approaches $\frac{Yr}{2}\left(1 - \frac{2b}{r}\right)^2$ as n becomes indefinitely large. Marginal revenue, $R_{n+1} - R_n$, is a positive decreasing function of n, which approaches zero as n becomes infinite:

$$R_{n+1} - R_n = \frac{1}{2n(n+1)} Yr \left(1 - \frac{2b}{r}\right)^2 \quad (n \geq 2), \ (r \geq 2b) \tag{11}$$

Total cost, na, is simply proportional to a; and marginal cost is a constant.

There are four possible kinds of solution n^*, of which Chart 5 illustrates only one. These are defined by the relation of the interest rate to volume and costs of transactions, as follows:

I. $a > \frac{1}{8} Yr \left(1 - \frac{2b}{r}\right)^2$. π_2 is negative. In this case, π_n will also be negative for all values of n greater than 2. The optimal number of transactions is zero, because π_0 is equal to zero.

II. $a = \frac{1}{8} Yr \left(1 - \frac{2b}{r}\right)^2$. π_2 is zero. In this case, π_n will be negative for all values of n greater than 2; n^* is indeterminate between the two values 0 and 2.

III. $\frac{1}{8} Yr \left(1 - \frac{2b}{r}\right)^2 > a > — Yr \left(1 - \frac{2b}{r}\right)^2$; $n^* = 2$. Here π_n is positive for $n = 2$ but negative for all greater values.

IV. $1/12 \ Yr \left(1 - \frac{2b}{r}\right)^2 \geq a$. The optimal number of transactions n^* is (or at least may be) greater than 2. This is the case illustrated in Chart 5.

3. The third step in the argument concerns the relation of the optimal number of transactions, n^*, to the rate of interest r. From (9) and (11) it is apparent that both the total and the marginal revenue for a given n will be greater the larger is r. If an increase in r alters n^* at all, it increases n^*. Now $\overline{B}_{n^*}$, average bond holdings, is for two reasons an increasing function of n. As is clear from (8), B_n for given n depends directly on r. In addition, B_n increases with n; and n^* varies directly with r.

[6] See Appendix for proof.

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Chart 5. — Determination of Optimal Number of Transactions

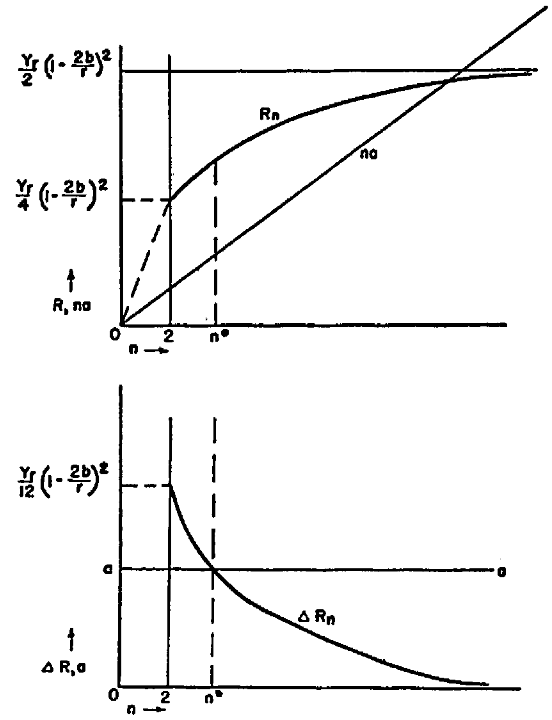
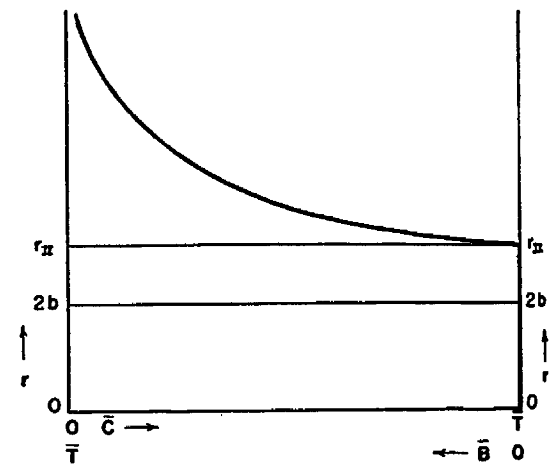


Chart 6. — Relation of Average Bond and Cash Holdings to Rate of Interest



Thus it is proved that the optimal share of bonds in a transactions balance varies directly, and the share of cash inversely, with the rate of interest. This is true for rates high enough in relation to transaction costs of both kinds to fall in categories II, III, and IV above. Within category I, of course, r can vary without affecting cash and bond holdings.

Chart 6 gives an illustration of the relationship: r_{II} is the level of the rate of interest which meets the condition of category II, which is also the boundary between I and III.

The ratio of cash to total transactions balances is not independent of the absolute volume of transactions. In equation (11), marginal revenue depends directly on Y, the amount of periodic receipts; but marginal cost a does not. Consequently n^* will be greater, for solutions in category IV, the greater the volume of transactions Y; and the ratio of cash holdings to Y will vary inversely with Y. Moreover, the range of values of r for which the demand for cash is sensitive to the interest rate (categories II, III, IV) is widened by increases in Y. Small transactors do not find it worth while even to consider holding transactions balances in assets other than cash; but large transactors may be quite sensitive to the interest rate. This conclusion suggests that the transactions velocity of money may be higher in prosperity than in depression, even if the rate of interest is constant. But it would not be correct to conclude that, for the economy as a whole, transactions velocity depends directly on the level of *money* income. It is the volume of transactions Y *relative to* transaction cost a that matters; and in a pure price inflation Y and a could be expected to rise in the same proportion.

APPENDIX

I. Suppose that $(1 - t_2)Y$ bonds are bought at time $t = 0$ and held until $t = t_2$. From t_2 until t_3, $(1 - t_3)Y$ bonds are held. In general, from t_{i-1} until t_i, $(1 - t_i)Y$ bonds are held, and finally from t_{n-1} until t_n, $(1 - t_n)Y$ bonds are held. After t_n, bond holdings are zero. Every dollar of bonds held from t_{i-1} until t_i earns interest in amount $(t_i - t_{i-1})r$. Since the total sales of bonds are the same as the initial purchase, $(1 - t_2)Y$, total transaction costs — ignoring those costs, na, which are fixed when the number of transactions is fixed — are $2b(1 - t_2)Y$. Consequently, revenue R_n is given by the following expression:

$$R_n = (1-t_2)Y\cdot t_2 r + (1-t_3)Y\cdot(t_3-t_2)r + \cdots + (1-t_n)Y(t_n-t_{n-1})r - 2b(1-t_2)Y. \quad \text{(A1)}$$

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It is convenient, for the purposes of this Appendix, to define t_1 as equal to $\frac{2b}{r}$. Then we may write R_n as follows:

$$R_n = Yr \sum_{i=2}^{n} (1 - t_i)(t_i - t_{i-1}). \qquad \text{(A2)}$$

The $t_i (i = 2, \ldots n)$ are to be chosen so as to maximize R_n. Setting the partial derivatives equal to zero gives the following set of equations.

$$\begin{aligned}
-2t_2 + t_3 \qquad\qquad\qquad &= -t_1 \\
t_2 - 2t_3 + t_4 \qquad\qquad &= 0 \\
t_3 - 2t_4 + t_5 \qquad &= 0 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad\qquad & \\
t_{n-2} - 2t_{n-1} + t_n &= 0 \\
t_{n-1} - 2t_n &= 1
\end{aligned} \qquad \text{(A3)}$$

The solution to (A3) is:

$$\begin{aligned}
t_2 &= t_1 + \frac{1 - t_1}{n} \\
t_3 &= t_1 + \frac{2}{n}(1 - t_1) \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
t_i &= t_1 + \frac{i - 1}{n}(1 - t_1) \\
&\cdot \quad \cdot \quad \cdot \\
t_n &= t_1 + \frac{n - 1}{n}(1 - t_1).
\end{aligned} \qquad \text{(A4)}$$

From (A4) we have:

$$t_i - t_{i-1} = \frac{1}{n}(1 - t_1). \qquad (i = 2, 3, \ldots n) \quad \text{(A5)}$$

$$1 - t_i = \frac{n - i + 1}{n}(1 - t_1). \quad (i = 2, 3, \ldots n) \quad \text{(A6)}$$

Substituting (A5) and (A6) in (A2) gives:

$$R_n = Yr \frac{(1 - t_1)^2}{n^2} \sum_{i=2}^{n} (n - i + 1) = Yr \frac{(1 - t_1)^2}{n^2} \cdot \frac{n(n - 1)}{2}. \quad \text{(A7)}$$

From (A7), substituting $\frac{2b}{r}$ for t_1, expression (9) in the text is easily derived. Equation (6) in the text is a special case of (9).

$\overline{B}_n$, average bond holding, is obtained from the definition (4) as follows:

$$\overline{B}_n = Y \sum_{i=2}^{n} (1 - t_i)(t_i - t_{i-1}) + Y(1 - t_2)t_1$$

$$\overline{B}_n = \frac{Y(1 - t_1)^2(n - 1)}{2n} + \frac{2Yt_1(1 - t_1)(n - 1)}{2n}$$

$$\overline{B}_n = \frac{Y(n - 1)(1 - t_1)^2}{2n} \quad . \qquad \text{(A8)}$$

Substituting $\frac{2b}{r}$ for t_1 in (A8) gives (8) in the text, of which (5) is a special case.

II. The model used in the present paper is much the same as that used by Baumol, and the maximization of my expression (10) gives essentially the same result as Baumol's equation (2), page 547, and his expression for R on page 549. There are, however, some differences:

1. I permit the number of transactions into cash, $n - 1$, to take on only positive integral values, while Baumol treats the corresponding variable, I/C, as continuous. Consequently, it is possible to duplicate Baumol's equation (2) exactly only by ignoring differences between $n - 1$, n, and $n + 1$.

2. The present paper proves what Baumol assumes, namely that cash withdrawals should be equally spaced in time and equal in size.

3. Baumol does not consider the possibility that, in the general case where the individual has both receipts and expenditures, the optimal initial investment is zero. Of the four kinds of solution mentioned in the present paper, Baumol considers only case IV. In part this is because he treats the decision variable as continuous and looks only for the regular extremum. But it is also because of his definition of the problem. Baumol's individual, instead of maximizing his earnings of interest net of transaction costs, minimizes a cost which includes an interest charge on his average cash balance. This definition of the problem leads Baumol to overlook the question whether interest earnings are high enough to justify any investment at all. Baumol's calculation of interest cost is rather difficult to understand. By making it proportional to the average cash balance, he is evidently regarding as "cost" the sacrifice of earnings as compared with a situation in which the full transactions balance, which declines gradually from T to zero during the period, is invested and cash is held no longer than the split second preceding its expenditure. Since this situation would require infinitely many financial transactions and therefore infinitely large transactions costs, it hardly seems a logical zero from which to measure interest costs.

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