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A GENERAL MODEL OF BANK DECISIONS

Alfred Broaddus

Federal Reserve Bank of Richmond

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I. Introduction

This paper presents a general theoretical model of individual bank balance sheet management under conditions of uncertainty. The model seeks to integrate and extend the existing body of microbanking theory, most notably the work of Klein [17], Bell and Murphy [4], Karehen [16], Morrison [20], Orr and Mellon [22], and Porter [23]. On the basis of the assumption that banks seek to maximize the return from their activities, solution of the model yields a bank's desired balance sheet position over a given time period and specifies the determinants of this desired position. By "desired balance sheet position" we refer to the bank's desired stocks of particular types of assets (such as loans, securities, and reserves) and particular types of liabilities (such as demand deposits and time deposits). We express these stocks as dollar balances and identify them as the bank's decision variables. Hence, solution of the model yields a desired balance sheet position of the following general form:

| <u>Asset Categories</u> | <u>Liability Categories</u> |
|-------------------------|-----------------------------|
| | (in dollars) |
| Asset 1 | Liability 1 |
| Asset 2 | Liability 2 |
| . | . |
| . | . |
| . | . |
| Asset N | Liability N |
| | Net Worth |

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This result, by its nature, specifies three distinguishable decisions: (a) the bank's desired operating scale, as measured by the dollar volume of total assets or total liabilities; (b) the bank's desired liability structure, as indicated by the proportion of total liabilities accounted for by each liability category; and (c) the bank's desired asset structure, as indicated by the proportion of total assets allocated to each asset category. A principal goal of the analysis is to demonstrate that if the bank attempts to maximize its return, these decisions are not independent of one another but are mutually interdependent. For example, it is shown that the asset and liability structures that maximize the bank's return are not invariant with respect to bank size, but vary systematically with bank size. As a second example, the bank's decisions regarding asset structure are systematically related to its decisions regarding liability structure, and conversely. Such interdependencies have not been comprehensively analyzed in the existing literature. In constructing and solving the model, we indicate the character of these interdependencies and specify why they exist.

The paper proceeds as follows. The next section outlines the general analytical framework to be employed. In subsequent sections the model is constructed and solved, and a set of conditions consistent with optimization by a bank of its balance sheet position is derived. The analysis involves a number of restrictive and often unrealistic assumptions. Such assumptions are necessary, however, in order to confine the analysis within manageable bounds.

Framework of the Analysis and General Assumptions

The object of the analysis is the individual bank, a financial institution which seeks and obtains funds from a variety of sources and

subsequently invests these funds in a variety of financial assets. We assume that the bank is not subject to legal restrictions of any sort. The controlling operational assumption of the model is that the bank acts to maximize expected additions to equity over a finite time span designated the "planning horizon."¹ The bank accomplishes this optimization by managing its balance sheet position over the course of the planning horizon. In reaching decisions, the bank is not influenced by events preceding the planning period in time or by expectations regarding events following its close.² In reaching its planning period decisions, the bank has certain knowledge of all relevant economic variables and parameters comprising its environment with the following three exceptions: (a) the level of deposit liabilities at any moment during the period, (b) the market value at any moment during the period of securities held as secondary reserves against deposit withdrawals, and (c) total repayment by borrowers of outstanding loans maturing during the period: i.e., the level of loan defaults. Although uncertain with respect to these variables, the bank is assumed to know the form and parameters of their probability distributions precisely. Having listed these variables, we must note the omission, at this point in the analysis, of a major element of uncertainty facing banks in the real world: unanticipated changes in loan demand. This omission is designed to simplify the model, since the basic goals of

¹As indicated below, the bank is assumed to operate under conditions of uncertainty. Therefore, a more general assumption would be that the bank maximizes utility, where utility is a function of both expected return and the variance of return. Such an assumption would greatly complicate the analysis while adding little to its ultimate conclusions. As in the Morrison [20] and Porter [23] models, the presence of uncertainty will affect the bank's decisions through its effects on the bank's expected cost and revenue flows during the planning period.

²This assumption, in one form or another, is characteristic of the theoretical microbanking literature. A thorough summary of its implications is given by Porter [23, pp. 325-326].

the analysis can be achieved without explicitly considering this aspect of bank operations. In an appendix to the paper, we indicate how the incorporation of uncertain loan demand in the model affects the results generated by the model. Throughout the paper itself, however, the term "bank liquidity" refers solely to the bank's ability to meet unexpected deposit withdrawals.

We have assumed that the bank maximizes additions to net worth over the planning horizon by managing its balance sheet. Therefore, the elements of the balance sheet are the bank's decision variables. We now describe the nature of these balance sheet elements and the manner in which their manipulation influences the planning period change in net worth.

Decision Variables: The Balance Sheet Elements

In reality, banks gain the use of funds by accepting liabilities of widely varying form. They subsequently allocate these funds among an equally wide variety of assets. Any theoretical analysis of bank operations must abstract from the complexity of real world financial instruments. We assume that the bank balance sheet consists of several categories of assets and liabilities and that the bank formulates decisions in terms of these categories. As indicated below, we shall assume that the instruments comprising each category are internally homogeneous with the exception of loans. The broad characteristics and analytical roles of each category are outlined here. Additional assumptions will be introduced when the model is constructed. Symbols used to refer to each category are in parentheses. (A complete list of symbols used in the paper is provided below.)

1. Loans (L). Loans are assets that pay an explicit return but present the risk of default. We assume that no loan outstanding during the planning period is marketable during the period and that no loan matures before the end of the period.

2. Bonds (B). We use the term bonds to represent long-term investments for income. Bonds pay the bank a constant explicit rate of return regardless of the quantity held: i.e., bonds are available to the bank in perfectly elastic supply. Bonds are free of default risk. We assume that the bank, in its decision process, does not contemplate selling bonds for any purpose during the planning period. Hence, the bank holds bonds solely for income purposes. They provide an alternative to loans in that they guarantee a constant and certain, although generally lower, average return.

3. Securities (S). Securities are assets that pay an explicit return and are free of default risk but whose market value at any moment during the planning period is a random variable. An organized market for these issues exists, and the bank can buy or sell in this market at any moment during the period without influencing whatever market price exists at that moment. The bank holds securities as a secondary reserve against unexpected deposit withdrawals. We shall define the security issue as a consol for analytical simplicity, but it will play the role of a short-term government instrument.

4. Reserves (R). Reserves are perfectly riskless assets that pay no explicit return. The bank holds reserves in order to meet unexpected deposit withdrawals.

5. Demand Deposits (DD). From the standpoint of the bank, demand deposit liabilities represent funds that may be withdrawn at any moment during

the period. In keeping with the generality of the model, we assume that the bank must pay explicit interest as well as service and promotional costs in order to attract demand deposits.

6. Time Deposits (TD). Time deposits, like demand deposits, are funds which may be withdrawn at any moment during the period and which cause the bank to incur interest, service, and promotional costs. We shall assume that the probability of a given time deposit outflow differs, in general, from the probability of an identical demand deposit outflow. Further, we shall assume that the functional relationship between deposit costs and deposit volume differs, in general, between the two deposit categories.

7. Borrowed Funds (BF). "Borrowed funds" are funds obtained from various nondeposit sources. Such funds cannot be withdrawn during the planning period.³ The bank must pay explicit interest in order to obtain these funds, but it does not incur any other costs for their use. This category denotes longer-term borrowing designed to support a sustained increase in the bank's lending and investing activity that is planned in advance by the bank's management. The category does not include borrowing to meet unanticipated liquidity needs.⁴

If the above balance sheet elements are to serve as decision variables, the model must be constructed in a manner that permits the

³For simplicity, the analysis omits liabilities of intermediate withdrawal risk: i.e. funds that may be withdrawn only after a warning period or, if called immediately, only by forfeiture of interest. The effect of including such liabilities can be inferred by generalizing the solution of the model as constructed.

⁴Borrowing of the latter variety does enter the model, however, as indicated below.

bank to control, quantitatively, the dollar stock each variable represents. The major difficulty in this respect concerns deposits. Clearly a bank does not control its deposit flows on a daily basis. It is equally clear, however, that through advertising and other means, banks attempt to influence at least the direction and approximate magnitude of net deposit flows over longer time periods. In the case of any particular bank, the time required to actually exert such influence is an empirical question.

In order to cope with this conceptual difficulty, we define the decision variables as the expected average values of the respective balance sheet stocks over the course of the planning horizon. More precisely, if we define a particular time path for any balance sheet stock over the planning horizon as:

$$(1) \quad O = O[t],$$

then, assuming continuity, the average value of the stock over the period is:

$$(2) \quad \bar{O} = \int_{t=a}^{t=b} \frac{1}{b-a} O[t] dt.$$

The value of a given stock may follow a number of different time paths due to fund inflow or outflow (e.g., deposits) or market price fluctuation (e.g., securities) during the period. In reaching its decisions, the bank is uncertain as to which path will appear as the planning period unfolds, but we assume it is able to attach a definite probability to every conceivable path. The average value of the stock over the period, \bar{O} , is then a random variable, the distribution of which the bank knows.

We define the decision variables as the respective means of these distributions. For those balance sheet elements such as borrowed funds and

bonds which present no possibility of withdrawal or market price change, the definitions are limiting cases of the general definition. We assume that the planning period is long enough to permit the bank to control all balance sheet decision variables as just defined. In the case of demand deposits, for example, we are assuming that the length of the period provides enough time for bank actions to control the parameters of the average demand deposit balance distribution over the course of the period. It seems reasonable to consider the period relatively short, perhaps two to three months in duration.

An Operational Equation for the Expected Change in Bank Net Worth

We have assumed that the bank's goal is to maximize the addition to net worth⁵ over the planning horizon. Hence, it is necessary to specify the manner in which bank actions influence the change in net worth. It will be useful to begin with accounting relationships and develop from these an operational relation between the change in net worth and the bank decision variables.

Let us indicate the last day of the previous period and the last day of the planning period by the subscripts $t-1$ and t , respectively. The balance sheet identities for each of these days are then:

$$(3) \quad L_{t-1} + B_{t-1} + S_{t-1} + R_{t-1} - DD_{t-1} - TD_{t-1} - BF_{t-1} - NW_{t-1} \equiv 0;$$

$$(4) \quad L_t + B_t + S_t + R_t - DD_t - TD_t - BF_t - NW_t \equiv 0,$$

where NW is bank net worth. By subtracting (3) and (4) and rearranging

⁵For simplicity, it is assumed that net worth consists entirely of capital stock (i.e., shareholder equity) and that the bank neither plans nor makes dividend payments during the planning period.

terms, we can write the following expression for the change in net worth over the planning period:

$$(5) \quad \Delta NW = [(L_t - L_{t-1}) + (B_t - B_{t-1}) + (S_t - S_{t-1}) + (R_t - R_{t-1})] \\ - [(DD_t - DD_{t-1}) + (TD_t - TD_{t-1}) + (BF_t - BF_{t-1})],$$

where $\Delta NW = NW_t - NW_{t-1}$. We proceed from accounting identity (5) to an operational relation as follows.

Step 1. Consider the final bracketed term on the right side of (5). Calculation of the magnitude of this term eliminates interliability substitution and gives the net inflow or outflow of total funds during the period.

Step 2. Consider now the first bracketed term on the right side of (5). Calculation of its magnitude eliminates substitution among assets and gives the net increase or decrease in total assets over the period.

Step 3. Subtract the result of step 1 from the result of step 2, as indicated by the right side of (5). This operation eliminates from the net change in total assets (step 2) that portion which results directly from the net change in total funds (step 1).

In general, a residual (net) change in total assets remains after performance of step 3. This residual represents that portion of the change in total assets which does not result from inflows or outflows of funds. Further, this residual equals ΔNW , the change in net worth over the period. For present purposes, we may assume that the residual consists of the following three elements: (a) loan defaults during the period; (b) the change in security portfolio value that results from market price fluctuation as opposed to sales or purchase transactions;

and (c) the net revenue flow during the period arising from the bank's operations.

In subsequent analysis, we shall treat loan defaults as a reduction of the rate of return on loans. Therefore, element (a) above will be absorbed by element (c). Schematically, we can write:

$$(6) \quad \Delta NW = \text{exogenous security portfolio change} + \text{net revenue flows.}$$

Equation (6) is the operational relation facing the bank at the beginning of the planning horizon. The equation is operational because the flow of net revenue over the period depends upon the decision variables, as we shall indicate in constructing the model. Because the decision variables have been defined as expectations, however, equation (6) must be rewritten as:

$$(7) \quad E(\Delta NW) = \underline{\text{expected}} \text{ exogenous security portfolio change} + \underline{\text{expected}} \text{ net revenue flows.}$$

As indicated below, the bank does not, in the probabilistic sense, expect security prices to change during the planning period. Therefore, equation (7) reduces to:

$$(8) \quad E(\Delta NW) = \text{expected net revenue flows.}$$

The model construction in the next section consists, essentially, of a detailed specification of (8) with particular attention to the dependence of expected net revenue on the decision variables.

II. Construction of the Model

In this section, we shall develop a detailed operational function that specifies the determinants of the bank's expected change in net worth during the planning period. This relation will serve as the bank's objective function. The procedure will be to consider each decision

variable (i.e., each balance sheet element) in turn, noting its contribution to the function. We shall then summarize by writing the complete function. Solution of the model in the following section consists of maximizing the complete function subject to a balance sheet identity constraint.

Loans

In addition to the assumptions already outlined, we place the following specific restrictions on the character of bank lending activities. (a) All loans outstanding on the day preceding the planning period mature on that date. If the bank chooses to renew a portion of these loans, it does so under contract terms prevailing at the beginning of the planning period. This assumption eliminates from the calculation of expected planning period revenues the analytically unnecessary complication of loan carry-over from previous periods at previously contracted rates. (b) The entire balance of each loan contracted during the planning period falls due at the end of the period, and any loan default occurs at that time. This assumption insures that all prospective defaults on loans contracted during the planning period enter the objective function of the model. (c) Noninterest loan terms, specifically loan size, are identical across loans and are exogenous to the bank.⁶

In standard microeconomic theory, the firm faces a demand curve which specifies the manner in which average revenue from product sales varies with total sales. We wish to introduce a similar relation for the

⁶The homogeneity assumption is for analytical simplicity. It could presumably accommodate balancing trade-offs among the noninterest terms of particular loans.

bank's lending activity. The analysis is complicated, however, by the fact that, from the standpoint of a bank, alternative borrowers differ with respect to prospective default (i.e., credit rating) and the duration and strength of their customer relationship with the bank. Banks, unlike many nonbank firms, are highly selective in choosing the particular customers to whom they "sell" their loan product, and they discriminate among customers in establishing prices for this product.⁷ We wish to avoid treating the complex process by which banks select the particular customers to whom they lend. To do so, we introduce the following final set of assumptions. (d) The bank faces a particular set of potential loan customers. The prospect of default associated with each customer is summarized by a probability distribution of total loan repayment. The character of these distributions varies from one customer to another. The parameters of each such distribution are exogenous to the bank but known by the bank. We assume that the bank lends to these customers in a fixed sequence determined by considerations outside the scope of the analysis. The bank extends loans in this sequence up to a point of its choice where it ceases lending altogether. The usefulness of these last assumptions will become clear as the analysis proceeds.

Loan revenue conditions facing the bank are summarized by the following expected average net rate of return function for loans:

$$(9) \quad r_L[L] = r'_L[L] - d_L[L] - c_L[L],$$

⁷Banks discriminate further by differentiating the character of the loan product among customers, that is, by varying noninterest lending terms. We have eliminated this difficulty by assumption (c) above. For general analyses of the determinants of bank lending behavior see Hester [11] and Jaffe and Modigliani [14]. For a detailed examination of the "customer relationship" and its effect on bank lending see Hodgman [13, pp. 97-144].

where:

$r_L[L]$ = the expected average net rate of return on loans as a function of total loans,

L = average total loans held by the bank during the planning period,⁸

r_L' = the average gross contract rate on loans as a function of total loans,

$d_L[L]$ = the expected average default rate on loans as a function of total loans, and

$c_L[L]$ = the average cost of loans (expressed as a percentage rate on the dollar) as a function of total loans.

Equation (9) plays an analytical role similar to that of a demand curve in standard theory. That is, (9) specifies how revenue from the bank's principal revenue-producing activity varies as the volume of lending activity changes.⁹ We now examine each of the component functions on the right side of (9).

1. $d_L[L]$. The construction of this function can be described with the aid of Figure 1, which plots total loan repayment on the vertical axis against the decision variable L on the horizontal axis. As indicated above, the bank knows the probability distribution of loan repayment for each customer. These individual distributions are marginal distributions of a joint distribution that we assume the bank also knows. This joint distribution may or may not exhibit some degree of covariance among borrowers. Given knowledge of this distribution and, by assumption (d), the sequence in which loans are extended, the bank can construct a probability

⁸ L is the bank's lending decision variable and refers to the dollar volume of loans the bank extends. Throughout the body of this chapter, the bank controls this particular decision variable with certainty. That is, the bank can select the precise volume of loans it wishes to extend, even though the level of repayment is uncertain. In the appendix, we analyze the implications of an alternative assumption.

⁹The function obviously differs from a conventional demand curve in that revenue is here defined net of lending costs.

distribution of total loan repayment at any particular level of total loans outstanding. Such a distribution is depicted by Figure 1 for loan level L_0 . This particular distribution is one member of a family of such distributions for various total loan levels. For simplicity, we assume that each distribution within this family is continuous and symmetric and that the respective means and limits of the distributions comprising the family form the continuous lines radiating from the origin in Figure 1.¹⁰

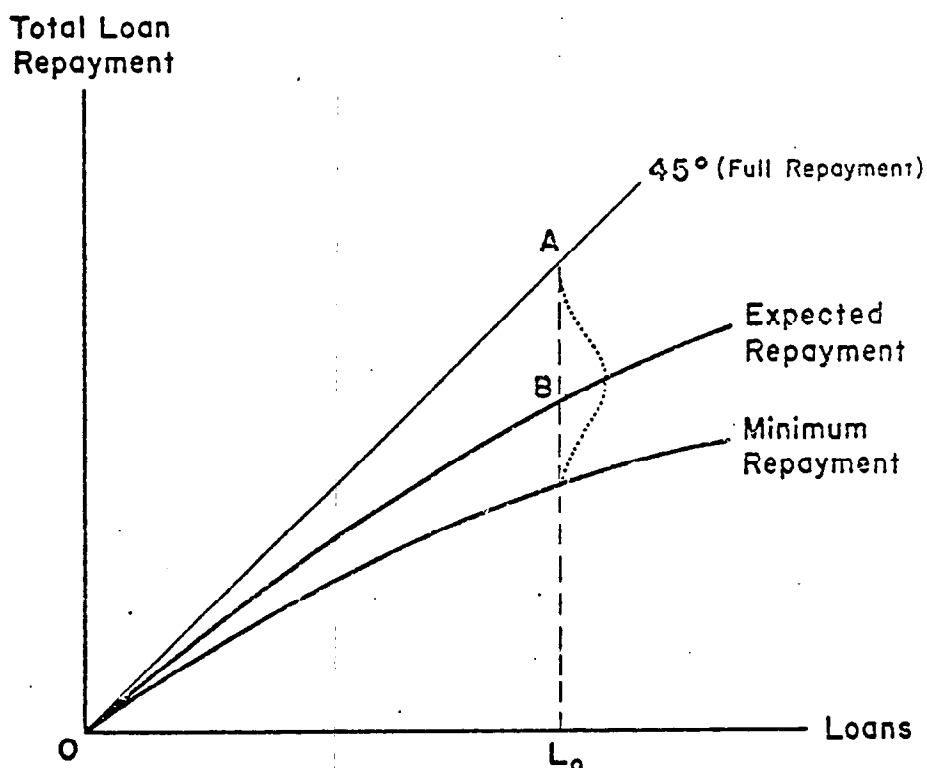


FIGURE 1

In terms of Figure 1, the variable d_L is (for loan level L_0) the ratio $\frac{AB}{OL_0}$. Clearly, the functional relation of d_L to L depends on the shape

¹⁰The upper limits (full repayment) of the distributions obviously form the continuous 45° line. We are assuming that each distribution exhibits a finite lower limit indicating the smallest total repayment to which the bank attaches a positive probability.

of the figure's "expected repayment" line which, in turn, reflects the manner in which each additional borrower in the fixed sequence of borrowers affects the distribution of total loan repayment facing the bank. For example, if the expected repayment line is linear, d_L is a constant. In this case, later borrowers in the sequence do not increase the expected default rate. As Figure 1 is drawn, later customers present a greater risk of default and cause d_L to rise.¹¹ In general terms:

$$(10) \quad d_L = d_L[L; \bar{z}],$$

where \bar{z} is a vector of parameters summarizing the default risk characteristics of the bank's loan customers. We place no specific restrictions on the mathematical form of (10). As a model parameter, however, the vector \bar{z} will affect the solution of the model for the optimal values of the decision variables.

2. $r'_L[L]$. This function specifies how the average gross contract rate varies with L in much the same way that a conventional demand curve specifies the relation of average revenue to sales. The mathematical form of this function in any specific case reflects the influence of two interrelated factors. First, the function reflects the credit risk characteristics of particular borrowers in our assumed fixed sequence, since these characteristics determine the risk premiums the bank seeks to extract from each customer. Second, the function reflects competitive conditions in the loan market within which the bank operates. In a manner similar to conventional demand analysis, competitive conditions affect the

¹¹The term "risk," as used here, refers only to the relationship of the first moment of the total loan repayment distribution to the level of loans outstanding. We can reasonably assume, however, that the variance of the repayment distribution increases if d_L increases with L .

form of the function through their determination of the bank's market power within the relevant loan market. Here, competitive conditions specifically affect the degree to which the bank can extract risk premiums from its borrowers. Therefore, we may write:

$$(11) \quad r'_L = r'_L[L; \alpha_L, \bar{z}],$$

where α_L is a vector of parameters summarizing the competitive structure of the bank's loan market. Again, we place no specific restrictions on the form of the function. It is worth noting that, in contrast to traditional demand theory, we may have $\frac{dr'_L}{dL} > 0$ if competitive conditions are such that the bank can fully compensate for the increased default risk associated with nonprime borrowers.

3. $c_L[L]$. This function indicates how the average cost to the bank of making and servicing loans, per dollar of loans, varies with total loan volume. We assume that the bank's real capital stock is fixed over the planning period. Hence, this function is analytically comparable to a short-run average cost function in the standard theory of the firm. There are, however, a number of conceptual difficulties in treating loan costs in this fashion. In the traditional theory of the firm under continuous production conditions, one derives the functional relationship between short-run average costs per unit of output and total output by minimizing costs at each level of output subject to (a) the technical constraint imposed by a production function and (b) factor supply conditions facing the firm. This procedure presupposes an unambiguous definition of the relevant output upon which costs depend.

In the present analysis, as the above cost function indicates, the "output" is the dollar amount of loans outstanding. This choice is

necessary because we are treating balance sheet stocks as decision variables. It is by no means clear, however, that this variable is the relevant output on which lending costs depend.¹² Broadly, a bank incurs lending costs due to (a) credit risk investigations and (b) administrative services surrounding the management of loan accounts. These activities constitute the physical output flows upon which lending costs directly depend.¹³ In general, one would not expect the volume of these services to exhibit any invariant relation to dollar loan volume. If for a given bank, however, all loan characteristics including loan size were identical across loans, a fixed relation would exist between the flow of loan services and total loan volume. Under these conditions, costs as a function of dollar loan volume would be a simple transformation of costs as a function of service output. In the present analysis, we have assumed that loans are identical with the exception of borrower default risk characteristics. Therefore, we may think of $c_L[L]$ as the sum of two independent components. The first component is a simple transformation of a standard cost function with all loan services other than those related to the credit risk of individual borrowers defined as output. The second comprises those costs, such as credit investigation costs, that are related to borrower risk. Because, by assumption (d), the sequence and risk characteristics of borrowers are predetermined, it follows that the relationship between the costs of investigating borrower credit standing and the bank's total loan volume is exogenous from the standpoint of the bank.

¹²In this regard see Broaddus [6, pp. 37-44].

¹³For a thorough discussion see Benston [5, pp. 522-534].

On the basis of these considerations, we may write:

$$(12) \quad c_L = c_L[L; \bar{K}_0, \bar{w}_0, \bar{z}],$$

where \bar{K}_0 is the bank's fixed capital stock, \bar{w}_0 is the constant wage rate facing the bank, and, again, \bar{z} is a vector summarizing the credit risk characteristics of the bank's borrowers. Following our earlier procedure, we impose no specific restrictions on these parameters.

Having discussed the components of the bank's loan revenue function, equation (9) can be rewritten in general form as:

$$(13) \quad r_L = r_L[L; \bar{K}_0, \bar{w}_0, \alpha_L, \bar{z}],$$

where, again, r_L is the expected average rate of return on loans net of default and loan costs. Because we have not restricted the parameters of (10)-(12), it follows that we have not restricted the parameters of (13). In what follows, we shall drop the parameter notation and write (13) as:

$$(14) \quad r_L = r_L[L].$$

From the preceding discussion, however, we know that the form of this function depends on the risk characteristics of the bank's loan customers, the competitive structure of the market in which the bank operates, and factor prices.

We can now write the bank's expected total net revenue from loans as:

$$(15) \quad ER_L = r_LL.$$

Equation (15) is the first component of the objective function of the model.

Bonds

The analytical role played by bonds was briefly outlined in an earlier section. The distinction we have introduced between "bonds" and "securities" is designed to separate, in a gross fashion, nonloan investments made for income purposes from securities held as secondary reserves to meet deposit withdrawals. In the real world, this distinction is not clear-cut.¹⁴ Further, the composition of an actual bank's earning asset portfolio reflects the term structure of interest rates and expectations with respect to the future course of interest rates. In our static, single-period model, these dynamic considerations play no role. As stated above, the bank views bonds as an alternative to loans because they are free of default risk and yield a fixed rate of return regardless of the dollar volume held.¹⁵

We introduce the following additional assumptions. (a) Bonds available to the bank are homogeneous consols. (b) The price of an individual bond is constant over the planning period.¹⁶ (c) Each bond pays the fixed coupon rate \bar{r}_B over the planning period. (d) Bond transactions are costless.

On the basis of these assumptions, the total planning period revenue from bonds is:

$$(16) \quad ER_B = \bar{r}_B(B),$$

¹⁴This is not to say that such distinctions are unrecognized within the banking industry. See, for example, American Bankers Association [1, pp. 270-271].

¹⁵The effect of asset supply conditions on bank decisions has been relatively neglected in the theoretical banking literature. See, however, Klein [18].

¹⁶Since, as indicated earlier, the bank does not contemplate bond liquidation during the planning period, it is analytically unnecessary to introduce uncertainty with respect to bond prices.

where B , the decision variable, is the average value of the bank's bond portfolio over the period. Equation (16) forms the second element of the objective function.

Securities

As stated earlier, the bank holds securities as a secondary reserve to meet unexpected deposit withdrawals. Securities are an alternative to reserves for this purpose, because we have assumed that securities, unlike reserves, pay an explicit return. We introduce the following additional assumptions regarding securities. (a) Securities available to the bank are homogeneous consols. (b) The price of a security at the beginning of the planning horizon is one dollar. (c) From the bank's standpoint, the average price of an individual security over the planning period is a random variable:

$$(17) \quad P_S = 1 + w,$$

where w is a uniformly distributed random variable with mean zero, and where $-a \leq w \leq a$, $0 < a < 1$.¹⁷ The homogeneity assumption implies that individual security prices are perfectly correlated. Hence the average value of the bank's total security portfolio over the period is also a uniformly distributed random variable. The expected value of this latter random variable, designated by the symbol S , is the bank decision variable with respect to security holdings. (d) Each security pays the coupon rate \bar{r}_S over the planning period. (e) Transactions in securities are costless.

On the basis of these assumptions, the bank does not expect any capital gain or loss on security holdings, and the total expected explicit

¹⁷Therefore, the distribution of P_S is $\phi(P_S) = \frac{1}{2a}$ over the range $1 - a \leq P_S \leq 1 + a$.

revenue from securities is:

$$(18) \quad ER_S = \bar{r}_S(S).$$

Equation (18) is the third element of the objective function.

We turn now to the liabilities side of the balance sheet, returning to reserves at a later point.

Deposits

As stated earlier, the bank accepts two types of deposit liabilities: (a) demand deposits and (b) time deposits. It is assumed that two characteristics of the bank's deposit accounts influence the bank's decisions: (a) stochastic deposit variability and (b) deposit costs. Since these factors play a critical role in the analysis, it is necessary to discuss each at some length.

Stochastic deposit variability

In reality, individual banks face continual inflows and outflows of funds due to depositor transactions. The pattern of these flows determines a bank's total deposit stock at any moment in time and the path followed by the balance through time. The time path of an actual bank's deposit stock reflects, ultimately, the behavior of the bank's depositors and the innumerable factors that condition this behavior. Banks themselves can influence depositor behavior to some degree through deposit interest payments (to the extent that such payments are permitted by regulatory authorities), services provided depositors, advertising, and promotional campaigns. Further, real world banks can predict, with considerable accuracy, deposit variation caused by cyclical and seasonal movements in income and in other economic variables. In addition to these partially predictable deposit movements, however, all

banks experience essentially random deposit fluctuations caused by a myriad of unsystematic and unpredictable conditions influencing their depositors.¹⁸ For simplicity, and in keeping with the single-period framework of the analysis, we assume in what follows that all factors influencing depositor behavior other than the bank's own actions are, in the bank's view, random.

We assume that the bank attracts demand and time deposits from a diverse group of individual depositors. The average balance of the i^{th} individual demand deposit account over the planning horizon can be expressed as:

$$(19) \quad DD_i = B_{DD,i} + u_i, \quad i = 1, \dots, N_{DD},$$

where N_{DD} is the total number of demand deposit accounts, and u_i is a random variable having zero mean and following an otherwise unspecified probability distribution. Since u_i has zero mean, it follows that $B_{DD,i}$ is, in the bank's view, the i^{th} depositor's expected average balance over the planning horizon. We further define:

$$(20) \quad DD_{TOTAL} = \sum_{i=1}^{N_{DD}} (B_{DD,i} + u_i).$$

DD_{TOTAL} is the bank's average total demand deposit stock over the planning horizon. Since we have not specified the joint distribution of the u_i , we cannot fully specify the mathematical form of the distribution of DD_{TOTAL} . We can, however, define the mean of this latter distribution as:

$$(21) \quad DD = \sum_{i=1}^{N_{DD}} B_{DD,i}.$$

¹⁸In this connection see Dewald and Dreese [8].

DD is the bank's expected average demand deposit stock over the planning horizon and is the bank's demand deposit decision variable. We assume that the bank controls DD through deposit interest payments,¹⁹ advertising, and other policies that influence the levels of the individual $B_{DD,i}$ and the total number of depositors maintaining demand deposit accounts at the bank.

Similarly, we express the average balance of the i' th time deposit account as:

$$(22) \quad TD_{i'} = B_{TD,i'} + v_{i'}, \quad i' = 1, \dots, N_{TD},$$

where N_{TD} is the number of time deposit accounts held by the bank, and $v_{i'}$, like u_i , is a random variable having zero mean but following an otherwise unspecified probability distribution. The bank's average total time deposit balance is:

$$(23) \quad TD_{TOTAL} = \sum_{i'=1}^{N_{TD}} (B_{TD,i'} + v_{i'}).$$

The mean of the unspecified distribution of TD_{TOTAL} is then:

$$(24) \quad TD = \sum_{i'=1}^{N_{TD}} B_{TD,i'}.$$

TD is the bank's time deposit decision variable.

Using (20) and (23), the bank's average total deposit balance (i.e., demand plus time deposits) is:

$$(25) \quad D_{TOTAL} = DD_{TOTAL} + TD_{TOTAL} = \sum_{i=1}^{N_{DD}} (B_{DD,i} + u_i) + \sum_{i'=1}^{N_{TD}} (B_{TD,i'} + v_{i'}).$$

¹⁹Since our analysis is general and abstract, the current prohibition of explicit interest payments on demand deposits in the United States is ignored. Time deposit rate ceilings are also ignored.

Our assumptions to this point do not permit us to indicate the form of the distribution of D_{TOTAL} ; however, these assumptions do permit specification of the distribution's mean as $DD + TD$. We now define an additional random variable that will play an important role in subsequent analysis:

$$(26) \quad U = \sum_{i=1}^{N_{DD}} u_i + \sum_{i'=1}^{N_{TD}} v_{i'}$$

U is the absolute deviation of the bank's average total deposit balance, D_{TOTAL} , from its mean value $DD + TD$. The above assumptions imply that U has zero mean; however, the form of its distribution cannot be specified further.

The analysis that follows focuses considerable attention on the risk of deposit variability faced by the bank during the planning period, as measured by the dispersion of the random variable U around its mean value.²⁰ Of great importance in the analysis, we shall assume that the degree of this risk depends systematically on (a) the bank's size and (b) the structure of the bank's liabilities. Further, we shall assume that the bank knows the quantitative character of these relationships and takes explicit account of them in reaching decisions.

Since we have postulated relationships between deposit variability, bank size, and liability structure, it is necessary to indicate the rationale for believing that such relationships exist. We now develop this rationale.

In the context of the above discussion, a useful measure of the deposit variability risk faced by the bank is the standard deviation

²⁰In what follows, the terms "deposit instability" or "stability" will always refer to the degree of this dispersion. It is deposit instability in this sense, rather than in the sense of high deposit turnover rates or velocity, that generates the bank's need for liquidity. See Morrison and Selden [21, p. 12].

of the random variable U . For simplicity, all individual deposit accounts, both demand and time, are assumed to have identical mean balances \bar{B} . It is further assumed that individual demand deposit accounts have identical variance $\text{var}(u)$, and that individual time deposit accounts have identical variance $\text{var}(v)$. Under these conditions, the standard deviation of U is:

$$(27) \quad \sigma_U = \left[N_{DD} \cdot \text{var}(u) + N_{TD} \cdot \text{var}(v) + \sum_{\substack{i, j \\ i \neq j}}^{N_{DD} N_{DD}} \text{cov}(u_i, u_j) \right. \\ \left. + \sum_{\substack{i', j' \\ i' \neq j'}}^{N_{TD} N_{TD}} \text{cov}(v_{i'}, v_{j'}) + \sum_{i, i'}^{N_{DD} N_{TD}} \text{cov}(u_i, v_{i'}) \right]^{\frac{1}{2}},$$

where $\text{cov}(u_i, u_j)$ is the covariance of the i^{th} and j^{th} demand deposit accounts, $\text{cov}(v_{i'}, v_{j'})$ is the covariance of the i'^{th} and j'^{th} time deposit accounts, and $\text{cov}(u_i, v_{i'})$ is the covariance of the i^{th} demand deposit account and the i'^{th} time deposit account. We can now indicate why σ_U is likely to vary with liability structure and bank size.

Liability structure. For present purposes, we define liability structure as the relative allocation of the bank's total deposits between demand and time deposits. For the moment, we assume that bank size, as measured by the expected average total deposit balance $DD + TD$, is constant. We further assume that the bank's depositors distinguish between demand and time deposits with respect to function. Specifically, it is assumed that depositors use demand deposits primarily as a means of payment, but that they use time deposits primarily as a store of wealth.²¹ On these grounds, it is reasonable to assume that the variance of individual

²¹This strong distinction is made for analytical convenience. In reality, the distinction is not absolute but a matter of degree. In this regard see Hicks [12].

demand deposit accounts exceeds the variance of individual time deposit accounts: i.e., $\text{var}(u) > \text{var}(v)$.²² Consider now the effects of a shift in the bank's liability structure to a greater proportion of time deposits. For simplicity, assume that this shift takes the form of individual demand depositors closing their demand deposit accounts and using the funds to open savings accounts. Such transfers reduce the magnitude of the first term in parenthesis on the right side of (27) and increase the magnitude of the second term. With $\text{var}(u) > \text{var}(v)$, the net change in these two terms is negative. This net change tends to reduce overall deposit variability as measured by σ_U . We cannot specify the effect of the postulated deposit transfers on the covariance terms in (27) without detailed knowledge of the underlying joint distribution of individual deposit deviations. In general, however, there is no a priori reason for supposing that resulting changes in these covariance terms will exactly offset the downward effect of the transfers on σ_U just specified. Therefore, we have established theoretical grounds for presuming that deposit variability, as measured by σ_U , varies with changes in the bank's liability structure. The precise character of this relationship in any given case depends on the form of the joint distribution of the bank's individual deposit accounts and on the manner in which changes in liability structure affect this distribution. The particular assumptions we have made suggest that, with total deposit volume $DD + TD$ constant, σ_U is inversely related to the ratio of time to total deposits.

²²The validity of this assertion is, of course, an empirical question. Limited evidence indicates that time deposits are, in fact, more stable than demand deposits. See Morrison and Selden [21, pp. 12-19].

Bank size. Equation (27) can also be used to analyze the relationship between deposit variability and bank size, where bank size is measured by the bank's expected average total deposit volume $DD + TD$.

In general, changes in the bank's expected deposit volume can result from (a) changes in the average balances held by existing individual demand and time depositors, (b) changes in the number of individual demand and time deposit accounts held by the bank, or (c) some combination of the above.²³ Suppose first that $DD + TD$ increases due to increases in existing individual deposit balances. The variance and covariance terms that comprise σ_U are measures of the dispersion of individual deposit balances around their respective means and of the codispersion of pairs of deposit balances around their respective means. The magnitudes of particular variance and covariance terms are likely to change following increases in the corresponding means of individual deposit balances. For example, the variance of a particular demand deposit account might increase following an increase in the account's mean balance if the increased balance is accompanied by unsynchronized receipt and payment transactions of greater absolute size. As indicated by (27), any such changes in individual variance and covariance terms directly affect the value of σ_U . It follows that deposit variability as measured by σ_U is likely to change following an increase in $DD + TD$ caused by increases in the mean balances of existing accounts. The exact functional relationship between σ_U and $DD + TD$ in any given case depends upon the precise manner in which the variance and covariance terms comprising σ_U change following

²³For the moment, we ignore changes in liability structure that may accompany changes in total deposit volume. This possibility will be considered below.

a given increase in deposit volume.

Alternatively, suppose that $DD + TD$ increases due to the opening of new demand or time deposit accounts at the bank. Such an occurrence adds a new variance term to σ_U for each of the new deposit accounts, and (if the total number of accounts held by the bank is at all sizable) a large number of new covariance terms. As (27) again indicates, these new variance and covariance terms directly affect the value of σ_U . Therefore, σ_U is likely to change following an increase in the bank's deposit volume caused by an increase in the number of deposit accounts the bank holds.

We have now indicated why it is reasonable to postulate that deposit variability faced by the bank, as measured by σ_U , varies with changes in the bank's expected average deposit volume. The exact quantitative character of this relationship in any given case depends ultimately on the nature of changes in the joint distribution of individual deposit account balances that accompany any given change in the bank's deposit volume.²⁴ The discussion above suggests, however, that σ_U is more likely to vary directly than inversely with deposit volume.

²⁴Our discussion has been concerned solely with the relationship between deposit volume and absolute deposit variability, as measured by σ_U . Equation (27) can also be used to show that relative deposit variability is likely to vary with bank size. For this purpose, an appropriate measure of relative variability is:

$$\Omega = \frac{\sigma_U}{DD+TD} .$$

Suppose that $DD + TD$ increases by some proportionate amount due to an increase in the number of accounts the bank holds. No a priori reason exists for expecting the accompanying change in σ_U discussed in the text to be proportionately equal to the change in deposit volume. Therefore, it is reasonable to presume that, in general, Ω varies with deposit volume. Most recent empirical studies of deposit variability focus on the relationship between relative deposit variability and bank size. Several of these studies suggest that relative deposit variability declines as bank size increases. See Gramley [10, pp. 41-53], Rangarajan [24], and Struble and Wilkerson [25].

Joint effects of liability structure and bank size. To this point, we have analyzed the effects of liability structure and deposit volume on σ_U separately. That is, we first analyzed the effect of a change in liability structure on σ_U under the assumption that total deposit volume was fixed. We then analyzed the effect of a change in deposit volume on σ_U without considering changes in liability structure likely to accompany the change in deposit volume. Unless a given change in deposit volume is caused by proportionately equal changes in demand and time deposits, some alteration of liability structure must accompany the change in deposit volume. We can reasonably presume that the quantitative effect on σ_U of a given change in deposit volume depends on the particular change in liability structure that occurs. This last proposition can be defended by a final example. Suppose that the bank's expected total deposit volume increases by some given amount. Further, assume that this increase results entirely from an increase in the number of demand deposit accounts the bank holds: that is, the increase in deposit volume is accompanied by a shift of liability structure in favor of demand deposits. The new demand deposits add additional variance and covariance terms to (27), causing, as indicated above, some change in σ_U . Alternatively, assume that the increase in deposit volume results entirely from an increase in the number of time deposit accounts: that is, the increase in deposit volume is accompanied by a shift of liability structure in favor of time deposits. The new time deposit accounts then add additional variance and covariance terms to σ_U in (27). As indicated above, it is reasonable to assume that the variance and covariance properties of demand deposit accounts differ systematically from the corresponding properties of time deposit accounts. It follows that these two alternative occurrences

will have systematically different quantitative effects on σ_U . For example, if the variance and covariance of time deposit accounts is less than the variance and covariance of demand deposit accounts, an increase in time deposits contributes less to the bank's deposit variability as measured by σ_U than a quantitatively equal increase in demand deposits. This argument can easily be extended to apply to any proportionate mixture of demand and time deposit change comprising a given change in total deposit volume.

We have now completed our defense of the assumption that the variability of the bank's average total deposit balance over the planning period depends systematically on the structure of the bank's liabilities and on the bank's size as measured by deposit volume. It should be emphasized that the discussion has not led to specific conclusions regarding either the directions or quantitative characteristics of these relationships. Rather, the discussion has suggested that the nature of these relationships depends on the exact form of the joint probability distribution of the bank's individual demand and time deposit account balances in any given case, and on the manner in which changes in either DD or TD affect this distribution. Some of the examples given in developing the discussion, however, were designed to suggest that under a wide variety of specific conditions, deposit variability as measured by σ_U is likely to increase with increases in deposit volume, but to increase at a slower rate following a given increase in the bank's time deposit stock than following an identical increase in the bank's demand deposit stock.

We must now express the postulated relationships between deposit variability, bank size, and liability structure in a form appropriate for

inclusion in the model. In the preceding discussion, the absolute dispersion of the bank's average total deposit stock distribution was measured by σ_U . Due to the inventory theoretic character of subsequent model construction, it is convenient to introduce an alternative measure of the deposit variability risk that the bank faces. This alternative measure is the maximum range of possible variation in the bank's average total deposit balance to which the bank attaches a nonzero probability. We assume that the expected average total balance $DD + TD$ is the midpoint of this range, and we define the width of the range as $2K$. To illustrate, if the expected average balance is \$100 and K is \$10, the probability distribution of the bank's average total balance (i.e., the distribution of the random variable D_{TOTAL} defined by (25)) has limits \$90 and \$110 or, equivalently, the probability distribution of U has limits $-\$10$ and $\$10$. As this illustration suggests, we assume that the range specified by any given value of K is generally less than the widest conceivable range of deposit variation. In this respect it will be useful to think of K as some multiple of σ_U .

Let us now specify K somewhat more formally. We define:

$$(28) \quad K = K[DD, TD].$$

As indicated above, K and $-K$ ²⁵ are limits to the distribution of the random variable U defined by (26). Equation (28) states that K is a function of the bank's expected demand and time deposit balances. Therefore, in keeping with the discussion of (27), K , like σ_U , is a function of both bank size, as measured by the bank's total deposit volume $DD + TD$, and liability structure, as measured by the relative magnitudes of DD and TD . We do not specify the explicit form of this function; however, we

²⁵The limit $-K$ is comparable to Richard C. Porter's "deposit low." See Porter [23, pp. 37-44].

assume that the bank knows the form of the function precisely.²⁶

This completes the discussion of deposit variability. We turn now to deposit costs.

Deposit costs

The same fundamental conceptual difficulty encountered earlier with respect to lending costs arises in treating deposit costs. That is, because we have defined the dollar stocks of demand and time deposits as bank decision variables, it is necessary to specify the functional relationship between deposit costs and these stocks. As in the case of loans, however, dollar volume is not, in general, the relevant "output" on which all costs directly depend.²⁷ We now outline a general procedure for coping with this problem analytically. Subsequently, we shall specify the bank's cost functions for demand and time deposits.

The types of costs an actual bank incurs in attracting and maintaining deposit accounts fall, roughly, into 4 categories: (a) operating and service costs, (b) promotional and advertising costs, (c) explicit interest payments, and (d) service charges, a negative increment to costs. In what follows, we shall ignore service charges. Empirically, service charges appear to be largely unrelated to actual bank costs or to the factors underlying bank costs.²⁸ Further, for simplicity, we shall

²⁶The discussion of equation (27) implied that the effects on deposit variability of a given change in either DD or TD may vary depending on whether the change in deposit volume results from changes in the average balances of existing accounts, changes in the number of accounts the bank holds, or some combination of the two. Therefore, the assumption that the bank knows the explicit form of (28) implies that the manner in which a given change in either DD or TD occurs is predetermined. We shall return to this point in our discussion of deposit costs.

²⁷See Broaddus [6, pp. 37-44].

²⁸See Bell and Murphy [3].

group categories (b) and (c) above into a single category. Operating and service costs arise from the bank's deposit maintenance activities. These costs depend on the technical characteristics of deposit service production and on competitive conditions in markets for the basic factors (labor and real capital) that the bank uses to produce these services. Both promotional and explicit interest costs, on the other hand, arise from the bank's efforts to attract deposits and reflect competitive conditions within deposit markets.²⁹ We now analyze each of these two remaining cost categories in turn.

Operating and service costs. In reality, the deposit services banks provide individual depositors vary both qualitatively and quantitatively from one deposit account to the next. Clearly, the qualitative character of deposit services differs between deposit categories such as demand and time deposits. Further, service flows vary quantitatively among individual deposit accounts within any deposit category due to differences in account activity (i.e., variations in the number of credit and debit transactions) and differences in account size. In order to abstract from these complexities, we continue to assume that all of the bank's deposit accounts have identical expected average balances \bar{B} . We further assume that the levels of individual account activity are identical across all demand deposits and time deposits, respectively, and that these activity levels are exogenous to the bank.³⁰ On these grounds,

²⁹It is of course true that banks can vary the level of account services, and therefore service costs, as a competitive move to attract deposits away from other institutions. As indicated below, however, we shall assume the services supplied each depositor by the bank are exogenously determined.

³⁰For an empirical index of account activity see Benston [5, pp. 515-516].

the bank is assumed to produce one "unit" of identical demand deposit services for each demand deposit account it holds over the planning period, and, similarly, one unit of identical time deposit services for each time deposit account. The bank produces these service flows using labor and capital inputs in accordance with the technical constraints of production functions for both demand and time deposits. For analytical convenience we assume that these production functions are mutually independent: that is, that demand and time deposit services are not jointly produced. Therefore, we can define the following short-run average service cost function for deposit category i :

$$(29) \quad c_{D_i}^{S'} = c_{D_i}^{S'} [N_{D_i}; \bar{K}_0, \bar{w}_0, \bar{A}_{D_i}, \bar{B}],$$

where:

$c_{D_i}^{S'}$ = the average service cost per "unit" of deposit category i service output,

N_{D_i} = the total output of category i service units = the total number of category i deposits,

\bar{K}_0 = the bank's fixed stock of real capital,

\bar{w}_0 = the (constant) wage rate, and

\bar{A}_{D_i} = an index of the identical account activity levels of category i deposits.

The preceding discussion of deposit variability indicated that the bank controls its expected average demand and time deposit balances by influencing both the number and average size of individual accounts. For expository convenience, assume that the (identical) average balances of individual accounts are exogenous to the bank. The bank then controls its expected average deposit stocks by acting to influence the number of accounts it holds. Therefore, N_{D_i} becomes the bank's deposit category i

control variable, and \bar{B} becomes a model parameter as indicated by (29).

Assume also, again for convenience, that \bar{B} equals unity. If we now

define:

$c_{D_i}^S$ = the average service cost per dollar of the bank's deposit category i balance, and

D_i = the expected average category i balance, our assumptions imply:

$$(30) \quad c_{D_i}^S = c_{D_i}^{S'} ;$$

$$(31) \quad D_i = N_{D_i} .$$

Under these circumstances, we can substitute $c_{D_i}^S$ for $c_{D_i}^{S'}$ and D_i for N_{D_i} in (29), obtaining:

$$(32) \quad c_{D_i}^S = c_{D_i}^S [D_i; \bar{K}, \bar{w}_o, \bar{A}_{D_i}, \bar{B}] .$$

Equation (32) expresses average service costs per dollar of category i deposits as a function of the category i deposit stock held by the bank.

On the basis of these specifications, we may presume that a service cost function of the general form (32) exists for both demand and time deposits. We write these functions, respectively, as:

$$(33) \quad c_{DD}^S = c_{DD}^S [DD; \bar{K}_o, \bar{w}_o, \bar{A}_{DD}, \bar{B}] ;$$

$$(34) \quad c_{TD}^S = c_{TD}^S [TD; \bar{K}_o, \bar{w}_o, \bar{A}_{TD}, \bar{B}] .$$

We do not specify the explicit form of either function. Total service cost functions are then:

$$(35) \quad SC_{DD} = c_{DD}^S [DD] (DD) ;$$

$$(36) \quad SC_{TD} = c_{TD}^S [TD] (TD) .$$

Equations (35) and (36) will enter the objective function of the model.

Promotional, advertising, and explicit interest expenses.³¹

This category of costs arises not from deposit service production but from the bank's attempt to attract deposit funds away from competing financial institutions and money market instruments. It is reasonable to presume that these costs depend directly on the total dollar volume of deposits the bank seeks to attract. We write the bank's average promotional-interest cost function for deposit category i in general form as:

$$(37) \quad r_{D_i} = r_{D_i} [D_i; \alpha_{D_i}],$$

where:

r_{D_i} = average promotional-interest costs per dollar of category i deposits,
and

α_{D_i} = a vector of parameters summarizing the competitive structure of the
category i deposit market within which the bank operates.

Equation (37) is analytically comparable to a factor supply function facing a firm in standard theory. The explicit form of (37) depends on competitive conditions facing the bank in the category i deposit market. These conditions are summarized by α_{D_i} , which is exogenous to the bank.

The bank faces average promotional-interest cost functions of the general form (37) for both demand and time deposits. We write these functions, respectively, as:

$$(38) \quad r_{DD} = r_{DD} [DD; \alpha_{DD}];$$

³¹The reader will recall that, in keeping with the generality of the model, we are ignoring real world restrictions on explicit deposit interest payments.

$$(39) \quad r_{TD} = r_{TD}[TD; \alpha_{TD}].$$

Again, we do not specify the explicit form of either function. Total promotional-interest costs for each category are then:

$$(40) \quad RC_{DD} = r_{DD}DD;$$

$$(41) \quad RC_{TD} = r_{TD}TD.$$

Equations (40) and (41) will enter the objective function of the model.

The Expected Loss Function: Implicit Returns to Reserves and Secondary Reserves and Implicit Deposit Costs

In reality, banks hold a variety of reserve assets (such as vault cash and reserve balances at the central bank) that pay no explicit return. Banks also hold so-called "secondary reserve" assets (such as short-term government securities) that commonly pay explicit returns well below the yields available on other assets. Banks typically hold both reserve and secondary reserve assets in amounts that exceed legal requirements. Rational banks behave in this fashion because, in a world of uncertainty, they attach positive economic value to the liquidity of these assets. Stated differently, reserve assets yield implicit flows of income to the banks that hold them. Previous theoretical analyses of individual bank behavior have employed several procedures to express this implicit reserve asset return in analytically explicit form. The most powerful approach to this problem yet devised is the direct application of formal inventory theory under conditions of uncertainty to a bank's demand for reserves. In what follows we shall expand on earlier work by

applying the inventory approach to secondary as well as primary reserves.³² We shall also apply the approach to deposits and develop certain implicit costs.

As indicated above, the bank of our model faces stochastic variation in its demand and time deposit balances over the planning horizon. We assume that the bank is required by law to meet all deposit withdrawals immediately with acceptable primary reserve assets. Hence, the bank is particularly concerned with the possibility of net deposit outflows during the planning period. Actual banks, of course, face this possibility continuously. For analytical convenience, the bank of the model is assumed to face the threat of deposit withdrawal at only one point in time, toward the end of the planning horizon. We designate this point in time the "moment of adjustment."

In an earlier section, the need to define the bank's decision variables as average values over the course of the planning period was indicated. Consequently, the various random variables we have introduced relating to security prices and the bank's demand and time deposit balances were also defined as planning period averages. For simplicity, we now assume that, in the bank's view, the probability distribution of each stochastic variable at the moment of adjustment is identical to the distribution of the average value of the variable over the entire planning period. That is, if the probability is .30 that the bank's average demand deposit balance over the planning horizon will be \$10 million, then the

³²Throughout this section, the terms "reserves" and "primary reserves" refer to assets (such as vault cash) that the bank can use to meet deposit withdrawals directly. The term "secondary reserves" refers to assets that the bank must first convert to primary reserves in meeting deposit withdrawals. In the present model, the bank's secondary reserves consist entirely of securities as defined earlier.

probability is also .30 that the instantaneous balance will be \$10 million at the moment of adjustment.

The reader will recall that the bank's expected average demand and time deposit stocks are bank decision variables subject to bank control. With this specification in mind, we define a net deposit withdrawal as deviation of the bank's total deposit balance below its expected level $(DD + TD)$ at the moment of adjustment. Suppose that the bank faces a net deposit withdrawal. Stated differently but equivalently, suppose that, at the moment of adjustment, the random variable U falls in the negative portion of its range. We assume that, faced with this situation, the bank can fulfill its deposit obligations in one of four ways: it can (a) meet the withdrawal directly with primary reserve assets such as vault cash, (b) sell securities in exchange for primary reserves, (c) borrow primary reserves at the constant penalty rate n , or (d) employ some combination of the above. In general, the costs associated with each of these alternatives differ.³³ If the bank uses primary reserves on hand, it incurs no loss or cost. If the bank borrows, it incurs a penalty cost at the rate n . The cost of using security sales is unknown to the bank due to the stochastic character of security prices. We assume that if, at the moment of adjustment, the price of a security is less than its initial one dollar value (i.e., if the random variable w defined by (17) falls in the negative portion of its range), the bank records a capital loss at the rate of w percent per withdrawal dollar met by security sales.³⁴

³³We refer here only to the direct and immediate costs of meeting a deposit withdrawal.

³⁴In contrast, it is assumed that the bank ignores the possibility of capital gains from security sales when making its decisions. This asymmetry seems reasonable since most actual banks are probably more concerned about the possibility of a capital loss than the possibility of a capital gain when they liquidate securities.

Therefore, security liquidation presents the bank with the possibility of capital loss at an uncertain rate.

Let us assume that, in meeting a deposit withdrawal, the bank always selects the least costly of the above alternatives first. Therefore, the bank always meets a deposit withdrawal initially with available primary reserves.³⁵ If primary reserve stocks are insufficient to cover the entire withdrawal, the bank meets the remainder by either selling securities or borrowing, whichever is least expensive. If the bank chooses to liquidate securities before borrowing, but primary reserve and security holdings together are insufficient to meet the entire withdrawal, the bank must and will resort to borrowing.

Since the bank is aware that a deposit withdrawal may occur at the moment of adjustment, it must introduce an expression into its objective function which captures, probabilistically, the possibility it may suffer a penalty cost or capital loss flow. We designate this expression the bank's expected loss function. Although somewhat formidable at first glance, it is a straightforward extension of similar functions employed in several of the previous studies cited at the beginning of this paper. We write the function and then discuss its meaning in detail:

$$\begin{aligned}
 (42) \quad EL\{S, R, DD, TD\} = & \int_{-a}^{-n} \int_{-K[DD, TD]}^{-R} n(-U-R)\phi(w)\theta(U)dUdw \\
 & + \int_{-n}^0 \int_{-(R+S(1+w))}^{-R} -w(-U-R)\phi(w)\theta(U)dUdw \\
 & + \int_{-n}^a \int_{-K[DD, TD]}^{-(R+S(1+w))} n(-U-(R+S(1+w)))\phi(w)\theta(U)dUdw,
 \end{aligned}$$

³⁵If $w \geq 0$, we assume the bank first meets a withdrawal with primary reserves.

where R is the bank's stock of primary reserves, $\phi(w)$ is the uniform distribution of the random variable w , $\theta(U)$ is the unspecified distribution of the random variable U , and all other symbols are as previously defined.³⁶

Equation (42) can be explained most conveniently by considering the three terms on the right side of the equation in turn. Each of these terms involves integration with respect to both w and U . For each term, the inner integration is with respect to U where U , as indicated earlier, is the deviation of the bank's total deposit balance from the expected value of the balance at the moment of adjustment. The outer integration for each term is with respect to w , where w is the moment of adjustment deviation of security prices from their expected value. Recalling our assumption that w varies over the range $-a \leq w \leq a$, the three terms can be most clearly interpreted as specifying the bank's expected loss due to deposit withdrawal over particular portions of the range of w : i.e., over particular portions of the range of possible security prices at the moment of adjustment. Let us now indicate the meaning of these terms in detail.

Suppose first that, at the moment of adjustment, (a) a net deposit withdrawal occurs (i.e., U is in the negative portion of its range), (b) security prices are below their expected value (i.e., w is in the negative portion of its range), and (c) $-w < -n$.³⁷ In this case, the first term of (42) is relevant, as indicated by its range of integration with respect to w . With $-w < -n$, borrowing is less costly

³⁶The distributions of w and U are assumed to be independent.

³⁷Throughout this discussion we assume $|a| > n$.

per dollar than security liquidation. Under these conditions, the bank will first give up primary reserves. If the withdrawal exhausts its primary reserve stock, the bank will subsequently borrow.³⁸ As the integrand and inner range of integration for this term indicate, the bank incurs penalty n for each dollar by which net withdrawals $-U$ exceed the bank's primary reserve holdings R .³⁹ The lower limit to the inner integration, $U = -K$, is the maximum withdrawal the bank faces under our deposit variability assumptions. Actual integration over this term yields the bank's expected loss due to deposit withdrawals for this position of the range of w .

Consider now the last two terms of (42). These two terms specify the bank's expected loss from net withdrawals for the portion of the range of w where the algebraic value of $w \geq -n$. Over this portion of the range of w , security liquidation is less costly than borrowing to meet deposit withdrawal obligations. Here, the bank will initially give up primary reserves. If the withdrawal exhausts primary reserves, the bank will then sell securities. If the withdrawal exhausts both primary and secondary reserves, the bank will borrow. The second term of (42) specifies the expected loss from security sales after primary reserves are depleted, and the third term specifies the expected loss from supplemental borrowing should the withdrawal exhaust both primary and secondary reserves.

³⁸If $w = -n$, the bank is assumed to sell securities before borrowing. Therefore, because definite integration is defined over a closed interval, specification of $-n$ as the upper limit of integration for the first term of (42) is not strictly accurate. We ignore this minor difficulty.

³⁹Throughout our discussion of (42) we shall be dealing with the negative portion of the range of U , within which $U < 0$ and $-U > 0$.

Consider the second term. The integrand and inner range of integration for this term indicate that the bank incurs capital loss penalty w for each dollar of deposit withdrawal exceeding the bank's primary reserve stock, up to the point where the bank's security portfolio as well as its primary reserve stock is exhausted: i.e., up to the point where $U = -(R+S(1+w))$.⁴⁰ For this term, integration with respect to w is restricted to the range $-n \leq w \leq 0$, because the bank does not incur a capital loss from security liquidation if $w > 0$.

Consider now the third term. The integrand and inner range of integration for this term indicate that the bank incurs borrowing penalty n for each withdrawal dollar exceeding $(R+S(1+w))$, where $(R+S(1+w))$ specifies the value of the bank's reserve and secondary reserve balance at the moment of adjustment. In contrast to the second term, integration with respect to w here is over the range $-n \leq w \leq a$, because the bank incurs penalty n when all primary and secondary reserves are exhausted, regardless of the prevailing market price of securities.

To summarize, (a) actual integration of the three terms comprising (42) under any particular specifications of the probability distribution $\theta(U)$ and the function $K = K[DD, TD]$, and (b) summation of the results of these integrations yields the bank's total expected loss due to the possibility of a net deposit withdrawal at the moment of adjustment. This total expected loss will enter the objective function of the model as a negative increment to the bank's expected change in equity over the planning horizon.

⁴⁰On the basis of our earlier assumptions, $S(1+w)$ is the value of the bank's total security portfolio at the moment of adjustment. Hence $(R+S(1+w))$ is the value of the bank's primary and secondary reserve holdings at the moment of adjustment.

We are now in a position to specify (a) the implicit returns to primary reserves and secondary reserves and (b) the implicit costs of demand and time deposits that arise from the possibility of a net withdrawal at the moment of adjustment. As indicated by (42), the total expected loss EL is a function of the four bank decision variables R , S , DD , and TD . Partial differentiation of (42) with respect to any one of these variables indicates the marginal change in EL resulting from a marginal adjustment of the decision variable in question. We cannot, in general, specify the signs of these partial derivatives. That is, we cannot generally indicate whether an increase in one of the decision variables increases EL , decreases EL , or leaves EL unchanged. The qualitative character of these effects depends in any given case on the explicit character of the unspecified functions $\phi(U)$ and $K[DD, TD]$ which appear in (42). We can, however, make reasonable presumptions regarding the respective directions of these effects that would be valid under a wide variety of explicit specifications of $\theta(U)$ and $K[DD, TD]$.

Consider first the effects on EL of marginal changes in primary reserves and securities, as given by $\frac{\partial EL}{\partial R}$ and $\frac{\partial EL}{\partial S}$, respectively. It is reasonable to presume that, ceteris paribus, an increase in the stock of either of these two assets would reduce the bank's expected loss due to net deposit withdrawals at the moment of adjustment. As an intuitive justification for this presumption, consider the effect of a marginal increase in the primary reserve stock R on each of the terms comprising EL in (42). With K unchanged, an increase in R reduces the range of integration with respect to U for both the first and the third terms. That is, an increase in R (a) reduces the range of possible deposit withdrawals over which such withdrawals force the bank to borrow and

(b) reduces the amount the bank would be forced to borrow to meet any given net withdrawal. With respect to the second term of (42), an increase in R does not reduce the extent of the range of possible withdrawals over which the bank would be forced to borrow; however, an increase in R shifts this range outward from the mean value of U to encompass larger (and therefore, under a variety of reasonable specifications of $\theta(U)$, less probable) net withdrawals. To summarize, this illustration suggests that a marginal increase in R would probably reduce the magnitudes of all three terms comprising EL and therefore reduce EL . A similar although more complicated argument can be given with respect to $\frac{\partial EL}{\partial S}$. We now define:

(43) - $\frac{\partial EL}{\partial R}$ = the implicit marginal return to primary reserves;

(44) - $\frac{\partial EL}{\partial S}$ = the implicit marginal return to securities.

On the basis of our presumption that $\frac{\partial EL}{\partial R} < 0$ and $\frac{\partial EL}{\partial S} < 0$, it follows that the implicit marginal returns to reserves and securities are positive.⁴¹

Consider now the effects on EL of marginal changes in the bank's expected average demand and time deposit balances as given, respectively, by $\frac{\partial EL}{\partial DD}$ and $\frac{\partial EL}{\partial TD}$. Our discussion in an earlier section indicated that we can expect increases in either DD or TD to increase total deposit variability as measured by σ_U and therefore by $|K|$, where K and $-K$ are the

⁴¹The argument just given requires $|R| < |K|$ and $|R+S(1+w)| < |K|$. These conditions simply state that neither the bank's primary reserve balance nor its total primary and secondary reserve balance exceeds the maximum withdrawal that the bank considers possible. We shall indicate below that where a local solution to the model exists, both of the marginal returns just defined must be positive.

limits to the distribution of U . On these grounds, we can reasonably presume that, ceteris paribus, an expansion of either DD or TD increases the bank's expected loss due to net deposit withdrawals at the moment of adjustment. Let us now defend this last presumption. An increase in either DD or TD has two distinguishable effects on the terms comprising (42). First, by increasing $|K|$ with R and S constant, an increase in either DD or TD extends the range of integration with respect to U for the first and third terms. That is, deposit expansion extends the range of possible net withdrawals over which the bank must borrow. This effect tends to increase EL . Second, increases in DD or TD alter the form of $\theta(U)$.⁴² It is likely that the increased range of U resulting from expanded deposit volume would reduce the probability of any given net withdrawal. Hence, this second effect of augmented deposit volume would probably tend to reduce EL . The total effect of an increase in DD or TD on EL represents the net result of the two opposing effects just outlined. We can reasonably presume that, under a variety of particular specifications of $K[DD, TD]$ and $\theta(U)$, the former effect would outweigh the latter effect, with the result that increases in DD or TD would cause EL to rise.

We define:

(45) $\frac{\partial EL}{\partial DD}$ = the implicit marginal cost of demand deposits;

(46) $\frac{\partial EL}{\partial TD}$ = the implicit marginal cost of time deposits.

⁴²From our earlier discussion of deposit variability, the reader will recall that the form of $\theta(U)$ depends on the joint distribution of the individual account deviations u_i and v_i , and that the form of this joint distribution changes with increases in deposit volume. Therefore, in general, the form of $\theta(U)$ varies with deposit volume. That is, $\theta(U)$ is itself a function of DD and TD.

Since we have presumed $\frac{\partial EL}{\partial DD} > 0$ and $\frac{\partial EL}{\partial TD} > 0$, both of these marginal costs are positive. One additional point should be made. In our earlier discussion of deposit variability we concluded that the respective effects of changes in (a) demand and (b) time deposit volume on deposit variability faced by the bank differ. In the present context, the implication of this conclusion is that the quantitative characteristics of $\frac{\partial K}{\partial DD}$ and $\frac{\partial K}{\partial TD}$ diverge. Since differentiation of EL involves differentiation of K, it follows that, in general, the quantitative characteristics of $\frac{\partial EL}{\partial DD}$ and $\frac{\partial EL}{\partial TD}$ diverge. That is, the implicit costs of demand and time deposits differ due to the non-identical effects of changes in the respective balances of the two deposit categories on deposit variability. We shall return to this point below.

We have now specified (a) the bank's expected loss due to the possibility of a net deposit withdrawal at the moment of adjustment and (b) implicit marginal returns and costs that accrue to the bank as a result of this expected loss. This completes the development of the objective function. The next section closes and solves the model.

III. Solution of the Model

From the discussion in the preceding section, we can rewrite objective function (8) in detailed form as:

$$\begin{aligned}
 (47) \quad E(\Delta NW) &= r_L[L; \alpha_L, \bar{Z}](L) + \bar{r}_B(B) + \bar{r}_S(S) \\
 &\quad - c_{DD}^S[DD; \bar{A}_{DD}, \bar{B}](DD) - c_{TD}^S[TD; \bar{A}_{TD}, \bar{B}](TD) \\
 &\quad - r_{DD}[DD; \alpha_{DD}](DD) - r_{TD}[TD; \alpha_{TD}](TD) \\
 &\quad - EL[S, R, DD, TD].
 \end{aligned}$$

The bank seeks to maximize (47) subject to the balance sheet identity constraint:

$$(48) \quad L + B + S + R = DD + TD + NW_{t-1}^{43}$$

We perform the optimization using the standard Lagrangian technique. Omitting model parameters for notational simplicity, the first-order conditions for a local maximum are:

$$\begin{aligned} \text{a)} \quad & \frac{dr_L[L]}{dL}(L) + r_L[L] = \lambda \\ \text{b)} \quad & \bar{r}_B = \lambda \\ \text{c)} \quad & \bar{r}_S - \frac{\partial EL[S, R, DD, TD]}{\partial S} = \lambda \\ \text{d)} \quad & - \frac{\partial EL[S, R, DD, TD]}{\partial R} = \lambda \end{aligned}$$

$$\begin{aligned} (49) \quad \text{e)} \quad & \left[\frac{dc_{DD}^S[DD]}{dDD}(DD) + c_{DD}^S[DD] \right] + \left[\frac{dr_{DD}[DD]}{dDD}(DD) + r_{DD}[DD] \right] \\ & + \frac{\partial EL[S, R, DD, TD]}{\partial DD} = \lambda \\ \text{f)} \quad & \left[\frac{dc_{TD}^S[TD]}{dTD}(TD) + c_{TD}^S[TD] \right] + \left[\frac{dr_{TD}[TD]}{dTD}(TD) + r_{TD}[TD] \right] \\ & + \frac{\partial EL[S, R, DD, TD]}{\partial TD} = \lambda \\ \text{g)} \quad & L + B + S + R - DD - TD - NW_{t-1} = 0. \end{aligned}$$

Equations (49a)-(49g) form a system in the six decision variables and the Lagrange multiplier λ . Solution of the system yields the bank's desired average balance sheet position over the planning period, given the values of the model's parameters and the explicit forms of the various unspecified functions that appear in objective function (47). The solution, in any given case, simultaneously establishes three

⁴³All elements in the identity are expected planning period averages. NW_{t-1} is the bank's equity at the beginning of the period.

fundamental characteristics of the bank's desired balance sheet: (a) the relative allocation of funds among alternative assets, (b) liability structure, and (c) the scale of the bank's operations as measured by total deposits. The interdependence of these decisions is directly implied by conditions (49). To illustrate, consider a model parameter shift that alters the bank's desired demand deposit balance. In general, such a shift would directly affect the bank's optimal operating scale and desired liability structure. Moreover, because we have specified the bank's expected loss due to deposit withdrawals as a function of demand deposit volume, the marginal implicit returns to reserves and securities which appear in conditions (49c) and (49d), and therefore the bank's optimal security and reserve balances, are functions of desired demand deposit volume. Therefore, a change in the optimal demand deposit stock would cause a corresponding change in the composition of the bank's desired asset portfolio.

System (49) possesses a clear economic interpretation. Considering each of the component equations in turn, the left side of (49a) is the marginal expected net return on loans. The left sides of (49b), (49c), and (49d) are the marginal returns to bonds, securities, and reserves, respectively. The marginal return to reserves, $-\frac{\partial EL}{\partial R}$, consists entirely of the implicit return derived and described in the preceding section. The marginal return to securities contains both an explicit component, the coupon rate \bar{r}_S , and the implicit component $-\frac{\partial EL}{\partial S}$. The left sides of (49e) and (49f) are the marginal costs of demand and time deposit balances, respectively. In both cases, the first two terms are the explicit marginal costs arising from service and promotional-interest expenses, and the last term is the implicit marginal cost arising from

the expected loss function. Equations (49a)-(49f) state that, at a maximum, the total marginal return of each asset and the total marginal cost of each liability all equal λ and hence are mutually equal. That is, the rational bank invests among alternative assets and acts to attract funds of alternative liability form so as to equate, at the margin, all return and cost flows that arise in connection with each individual asset and liability during the planning period. This is the basic general result of the model and is analytically comparable to the equilibrium conditions in the standard theory of the firm.

Given restrictions on the form of objective function (47), any number of comparative statics experiments are conceivable. Such experiments would analyze the effects of specific parameter changes on the solution of system (49). In this connection, we might briefly indicate the relationship of the present model to the money supply function literature by pointing out that the functional relationship between the solution value for DD (the bank's desired demand deposit balance) and the parameters of the model is a conceptually proper microsupply function for money.⁴⁴

Before proceeding to second-order conditions, we briefly note certain additional characteristics of system (49). Since the bond return \bar{r}_B is positive, (49b) implies $\lambda > 0$. With $\lambda > 0$, (49d) implies that where a local maximum exists, the implicit return to primary reserves, $-\frac{\partial EL}{\partial R}$, must also be positive. Further, if, as one would expect, the coupon yield \bar{r}_S on securities is less than \bar{r}_B , (49c) implies that the implicit return to securities, $-\frac{\partial EL}{\partial S}$, is also positive.

⁴⁴See Kareken [16, pp. 1709-1710].

Second-order conditions insuring that the solution of (49) is indeed a maximum can be expressed as restrictions on the algebraic signs of a sequence of bordered Hessian determinants. The determinants in this sequence are of continuously increasing dimension and involve second-order partial derivatives of the Lagrangian expression used to solve the model. We shall not attempt a full analysis of second-order conditions; however, the first determinant of the sequence can be used to derive several restrictions relevant to economic interpretation of the model's solution. This first determinant, of dimension 3, has the following general form:

$$(50) \quad |V| = \begin{vmatrix} V_{ii} & V_{ij} & g_i \\ V_{ji} & V_{jj} & g_j \\ g_i & g_j & 0 \end{vmatrix},$$

where the V_{ij} are second-order partial derivatives of the Lagrangian expression with respect to the i^{th} and j^{th} decision variables, $i \neq j$. For a maximum, we must have $|V| > 0$. Since i and j may refer to any decision variable, we can derive restrictions by selecting decision variables in pairs and computing the resulting determinant $|V|$. Using this procedure we have:

$$(51) \quad \frac{d \left[\frac{dr_L[L]}{dL}(L) + r_L[L] \right]}{dL} < 0.$$

Condition (51) states that, at a maximum, the marginal return on loans must be a declining function of loan volume. Similar conditions hold for the marginal returns to reserves and securities. That is:

$$(52) \quad \frac{\partial \left[\bar{r}_S - \frac{\partial EL}{\partial S} \right]}{\partial S} < 0;$$

$$(53) \quad \frac{\partial \left[-\frac{\partial EL}{\partial R} \right]}{\partial R} < 0.$$

Further:

$$(54) \quad \frac{\partial \left[\frac{dc_{DD}^S[DD]}{dDD}(DD) + c_{DD}^S[DD] + \frac{dr_{DD}[DD]}{dDD}(DD) + r_{DD}[DD] + \frac{\partial EL}{\partial DD} \right]}{\partial DD} > 0.$$

Condition (54) states that, at a maximum, total marginal demand deposit costs must be an increasing function of demand deposit volume. A similar condition holds for time deposits.

With these preliminary remarks concerning the solution, we may develop the economic content of the solution in somewhat greater detail. Because of the model's generality, it possesses a variety of implications regarding (a) the operational practices of actual banks and (b) public regulatory policies toward banks. Many of these implications can only be derived by first restricting the model and introducing explicit assumptions concerning the form of its component functions. Some representative experiments of this nature are carried out in the broader study underlying this paper.⁴⁵ The discussion below is confined to several results that follow directly from first-order conditions (49).

Marginal Deposit Costs

In the preceding section, we postulated certain explicit and implicit cost flows arising from the bank's deposit activities. We divided the explicit cost flows for both demand and time deposits into

⁴⁵See Broaddus [6, Chs. 4-6].

two components: (a) operating-service costs and (b) promotional-interest costs. Further, we specified (3) implicit deposit costs arising from the possibility of net deposit withdrawals during the planning period. It is well known that explicit interest rates paid by actual banks in the United States on demand and time deposits, respectively, differ due to legal restrictions on such payments. Available data suggest that systematic differences also exist between total service-promotional-interest outlays by banks for the two types of deposits. That is, the sum of cost flows (a) and (b) above generated by demand deposits differs systematically from the corresponding flow generated by time deposits.⁴⁶ The present model suggests several factors that might account for these differences, including dissimilar competitive conditions in the two deposit markets⁴⁷ and characteristically different average account activity levels. In terms of the model, such divergences are captured by differences between corresponding parameters appearing in cost functions (33)-(34) and (38)-(39), respectively.

Our model also suggests an additional factor that might account for variations in deposit expenditures: namely, systematic differences between demand and time deposit variability. To isolate this factor, assume for the moment that (a) average service cost functions (33) and

⁴⁶This difference has been established using data developed through the Federal Reserve Functional Cost Analysis Program. Specifically, Klein [17, pp. 216-217] cites 1967 data from this source which indicate that, for 769 small banks, total cost rates net of service charges but including all permitted interest payments averaged 1.6 percent for demand deposits and 4.3 percent for time deposits. Comparable data compiled by the Federal Reserve Bank of Cleveland for small banks in the Fourth Federal Reserve District during 1966 yielded rates of 2.2 percent and 4.0 percent, respectively.

⁴⁷See Klein [17, p. 217].

(34) are identical and (b) average promotional-interest cost functions

(38) and (39) are identical. That is:

$$(55) \quad c_{DD}^S [DD] \equiv c_{TD}^S [TD];$$

$$(56) \quad r_{DD} [DD] \equiv r_{TD} [TD].$$

These assumptions eliminate the possibility of divergent costs due to differences in deposit market structure or account activity.

Let us now focus our attention on the implicit marginal deposit costs $\frac{\partial EL}{\partial DD}$ and $\frac{\partial EL}{\partial TD}$ that appear in first-order condition equations (49e) and (49f). It was suggested above that the variability of individual demand deposit accounts held by the bank differs from the variability of individual time deposit accounts. To capture this distinction formally, assume that each of the random variables u_i defined by (19) is uniformly distributed on the interval $-k \leq u_i \leq k$, where k is a constant. Assume further that each of the random variables v_i is uniformly distributed on the interval $-pk \leq v_i \leq pk$, where p is a constant, $0 < p < 1$. These assumptions imply that (a) all demand deposit accounts have identical ranges of variation, (b) all time deposit accounts have identical ranges of variation, and (c) the range of time deposit account variation is less than the range of demand deposit account variation. If, for convenience, we continue to assume that all deposit accounts have mean balance $\bar{B} = 1$, it follows that the random variable U defined by (26) varies on the range:

$$(57) \quad -k(DD + pTD) \leq U \leq k(DD + pTD).$$

We have previously noted that K and $-K$ are the limits to the distribution of U . Therefore, (57) specifies the form of the heretofore unspecified function (28) as:

$$(58) \quad K = k(DD + pTD).$$

Hence, we can now substitute the expression on the right side of (58) for K in expected loss function (42).

In order to compare $\frac{\partial EL}{\partial DD}$ and $\frac{\partial EL}{\partial TD}$, it is necessary to specify the form of the distribution $\theta(U)$ that appears in the expected loss function. Let us pick two extreme alternatives. First, if all of the individual deposit deviation variables u_i and v_i are perfectly correlated, then, using our assumptions in the preceding paragraph, U follows the uniform distribution:

$$(59) \quad \theta(U) = \frac{1}{2K} = \frac{1}{2k(DD + pTD)}.$$

Alternatively, if the u_i and v_i are mutually independent, then U follows the normal distribution:

$$(60) \quad \theta(U) = \frac{1}{\sigma_U \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{U}{\sigma_U} \right)^2},$$

where:

$$(61) \quad \sigma_U = [DD + p^2 TD]^{\frac{1}{2}} \left(\frac{K}{\sqrt{3}} \right).$$

We can now derive the results we are seeking. If we substitute (a) the right side of (58) for K in expected loss function (42) and (b) either the distribution (59) or the distribution (60) for $\theta(U)$ in the same function, the composite-function rule for differentiation implies:

$$(62) \quad \frac{\partial EL}{\partial DD} > \frac{\partial EL}{\partial TD}$$

for all values of DD and TD .⁴⁸ This result states that marginal increases in the bank's demand deposit balance increase EL at a faster rate than equivalent marginal increases in the bank's time deposit balance. This

⁴⁸This assertion is proved in Broaddus [6, appendix C].

result follows directly from the assumption in the present discussion that demand deposits are less stable than time deposits. Since (62) holds under either of the extreme assumptions represented by (59) and (60), we can expect it to hold under a variety of other specifications of $\theta(U)$.

Where (62) holds, first-order conditions (49e)-(49f) imply:

$$\left[\frac{dc_{TD}^S}{dT D} [TD^*] (TD^*) + c_{TD}^S [TD^*] + \frac{dr_{TD}}{dT D} [TD^*] (TD^*) + r_{TD} [TD^*] \right] >$$

$$\left[\frac{dc_{DD}^S}{dD D} [DD^*] (DD^*) + c_{DD}^S [DD^*] + \frac{dr_{DD}}{dD D} [DD^*] (DD^*) + r_{DD} [DD^*] \right],$$

where DD^* and TD^* are solution values for the bank's deposit decision variables DD and TD , and the terms on the left and right sides of the inequality are the marginal costs arising from service-promotional-interest expenses for time and demand deposits, respectively. In conjunction with (55)-(56), result (63) states that, at an optimal balance sheet position, a bank operating under the conditions outlined in this section would be willing to incur higher marginal service-promotional-interest expenses for time deposits than for demand deposits, even where the underlying service and promotional-interest cost functions characterizing the two deposit categories are identical. This result follows directly from the assumption that the bank anticipates greater stability in its time deposit accounts than in its demand deposit accounts during the planning period. As implied by (62), the greater stability of time deposit accounts means that time deposits present the bank with a less compelling inducement to hold primary and secondary reserve assets having low yields. In this sense, time deposits are more "productive" than

demand deposits from the standpoint of the bank. Consequently, the bank can afford to incur higher marginal costs to attract them.

Nondeposit Sources of Funds

To this point, the sources of bank funds have been restricted to demand and time deposits. This is obviously unrealistic. Banks in the real world obtain funds from a variety of nondeposit sources. In recent years these sources have included commercial paper issued through holding company affiliates, a variety of other domestic financial instruments, and Eurodollar borrowings. For our purposes, the distinguishing characteristic of these liabilities is that, in contrast to deposits, they do not generally present the risk of unanticipated withdrawal. For simplicity, we group these nondeposit liabilities under the heading "borrowed funds" and denote this liability category by the symbol BF. It is assumed that the bank is not required to repay funds in this category until some point in time following the close of the planning period. BF can be treated as an additional bank decision variable. For convenience, we assume that the only expense the bank incurs in obtaining borrowed funds is an explicit interest charge paid to the lender. We further assume that the average interest charge is a function of the amount borrowed:

$$(64) \quad r_{BF} = r_{BF}[BF; \alpha_{BF}],$$

where α_{BF} is a vector of parameters summarizing competitive conditions facing the bank in the market or markets for borrowed funds. Total costs of borrowed funds are then:

$$(65) \quad RC_{BF} = r_{BF}BF.$$

It is a simple matter to add borrowed funds to the model by adding (65) as a negative increment to objective function (47) and constraint (48). Solution of the augmented model yields first-order conditions consisting of system (49) plus the additional equation:

$$(66) \quad \left[\frac{dr_{BF}[BF]}{dBF}(BF) + r_{BF}[BF] \right] = \lambda.$$

The left side of (66) is simply the marginal cost of borrowed funds. Together with the original first-order conditions (49), (66) implies that to maximize its planning period return the bank assumed nondeposit liabilities up to the point where their marginal cost equals the marginal cost of deposit liabilities and the marginal return to assets.

Adding borrowed funds to the model produces two interesting results. First, from (66), (49e), and (63), we have:

$$(67) \quad \left[\frac{dr_{BF}[BF^*]}{dBF}(BF^*) + r_{BF}[BF^*] \right] > \left[\frac{dc_{TD}^S[TD^*]}{dTD}(TD^*) + c_{TD}^S[TD^*] + \frac{dr_{TD}[TD^*]}{dTD}(TD^*) + r_{TD}[TD^*] \right] > \left[\frac{dc_{DD}^S[DD^*]}{dDD}(DD^*) + c_{DD}^S[DD^*] + \frac{dr_{DD}[DD^*]}{dDD}(DD) + r_{DD}[DD^*] \right].$$

This result states that the bank is willing to pay more in interest charges for borrowed funds at the margin than it is willing to pay in service-promotional-interest outlays for either time or demand deposits. The bank accepts higher marginal costs for nondeposit liabilities because funds derived from these liabilities cannot be withdrawn during the planning period and therefore do not contribute to the expected losses specified by

(42). Inequality (67) states that, at a maximum, the marginal costs that the bank actually pays out for alternative liabilities stand in inverse relation to the marginal contribution of each liability to expected losses.

Second, the introduction of borrowed funds changes the optimal scale of the bank's operations. This can be seen by studying the first-order conditions before and after the introduction of borrowed funds. Because the bond return \bar{r}_B is constant and BF does not enter any first-order equation of the augmented solution other than (66), the two solutions are identical except that in the augmented solution optimal bond holdings increase by an amount equal to the volume of borrowed funds added to the balance sheet. In addition to the change in scale, this result also implies that the optimal ratio of total loan and investment assets ($L^* + B^*$) to total reserve and secondary reserve assets ($S^* + R^*$) increases, a result consistent with the reduced withdrawal risk per dollar of total liabilities. In this sense, the introduction of non-deposit liabilities is similar to a technical innovation in the standard theory of the firm.

Lending Behavior of the Bank

Students of banking and bank regulatory agencies are particularly concerned with bank lending activity because bank loans constitute a significant portion of total credit available to individual consumers and small business firms. The conditions that determine the volume of bank lending are of obvious interest to policymakers, since it may be possible to affect bank lending by influencing these conditions.

We can use first-order conditions (49) to derive the determinants of a bank's desired loan volume in the context of the present model. Conditions (49a) and (49b) indicate that the bank allocates available resources to loans up to the point where marginal loan revenue equals the constant bond return \bar{r}_B .⁴⁹ This condition is depicted graphically by Figure 2, where the downward sloping curve is the bank's

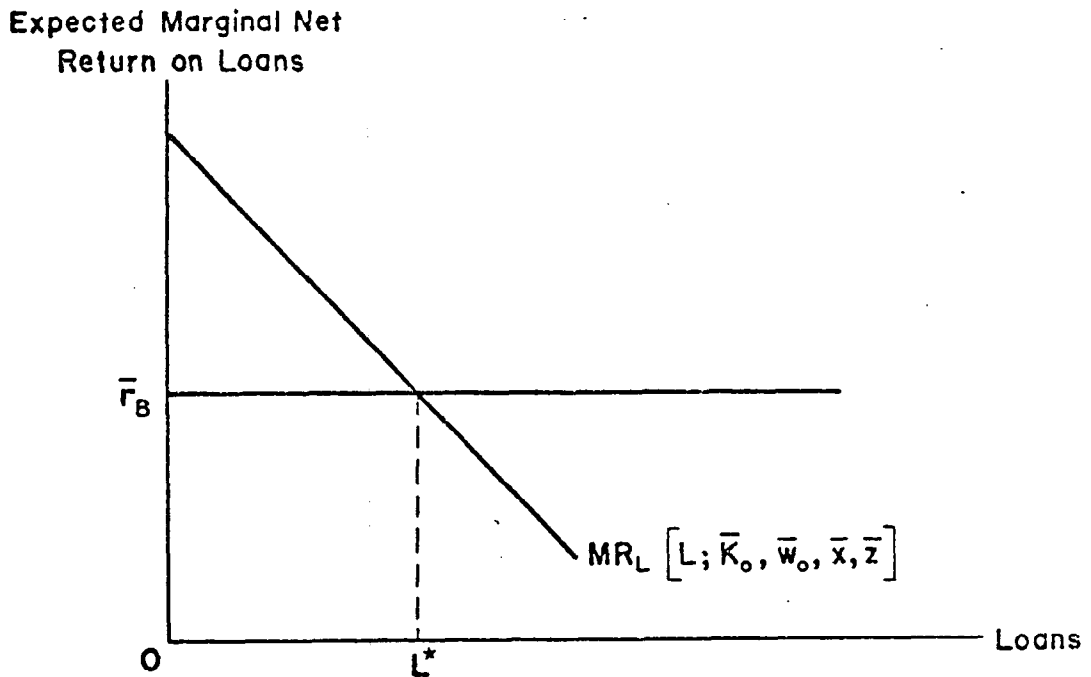


FIGURE 2

marginal loan revenue, and the horizontal line represents the constant bond yield. The bank's desired loan volume L^* is established by the intersection of these two lines. Figure 2 implies that L^* can be altered (a) by policies that influence \bar{r}_B or (b) by policies that affect the

⁴⁹The reader will note that no decision variable other than L appears in first-order conditions (49a)-(49b). Therefore, changes in the optimal scale of the bank's operations occasioned by changes in desired liability stocks have no effect on the bank's desired loan volume. This result follows from the assumption that bonds are in perfectly elastic supply to the bank.

parameters of the marginal loan revenue curve and hence the position of the curve.

As an example, consider a policy that might affect the bank's lending activity by influencing the parameter \bar{z} in the marginal loan revenue function. The reader will recall from the discussion of (10) that \bar{z} specifies the default risk characteristics of the bank's loan customers. Consider a bank facing loan applications for the purpose of home improvements from several isolated potential borrowers, all of whom reside in a given low income neighborhood. The bank is likely to consider the default risk associated with these applications relatively high and scale its lending accordingly. Under these circumstances, several alternative government policies might alter the bank's assessment of the risk it would incur by granting the loans. Obviously, the bank's risk would decline if a government agency agreed to insure the loans. As an alternative to loan insurance, a policy might be designed to coordinate rehabilitation throughout the neighborhood. Such a policy, by reducing externalities, might increase the probability that individual home improvements would produce increased property values. Under these conditions, the bank might consider the default risk associated with individual loan applications less than in the absence of such a policy. In terms of the model, the result would be a change in the parameter \bar{z} , an upward shift of the marginal loan revenue function, and an increased volume of lending.⁵⁰

⁵⁰Broaddus [6, Ch. 6] analyzes in detail the effects of policies designed to influence competitive conditions in loan markets as represented by the parameter α_L in (11).

IV. Conclusion

In this paper we have constructed and solved a general, static model of individual bank balance sheet management. Under the assumption that the bank acts to maximize the return to equity, solution of the model indicated that the external conditions specified by the model's parameters simultaneously determine the bank's desired asset and liability structures and the optimal scale of bank operations. The interdependence of these decisions resulted largely, although not entirely, from two related aspects of the model's construction: (a) the fact that the risk of net deposit withdrawals during the planning period, as measured by K , is functionally dependent on total deposit volume and deposit structure, and (b) the fact that the bank's expected loss due to the possibility of withdrawals is functionally dependent on both deposit volume and the bank's reserve and secondary reserve balances.

The model obviously has limited operational value in its present highly abstract form. It would have to be modified extensively to serve as the basis for detailed analysis of particular banking issues. The model has the virtue, however, of treating a number of diverse bank decisions within a unified analytical framework. Further, the model demonstrates that these decisions are related, at least in principle, on the basis of generally accepted optimization criteria. Only recently has the individual bank as an economic unit begun to receive the micro-theoretic attention it deserves in view of its pivotal role in modern economies. It is hoped that the model developed here may suggest a useful approach to further research in this field.

APPENDIX

The development of the model in this paper excluded an important element of uncertainty faced by actual banks: namely, uncertainty regarding the volume of future loan demand. In this paper we introduced a net loan revenue function similar to the demand function of standard theory. In constructing this function, we assumed that the bank extends loans to individual customers in a predetermined sequence up to some point where it ceases lending.¹ This approach was useful due to its similarity to the treatment of demand in the standard theory of the non-financial firm.

In reality, however, banks usually attempt to meet as many reasonable requests for loans as possible, particularly from established customers. In this connection, actual bankers use the term "liquidity" to refer to a bank's ability to meet unanticipated loan demand as well as unanticipated deposit losses. Since our model is a stochastic theory of individual bank behavior, it is necessary to consider how the model might be altered to permit explicit treatment of uncertain loan demand under the assumption that the bank seeks to meet all or nearly all loan requests. This appendix develops a procedure for incorporating uncertain loan demand in the bank's objective function and indicates the effect of this modification on the model's solution.

We assume that the model construction in the paper remains in effect except for the portion pertaining to bank lending.² For simplicity,

¹Solution of the model indicated that the bank ceases lending at the point where marginal loan revenue equals the constant bond rate.

²See pp. 11-17.

we continue to assume that all loans outstanding on the day preceding the beginning of the planning period mature on that date, and that all noninterest loan terms including loan size are identical across loans and exogenous to the bank. The new assumptions are as follows. First, the bank faces a finite set of borrowers. Second, the bank attempts to satisfy all loan requests received during the planning period. Third, the loan demand of each borrower is, from the bank's standpoint, a random variable.

We write the loan demand of the i^{th} borrower as:

$$(A1) \quad L_i^D = B_{L,i} + g_i, \quad i = 1, \dots, N_L,$$

where N_L is the number of borrowers, and g_i is a random variable having zero mean but following an otherwise unspecified probability distribution. It follows that $B_{L,i}$ is the amount the bank expects the i^{th} borrower to demand during the planning period. Aggregate planning period loan demand is then:

$$(A2) \quad L_{\text{TOTAL}}^D = \sum_{i=1}^{N_L} (B_{L,i} + g_i),$$

where L_{TOTAL}^D is a random variable. Because we have not specified the form of the joint probability distribution of the g_i , we cannot specify the distribution of L_{TOTAL}^D . We can, however, define the mean of the distribution as:

$$(A3) \quad L = \sum_{i=1}^{N_L} B_{L,i}.$$

L is the bank's loan decision variable under the new specifications. The bank controls L by inducing changes in the individual $B_{L,i}$ through loan rate manipulation. That is, L is a function of the loan rate, r_L :

$$(A4) \quad L = L[r_L; \alpha_L],$$

where α_L is a parameter summarizing the competitive structure of the loan market. In constructing the modified objective function, it will be convenient to treat r_L as a function of L . We assume $L[r_L; \alpha_L]$ is monotonic decreasing and write its inverse as:

$$(A5) \quad r_L = r_L[L; \alpha_L].$$

Expected total loan revenue is then:

$$(A6) \quad ER_L = r_L[L; \alpha_L](L).$$

Equation (A6) will enter the modified objective function.

The reader has undoubtedly recognized the similarity of the above specifications to the treatment of deposit variability in the body of the paper.³ In a manner also similar to that treatment we define:

$$(A7) \quad G = \sum_{i=1}^{N_L} g_i.$$

G is the random deviation of the aggregate demand for the bank's loans from its mean value L . If G is in the positive portion of its range, the bank faces unanticipated loan demand; if G is in the negative portion of its range, loan demand is less than expected. Like the individual g_i , G has zero mean; however, we cannot specify the form of its distribution further. G is comparable to the deposit deviation variable U . It is assumed that the distribution of G has limits H and $-H$ and that these limits are functionally related to expected loan volume. That is:

$$(A8) \quad H = H[L].$$

³See pp. 20-31.

The limit variable H is comparable to the limit variable K in the treatment of deposit variability.

We assume that all unanticipated loan requests are presented to the bank at the same "moment of adjustment" at which unexpected deposit withdrawals occur. We further assume that, at this moment, the bank first satisfies all deposit withdrawals in the manner described in the paper.⁴ Once this is accomplished, the bank follows an identical procedure to satisfy unanticipated loan demand.⁵ That is, after all deposit withdrawals are met, the bank first uses any remaining reserves to make loans. If reserves are exhausted, the bank meets whatever loan demand remains by either selling securities or borrowing, whichever is least costly. On the basis of these assumptions, we can write an expected loss function which captures the bank's expected loss due to unanticipated loan demand. This function can then be added to the expected loss function for random deposit deviation in the objective function of the model. The expression is algebraically complicated because, under our assumption that the bank meets unanticipated deposit withdrawals before meeting unexpected loan demand, the expression must take account of possible movements in three random variables: w (random security price deviation), U (random deposit deviation), and G (random loan demand deviation). Nonetheless, the expression merely extends the logic used to develop the expected loss function for random deposit flows to random movements in loan demand.

⁴See pp. 37-38.

⁵We assume that if G is in the negative portion of its range, so that loan demand is less than expected, the bank costlessly shifts a portion of the funds it had planned to use for lending to securities or other assets.

The expression is:

$$\begin{aligned}
 (A9) \quad E^L [L, S, R, DD, TD] &= \int_{-a}^{-n} \int_{-R}^0 \int_{R+U}^H [L] n(G-(R+U)) \phi(w) \theta(U) \psi(G) dw dU dG \\
 &+ \int_{-a}^{-n} \int_{-K[DD, TD]}^{-R} \int_0^H [L] nG \phi(w) \theta(U) \psi(G) dw dU dG \\
 &+ \int_{-n}^0 \int_{-R}^0 \int_{R+U}^{(R+U)+S(1+w)} -w(G-(R+U)) \phi(w) \theta(U) \psi(G) dw dU dG \\
 &+ \int_{-n}^a \int_{-R}^0 \int_{(R+U)+S(1+w)}^H [L] n(G-((R+U)+S(1+w))) \phi(w) \theta(U) \psi(G) dw dU dG \\
 &+ \int_{-n}^0 \int_{-(R+S(1+w))}^{-R} \int_0^{(R+U)+S(1+w)} -wG \phi(w) \theta(U) \psi(G) dw dU dG \\
 &+ \int_{-n}^a \int_{-(R+S(1+w))}^{-R} \int_{(R+U)+S(1+w)}^H [L] n(G-((R+U)+S(1+w))) \phi(w) \theta(U) \psi(G) dw dU dG \\
 &+ \int_{-n}^a \int_{-K[DD, TD]}^{-(R+S(1+w))} \int_0^H [L] nG \phi(w) \theta(U) \psi(G) dw dU dG,
 \end{aligned}$$

where all variables are as previously defined, and $\psi(G)$ is the unspecified distribution of the loan demand deviation variable G .

The explanation of expected loss function (A9) is similar to the explanation of expected loss function (42).⁶ Each term of (A9) gives the adjustment cost the bank incurs in meeting unanticipated loan demand when the three random variables w , U , and G fall in specified portions of their respective ranges.

The first two terms cover the case where $-w < -n$. Under these circumstances, the bank prefers to meet unanticipated loan demand by borrowing rather than by liquidating securities. If deposit withdrawals do not exhaust all primary reserves, so that some reserves are left over to meet unexpected loan demand, the first term is relevant. On the other hand, if deposit withdrawals exhaust all reserves, the second term is relevant.

The last five terms as a group cover the case where security liquidation is less costly than borrowing. Under these conditions, the bank sells securities prior to borrowing. The third and fourth terms are relevant where deposit withdrawals do not exhaust primary reserves. Under these circumstances, the bank first meets unanticipated loan demand by exhausting primary reserves that remain. Subsequently, the bank sells securities. If this security sale does not exhaust the bank's stock of securities, the third term is relevant. If securities are exhausted, the bank must then borrow to meet remaining loan demand, and the fourth term is relevant. The fifth and sixth terms cover the case where deposit withdrawals exhaust primary reserves and consume part but not all of the bank's securities. The fifth term is relevant where enough securities remain to cover unanticipated loan demand. The sixth term is relevant where loan

⁶See pp. 38-45.

demand exhausts remaining securities, forcing the bank to borrow. Finally, the seventh term covers the case where the bank depletes its entire stocks of primary reserves and securities in meeting deposit outflows, making it necessary to meet all unexpected loan demand through borrowing.

We can now reformulate the bank's objective function on the basis of the modified specifications introduced in this appendix. Under our new assumptions the function becomes:

$$(A10) \quad E(\Delta NW) = r_L[L; \alpha_L](L) + \bar{r}_B(B) + \bar{r}_S(S) \\ - c_{DD}^S[DD; \bar{A}_{DD}, \bar{B}](DD) - c_{TD}^S[TD; \bar{A}_{TD}, \bar{B}](TD) \\ - r_{DD}[DD; \alpha_{DD}](DD) - r_{TD}[TD; \alpha_{TD}](TD) \\ - EL^D[S, R, DD, TD] - EL^L[L, S, R, DD, TD],$$

where EL^D is the expected loss due to unanticipated deposit withdrawals, and EL^L is the expected loss due to unanticipated loan demand.⁷ Maximization of (A10) subject to the balance sheet identity constraint yields the following modified first-order conditions:

$$a) \quad \frac{dr_L[L]}{dL}(L) + r_L[L] - \frac{\partial EL^L[L, S, R, DD, TD]}{\partial L} = \lambda$$

$$b) \quad \bar{r}_B = \lambda$$

$$c) \quad \bar{r}_S - \frac{\partial EL^D[S, R, DD, TD]}{\partial S} - \frac{\partial EL^L[L, S, R, DD, TD]}{\partial S} = \lambda$$

$$d) \quad - \frac{\partial EL^D[S, R, DD, TD]}{\partial R} - \frac{\partial EL^L[L, S, R, DD, TD]}{\partial R} = \lambda$$

$$e) \quad \frac{dc_{DD}^S[DD]}{dDD}(DD) + c_{DD}^S[DD] + \frac{dr_{DD}[DD]}{dDD}(DD) + r_{DD}[DD]$$

⁷All variables appearing in (A10) are as previously defined.

$$\begin{aligned}
 & + \frac{\partial EL^D [S, R, DD, TD]}{\partial DD} + \frac{\partial EL^L [L, S, R, DD, TD]}{\partial DD} = \lambda \\
 \text{f)} \quad & \frac{dc_{TD}^S [TD]}{dT} (TD) + c_{TD}^S [TD] + \frac{dr_{TD} [TD]}{dT} (TD) + r_{TD} [TD] \\
 \text{(All)} \quad & + \frac{\partial EL^D [S, R, DD, TD]}{\partial TD} + \frac{\partial EL^L [L, S, R, DD, TD]}{\partial TD} = \lambda \\
 \text{g)} \quad & L + B + S + R - DD - TD - NW_{t-1} = 0
 \end{aligned}$$

These conditions are identical to conditions (49) except for the addition of partial derivatives of the new expected loss function EL^L with respect to the various decision variables. The economic content of these derivatives is similar to that of the corresponding derivatives of EL^D .⁸ That is, each derivative of EL^L indicates the marginal change in the bank's expected loss due to unanticipated loan demand that results from a marginal change in one of the bank's decision variables. Hence, these derivatives can be viewed as marginal revenues and costs just as the derivatives of EL^D were viewed as marginal revenues and costs. Therefore, the modified first-order conditions (All) yield the same broad result as the original conditions (49): that is, the profit-maximizing bank selects the balance sheet position that equates the marginal revenues and costs associated with the various assets and liabilities the bank holds. The modified conditions merely incorporate within this result the marginal revenues and costs arising from the bank's expected loss due to unanticipated loan demand.

This appendix has demonstrated how the model of this paper can be altered to take account of uncertain loan demand. The modification was accomplished at the cost of increased complexity. Nonetheless, it should

⁸See pp. 41-45.

be clear that the model is fully capable of dealing with this important aspect of actual bank operations.

LIST OF SYMBOLS*

Bank Decision Variables

| | |
|----|---|
| B | average total bond balance |
| BF | average "borrowed funds" balance |
| DD | expected average total demand deposit balance |
| L | average total loan balance |
| R | average total reserve balance |
| S | expected average total securities balance |
| TD | expected average total time deposit balance |

Other Variables and Parameters

| | |
|----------------------|--|
| a,-a | limits to the distribution of w |
| \bar{A}_{DD} | an index of demand deposit account activity |
| \bar{A}_{TD} | an index of time deposit account activity |
| $B_{DD,i}$ | expected average balance of the i^{th} individual demand deposit account |
| $B_{L,i}$ (appendix) | expected i^{th} borrower loan demand |
| $B_{TD,i}$ | expected average balance of the i^{th} individual time deposit account |
| c_{DD} | average total service-maintenance and promotional-interest costs per demand deposit dollar |
| $\frac{S}{c_{DD}}$ | average service-maintenance costs per demand deposit dollar |
| c_L | average lending cost |

*This list is restricted to principal variables and parameters. The word "average" is used in two senses in these definitions. Where the symbol denotes a stock, the word means average quantity over the planning period. (See pp. 6-8) Where the symbol denotes a flow, the word is used in the usual sense of economic theory to refer to average flow per relevant unit: for example, average loan return per loan dollar.

| | |
|-------------------|---|
| c_{TD} | average total service-maintenance and promotional-interest costs per time deposit dollar |
| c_{TD}^S | average service-maintenance costs per time deposit dollar |
| d_L | expected average default rate on loans |
| EL | expected loss due to unanticipated deposit withdrawals |
| EL^D (appendix) | expected loss due to unanticipated deposit withdrawals |
| EL^L | expected loss due to unanticipated loan demand |
| G (appendix) | random "moment of adjustment" deviation of the bank's total loan demand from its expected value |
| g_i (appendix) | random "moment of adjustment" deviation of the i^{th} borrower's loan demand from its expected value |
| H, -H (appendix) | limits to the distribution of G |
| K, -K | limits to the distribution of U |
| k, -k | limits to the distributions of the random variables u_i |
| n | penalty rate for reserve deficiencies |
| P_S | average price of an individual security |
| \bar{r}_B | constant coupon rate paid on bonds |
| r_{BF} | interest cost of "borrowed funds" |
| r_{DD} | average promotional, advertising, and explicit interest expenses per demand deposit dollar |
| r_L | expected average net rate of return on loans |
| r'_L | average gross contract rate on loans |
| \bar{r}_S | constant coupon rate paid on securities |
| r_{TD} | average promotional, advertising, and explicit interest expenses per time deposit dollar |
| U | random "moment of adjustment" deviation of the bank's average total deposit balance from its expected average value |
| u_i | random "moment of adjustment" deviation of the i^{th} demand deposit account balance from its expected average value |

| | |
|---------------|---|
| v_i | random "moment of adjustment" deviation of the i -th time deposit account balance from its expected average value |
| w | random "moment of adjustment" deviation of P_S from its expected average value |
| \bar{z} | a vector of parameters specifying the default risk characteristics of the bank's customers |
| α_{BF} | a vector of parameters summarizing the competitive structure of the market for borrowed funds |
| α_{DD} | a vector of parameters summarizing the competitive structure of the bank's demand deposit market |
| α_L | a vector of parameters summarizing the competitive structure of the bank's loan market |
| α_{TD} | a vector of parameters summarizing the competitive structure of the bank's time deposit market |
| σ_U | standard deviation of U |

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