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ALTERNATIVE OPTIMAL OPEN MARKET STRATEGIES
A CLASSICAL OPTIMIZATION
CERTAINTY EQUIVALENCE APPROACH

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The Federal Reserve System has historically been criticized for placing too much emphasis on insuring the short-run stability of financial markets and too little emphasis on real sector goals. Critics generally contend that the Federal Reserve is often misled by concentrating on money market indicators, that it formulates strategy in terms of short-run targets, and therefore reacts improperly to changing economic conditions. Guttentag, for example, finds that, in the period before 1966, the strategy of monetary policy was incomplete, failing to include "specific quantitative target values . . . for the money supply or [some] other strategic variable that could serve as a connecting link between open market operations and System objectives."[1]

Implicit in these criticisms is a set of requirements for specifying the nature of the policy process and for evaluating current policy. These requirements include a theory of policy formulation, a theory of how the effects of policy actions are transmitted through the economy, and precisely quantified measures of the goals of policy. These requirements may be met by specifying a policy regime and strategy. A policy regime is defined as a policy model of the economy, consisting of an instrument proxy and a theory of monetary policy transmission. A policy strategy is a framework for policy determination, consisting of a set of target variables and a theory of policy formulation.[2]

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[1]Jack M. Guttentag, "The Strategy of Open Market Operations," _Quarterly Journal of Economics_, 80 (February 1966), 1-30.

[2]Joseph M. Crews, "Alternative Optimal Open Market Strategies: A Simulation Approach," Unpublished doctoral dissertation, University of North Carolina, Chapel Hill, 1972.

The purpose of this paper is to present an optimal control framework for the analysis of monetary policy problems, using the target/indicator problem for illustration. This involves accepting as given two of the elements listed above--a theory of policy formulation and a theory of policy transmission. The remaining elements, the instrument proxy and target variables, are altered under controlled conditions to simulate the effects of alternative policy actions on the economy.

OPTIMAL CONTROL

The theory of policy formulation adopted is the "theory of quantitative economic policy" developed by Theil.[3] According to this approach, the policy maker maximizes a social preference function, in terms of targets and instruments, subject to the constraints of the economic structure, represented by an econometric model. This constrained optimum problem is transformed into a system of simultaneous equations by the Lagrangean multiplier technique. This approach is usually stated in terms of a linear model and quadratic preferences. Since most large-scale econometric models in use today are non-linear, this approach should be extended to cover this case. This paper presents a computational algorithm based on Theil's non-linear case.

Central to extension of this approach to non-linear models is the problem of uncertainty. Theil's theory of policy formulation recognizes that the constraining model may not be a true representation of the economic structure and that the preference function may not truly represent the preferences of the decision maker. Uncertainty may exist in the multiplicative structure of the model, in its predetermined structure, or in specification of the preference (or loss) function. These difficulties may be overcome by

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[3]Henri Theil, Optimal Decision Rules for Government and Industry, Amsterdam: North-Holland, 1964.

assuming that the decision maker seeks to minimize the expected value of the loss function. In this case Theil develops a certainty equivalence theorem: If the loss function is quadratic and the constraining model is linear and stochastic only by additive random disturbances that are independent of the instruments and whose expected values are zero, then the optimal values of the instruments are the same as if there were no uncertainty.[4]

A number of extensions of the certainty equivalence theorem are required if it is to apply generally. First, Theil demonstrates that when uncertainty exists in the multiplicative structure, a "first order certainty equivalence" results in the linear static and multi-period cases.[5] He also extends these results to the non-linear static case.[6]

Concerning the non-linear dynamic case the picture remains unclear. Malinvaud demonstrates that the first order certainty equivalence is general and applies "with exceptions for singular cases, as long as the various functions involved are twice differentiable."[7] In a recent dissertation, however, Porter shows that Malinvaud's results are more limited, applying only in cases where the degree of uncertainty is very small.[8]

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[4]Henri Theil, "Econometric Models and Welfare Maximization," Weltwirtschlaftliches Archiv, 72 (1954), 19-70.

[5]Henri Theil, Optimal Decision Rules, p. 59 and pp. 72-74.

[6]Henri Theil, Economic Forecast and Policy, Second Edition, Amsterdam: North-Holland, 1965, pp. 405-424, 510-513.

[7]E. Malinvaud, "First Order Certainty Equivalence, Econometrica, 37 (October 1969), 706-718.

[8]Richard D. Porter, "Strategies for Discrete-time Decision Models," Unpublished doctoral dissertation, University of Wisconsin, Madison, 1971, pp. 133-151.

Given the state of these generalizations, the results reported here are based on the non-linear static case. The policy maker is viewed as planning policy in a given period with future goals in mind, but revising his plans at the beginning of each new period as additional information becomes available.

The solution to the non-linear constrained optimum problem may be derived in the following manner. Assume that the economic structure is represented by the model:[9]

$$Y_j = f_j(Y_1,\dots,Y_{j-1}, Y_{j+1},\dots,Y_N, X|Z) \qquad j = 1,\dots,N \qquad (1)$$

where Y_j are N noncontrolled endogenous variables, X is an M element vector of instruments, and Z is a p element vector of predetermined variables.

Assume also that the preferences of the policy maker can be represented by the general quadratic loss function (2).

$$\begin{aligned} W = -\tfrac{1}{2}\Big[&\sum_{i=1}^{n} w_i(Y_i-Y_i^*)^2 + \sum_{k=1}^{m} w_k(X_k-X_k^*)^2 \\ &+ 2\sum_{i\neq j} w_{ij}(Y_i-Y_i^*)(Y_j-Y_j^*) + 2\sum_{k\neq h} w_{kh}(X_k-X_k^*)(X_h-X_h^*) \\ &+ 2\sum_{i=1}^{n}\sum_{k=1}^{m} w_{ik}(Y_i-Y_i^*)(X_k-X_k^*)\Big] \end{aligned} \qquad (2)$$

where there are n target variables and m instruments entering the preference function.

To determine the optimal instrument vector X^0 using the Lagrangean transformation, proceed as follows:

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[9]This development parallels Crews, op. cit., pp. 37-40.

(1) Establish the Lagrangean function:

$$L = w(X,Y|Z) + \sum_{j=1}^{N} \lambda_j \, g_j(X,Y|Z) \tag{3}$$

where the $g_j(X,Y|Z)$ are the constraints (1) expressed in the form:

$$g_j(X,Y|Z) = Y_j - f_j(Y_1,\ldots,Y_{j-1},Y_{j+1},\ldots,Y_N,\ X|Z) = 0 \qquad j=1,\ldots,N \tag{4}$$

(2) Differentiating (3) partially with respect to each endogenous variable and setting the result to zero:

$$\frac{\partial L}{\partial Y_i} = \frac{\partial W}{\partial Y_i} + \sum_{j=i}^{N} \lambda_j \frac{\partial g_j}{\partial Y_i} = 0 \qquad i=1,\ldots,N \tag{5}$$

Since each $\dfrac{\partial g_j}{\partial Y_j} = 1$, each equation (5) can be normalized on a unique λ_j:

$$\lambda_j = -\frac{\partial W}{\partial Y_j} - \sum_{i=1}^{j-1} \lambda_i \frac{\partial g_i}{\partial Y_j} - \sum_{i=j+1}^{N} \lambda_i \frac{\partial g_i}{\partial Y_j} \qquad j=1,\ldots,N \tag{6}$$

(3) Differentiating (3) partially with respect to each instrument and setting the result to zero:

$$\frac{\partial L}{\partial X_k} = \frac{\partial W}{\partial X_k} + \sum_{j=1}^{N} \lambda_j \frac{\partial g_j}{\partial X_k} \qquad k=1,\ldots,m \tag{7}$$

Each of these equations may be normalized on a unique instrument. This necessitates solving for the partial derivative of the loss function (2) with respect to each instrument:

$$\frac{\partial W}{\partial X_k} = - w_k(X_k - X_k^*) - \sum_{h \neq k} w_{kh}(X_h - X_h^*) - \sum_{i=1}^{n} w_{ik}(Y_i - Y_i^*) \qquad k=1,\dots,m \tag{8}$$

Substituting (8) into (7):

$$\frac{\partial L}{\partial X_k} = - w_k(X_k - X_k^*) - \sum_{h \neq k} w_{kh}(X_h - X_h^*) - \sum_{i=1}^{n} w_{ik}(Y_i - Y_i^*) + \sum_{j=1}^{N} \lambda_j \frac{\partial g_j}{\partial X_k} = 0 \qquad k=1,\dots,m \tag{9}$$

Normalizing (9) on X_k:

$$X_k = X_k^* - \frac{1}{w_k} \left(\sum_{h \neq k} w_{kh}(X_h - X_h^*) + \sum_{i=1}^{n} w_{ik}(Y_i - Y_i^*) - \sum_{j=1}^{N} \lambda_j \frac{\partial g_j}{\partial X_k} \right) \qquad k=1,\dots,m \tag{10}$$

(4) Finally, differentiating (3) partially with respect to λ and setting the result equal to zero determines the original equation system (4).

$$\frac{\partial L}{\partial \lambda_j} = g_j(X,Y|Z) = 0 \qquad j=1,\dots,N \tag{4}$$

The resulting sets of equations (6), (10), and (4) constitute a system of simultaneous equations, the solution of which establishes the first-order conditions for the optimal policy vector X^0, the corresponding real vector Y^0 and the vector of Lagrangean multipliers λ. The second order conditions are presented by Theil.[10]

The Guass-Seidel algorithm, used in solving many large-scale econometric models, may be extended slightly for use in solving optimizing models. This solution technique is discussed by Evans[11] and in the Appendix to this paper. Optimal simulation--in which the policy instrument is determined by the equation system rather than being read in as an exogenous variable--is therefore feasible with relatively straightforward extensions of current methodology.

The theory of policy transmission utilized in this study is contained in the most recently published version of the FRB-MIT econometric model.[12] This model was specifically designed to capture econometrically the effects of monetary policy actions on the real sectors of the economy. Its financial sector is highly developed and its financial-real linkage includes three separate channels--the cost-of-capital, the wealth effect, and credit availability. The model is based on a neo-Keynesian interest sensitivity of investment theory, with particular expenditure flows related to appropriate interest rates. Its portfolio adjustment mechanism is broadly inclusive.

While particular theories of policy formulation and policy transmission are accepted as given for purposes of this study, alternative instrument proxies and policy targets are assumed in the experimental design. The next sections develop the target/indicator problem and establish particular variables for evaluation.

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[10] Henri Theil, Optimal Decision Rules, pp. 37-40.

[11] Michael Evans, Computer Simulation of Non-Linear Econometric Models, Discussion Paper Number 97, Philadelphia: The Wharton School of Finance and Commerce, 1968, pp. 5-7.

[12] Frank de Leeuw and Edward M. Gramlich, "The Channels of Monetary Policy: A Further Report on the Federal Reserve-MIT Model," Journal of Finance, 24 (May 1969), 265-290.

THE TARGET PROBLEM

Monetary policy is conducted in an atmosphere of uncertainty. Knowledge of the economic structure is incomplete, the chain of causation from policy action to ultimate goals is long, the speed of monetary impulses is slow and variable, and information regarding current policy and economic conditions is available only after a time lapse. In view of these uncertainties, the policy maker finds it useful to direct his actions toward intermediate variables, closer in time and under more positive control than ultimate goals. The function of intermediate targets is to facilitate control over a sequence of successively longer-term targets so that ultimate goals may be achieved. This suggests several criteria by which intermediate targets may be evaluated. The target should (1) be readily observable with minimal lag, (2) bear some relation to the transmission of policy, as reflected in a stated structural hypothesis, (3) be sensitive to, but not necessarily dominated by policy actions, and (4) be strongly correlated with longer-term goals. Economic literature abounds with evaluations of suggested targets.[13] Six alternative quarterly target candidates, which meet the above criteria, were chosen for this study. First, for comparative purposes two money market targets, free reserves RF and the Treasury bill rate RTB are included to reflect "incomplete" strategies. Longer-term targets include: (1) total reserves RT, which constitutes the base upon which the banking system generates money and credit, (2) the money supply MS, a strategic variable in both the neo-Keynesian and Monetarist views of the transmission process, (3) bank credit BC, the commercial bank asset

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[13]For example, see Thomas R. Saving "Monetary Targets and Indicators," <u>Journal of Political Economy</u>," 75 (August 1967), 446-456.

counterpart of money supply creation, and (4) long-term interest rates, specifically the corporate bond rate RC, especially critical in a neo-Keynesian cost-of-capital transmission channel. These six alternative intermediate targets, together with a non-optimal control solution, provide seven strategies to be evaluated.

THE INDICATOR PROBLEM

Indicators are variables used by market participants to separate the impact of current policy actions from concurrent forces operating in financial markets. Within the context of econometric models, indicators are conceptually equivalent to instrument proxies.[14] The literature suggests the following criteria for indicators. They should (1) be readily observable, (2) be important links in the transmission process, (3) reflect the impact of policy action apart from all other forces affecting the target, and (4) provide reliable information regarding current and future movements in economic activity. Recent controversy on the indicator question has centered on criterion (4), the exogeneity problem.[15] This controversy narrows the question to whether the monetary base or one of its components is more nearly exogenous. Accordingly, this study utilizes the following instruments: (1) the monetary base MB, defined as to its uses as total reserves plus currency, (2) the adjusted base--nonborrowed reserves plus currency BA, (3) total reserves, and (4) free reserves. These instrument proxies and the

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[14] Thomas J. Sargent, "Framework of the Economic System--Discussion," American Economic Review, 60 (May 1970), 57-58.

[15] Frank de Leeuw and John Kalchbrenner, "Monetary and Fiscal Actions: A Test of Their Relative Importance in Economic Stabilization--Comment," Review, Federal Reserve Bank of St. Louis, 51 (April 1969), 6-11.

FRB-MIT model, modified slightly as required, constitute the four policy regimes to be evaluated.

THE EXPERIMENTS

The experimental design that incorporates the alternative regime/strategy framework of this analysis is presented in Table 1. The four regimes are specified as rows. The columns include non-optimal control solutions representing actual economic developments, two incomplete strategies, and four complete strategies. In each of these 28 simulation experiments the FRB-MIT model was solved in optimizing mode over a 16 quarter period from 1959 to 1962. This period was chosen because the version of the model used is not capable of handling the rapid inflation of later years.[16]

TABLE 1

AVERAGE WELFARE LOSS RESULTING
FROM ALTERNATIVE REGIMES AND STRATEGIES

	Strategy							
	Nonoptimal	Incomplete		Complete				
Regime	Control	RF	RTB	RT	MS	BC	RC	Average
RT	40.74	42.41	53.79	40.28	31.34	33.60	44.94	41.01
RF	40.46	40.65	48.83	36.91	32.35	33.22	37.61	38.45
MB	43.98	41.25	51.43	41.05	34.84	33.99	41.67	41.17
BA	40.35	40.66	48.75	42.19	32.32	34.83	32.22	39.48
Average	41.38	41.24	50.70	40.11	32.71	33.91	40.36	40.04

[16]Robert H. Rasche and Harold T. Shapiro, "The FRB-MIT Econometric Model: Its Special Features," American Economic Review, 58 (May 1968), 123-149.

Optimal policy levels were obtained by constraining alternative policy-determining loss functions by the model. Each policy-determining loss function includes as arguments the assumed instrument, the intermediate target of the specified strategy, a financial market stability target and ultimate goals. Typical is the total reserves/money supply case.

$$\begin{aligned} W_{ij} = -\tfrac{1}{2}\,[&(RT-RT^*)^2 + (MS-MS^*)^2 + (RCP-RCP^*)^2 \\ &+ (GNP-GNP^*)^2 + (P-P^*)^2 + 2(RT-RT^*)(MS-MS^*) \\ &+ 2(RT-RT^*)(RCP-RCP^*) + 2(MS-MS^*)(RCP-RCP^*)] \end{aligned}$$

Where * indicates a target desired level, RCP is the commercial paper rate, serving as a financial stability proxy that is not altered experimentally, GNP is gross national product, and P is the price level. Since GNP* is "potential GNP," which implies an unemployment target, no separate employment goal is specified.

The policy-determining loss function changes for each regime/strategy case, and the values produced are not comparable with any other. This problem is solved by assuming that the policy maker determines policy with regard to the particular loss function specified for the strategy being studied, but policy performance is evaluated in terms of all intermediate and ultimate objectives together.[17] The ultimate loss function used to evaluate all regime/strategies in this study is of the form:

$$\begin{aligned} U_{ij} = -\tfrac{1}{2}\,[&(GNP-GNP^*)^2 + (P-P^*)^2 + (RCP-RCP^*)^2 \\ &+ (RT-RT^*)^2 + (MS-MS^*)^2 + (BC-BC^*)^2 + (RC-RC^*)^2] \end{aligned}$$

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[17] Gary Fromm and Paul Taubman, Policy Simulations With an Econometric Model, Washington: The Brookings Institution, 1968, pp. 106-123.

EVALUATING THE RESULTS

This study focuses on two economic questions: (1) Which of four instrument proxy candidates best measures the thrust of monetary policy? (2) Which of six monetary targets provides for maximum effectiveness of open market policy?

These questions, together with the 4 x 7 experimental design, enables us to formulate the following hypotheses:

I. There is no difference among instrument regimes (Row effect).

II. There is no difference among policy strategies (Column effect).

III. The performance of the Federal Reserve under alternative strategies is independent of the instrument regime (Interaction effect).

In addition, the inclusion of a non-optimal control solution, representing the actual time path of the economy in the period of study, allows the testing of the general hypothesis:

IV. The Federal Reserve responds in a systematic manner to intermediate targets.

For each of the 28 regime/strategy combinations the ultimate loss function is evaluated. The results are summarized in Table 1, which reports the average welfare loss (cell means) for each case. The regimes rank in order from lowest (best) to highest as: RF, BA, RT, MB. Similarly, the strategies are ranked in the order: MS, BC, RT, RF, CONTROL, and RTB. The data was subjected to analysis of variance using a randomized bloc, two variables of classification, with replication model. There are sixteen blocks (time periods), four replications of rows (regimes) and seven columns (strategies) in the problem. Individual differences among regimes and strategies were further tested using a least significant difference test.[18]

[18]Ya-lun Chou, Statistical Analysis, New York: Holt, Rinehart and Winston, Inc., 1969, pp. 407-409, 417-423.

The results may be briefly summarized. Hypothesis I: There are no differences among regimes. The regimes produce significantly different welfare levels, and fall into two sets, [RF, BA] and [RT, MB], whose members are not statistically distinguishable from each other. The former set generates smaller welfare losses. These results are consistent with those of de Leeuw and Kalchbrenner, who argue that the monetary base is made up of endogenous components and is, therefore, also endogenous.[19] Since the present analysis is in terms of an open market proxy only, leaving the discount rate and reserve requirements aspects of monetary policy as given, RF and BA are expected to perform best. The results are consistent with these expectations.

Hypothesis II: There are no significant differences among strategies. This test concerns the choice of the optimal intermediate target. Preliminary analysis of the target candidates indicated that incomplete strategies provide closer control over short-term targets, while complete strategies provide closer control over intermediate and ultimate targets. These results were confirmed by the statistical tests. The strategies were found to separate into three sets whose members are indistinguishable from each other. One set [CONTROL, RF, RT, RC] shows a consistency between the Federal Reserve's actual behavior and the other members of the set. The set [MS, BC] represents the general Monetarist position that a money supply target should be adopted by the Federal Reserve. But this is not the "money supply rule" of Monetarist advocacy. It is, rather, an anti-cyclical use of the money supply as a target reflecting the use of Hendershott's neutralized money stock to establish target

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[19] de Leeuw and Kalchbrenner, op. cit., pp. 8-9.

values.[20] The RTB strategy is, as expected, the least effective strategy. It is included in the experimental design as a typical incomplete strategy for comparative purposes only. We conclude that the Federal Reserve's performance is improved by adopting a money supply target.

Hypothesis III: The performance of the Federal Reserve under alternative strategies is independent of the instrument regime. This hypothesis of an interaction effect is rejected, implying independence of the target and indicator concepts.

The general hypothesis of this study, Hypothesis IV, is that the Federal Reserve responds in a systematic manner to intermediate targets. The evidence is consistent with this hypothesis in the sense that several strategies are not significantly different from the CONTROL solution. The RF strategy is the subject of Guttentag's criticism that free reserves is too short-run a variable to be a proper policy guide. Guttentag and Brunner and Meltzer agree, however, that if the control period is three months or longer free reserves may perform better as a target.[21] Our results show that if quarterly targets are specified, there is no significant difference between the RF and RT strategies; and, in fact, the Federal Reserve has been behaving in a manner consistent with these strategies. The CONTROL solution is also not significantly different from the RC strategy, implying that the Federal Reserve holds a neo-Keynesian view of the monetary process, as distinct

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[20] Patric H. Hendershott, The Neutralized Money Stock, Homewood, Illinois; Richard D. Irwin, Inc., 1968.

[21] Karl Brunner and Allan Meltzer, "Genesis and Development of the Free Reserves Conception," Readings in Money, National Income, and Stabilization Policy, ed. Smith & Teigen, 1965, pp. 197-210.

from a Monetarist view. This evidence is consistent with the view that the Federal Reserve responds systematically to the targets which, in its view, are important in transmitting the impact of policy to ultimate objectives.

Finally, aside from the results of this particular set of experiments, the optimal simulation approach presented offers a method for studying the problems of economic policy using non-linear models. While general proofs are as yet available only for the non-linear static case, the implicit assumption that the policy maker revises his plan when new information becomes available is not unrealistic.

APPENDIX

SOLUTION METHODS

Of the three sets of equations derived above, only the set (6) is generally linear. Sets (10) and (4) are generally non-linear. Two alternative solution methods are discussed.

MATRIX INVERSION

System (6) is linear in the partial derivatives as coefficients and can be solved either by generalized Gauss-Seidel interactive technique, as discussed in the next section, or by the following matrix technique. Express (5) as:

$$\sum_{j=1}^{N} \frac{\partial g_j}{\partial y_i} \cdot \lambda'_j = - \frac{\partial w}{\partial y_i} \qquad i=1,\ldots,N \tag{11}$$

or, in matrix notation:

$$A\lambda' = B$$

where A is an NxN partial derivative matrix, λ' is an N element Lagrangean multiplier vector, and B is the negative of the $\frac{\partial w}{\partial y_i}$ vector. System (11) can be solved with ease by any one of several matrix manipulation programs available in computer libraries.[1] However, these programs involve some type of matrix inversion and are extremely time-consuming. On the other hand, the next section shows that the Gauss-Seidel solution method requires initial

[1]The Present research uses PROGRAM SIMQ, described in System 1360 Scientific Subroutine Package (360A-CM-03X) Version III Programmers Manual H20-0205-3. (International Business Machines Corporation, 1966), p. 120.

estimates of the endogenous variables. As a matter of computational feasibility and efficiency, therefore, the two procedures are used in combination. The matrix inversion procedure is used on the first iteration to obtain initial estimates of the λ vector. On all subsequent iterations the entire equation system (4), (6), and (10) is solved by the Gauss-Seidel procedure.

THE GUASS-SEIDEL ITERATIVE PROCEDURE

Computer simulation of large-scale econometric models requires a numerical solution technique for systems of simultaneous equations. If the model is linear, a matrix method such as that discussed above may be applied. In addition to the time-costliness of these matrix methods, the non-linear nature of many current models of the U. S. economy requires rapid non-linear solution methods. Until recently these models were solved by some variant of the Newton iterative method, which essentially linearizes each equation of the system by a Taylor series expansion about the trial solution vector and solves the resultant linear system by matrix inversion.[2] The several variants of the Newton method have been found to be extremely costly in terms of computer time because inversion of large matrices and iterative solution-seeking are both involved.

To overcome these difficulties, most large-scale econometric models are currently solved by the Gauss-Seidel method.[3] This is a straightforward iterative technique. After a first trial solution is assumed, each successive iteration adopts the previous trial solution as a starting estimate. Iteration

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[2] G. Hadley, Nonlinear and Dynamic Programming, pp. 56-57. C. C. Holt et al., Program Simulate II (Madison, Wisconsin: University of Wisconsin, 1967), Sections 9.3-9.4.

[3] Ibid., Section 9.5. See also, Michael K. Evans, Computer Simulations of Non-Linear Econometric Models, pp. 5-7; Jorge J. More, A Class of Nonlinear Functions and the Convergence of Gauss-Seidel and Newton-Guass-Seidel Iterations.

continues until successive solutions agree to the preassigned degree of precision.

Algebraically, the method may be expressed as follows: Let the jth equation of the system be represented as:[4]

$$y_j = f_j\ (y_1, y_2, \ldots, y_{j-1}, y_{j+1}, \ldots, y_N | X, Z) \qquad j=1,\ldots,N$$

where Y is endogenous, X is a policy vector and Z is exogenous. Assume an initial trial value for each endogenous variable, denoted $Y_j^{(o)}$. Evaluate the equation system:

$$y_j^{(1)} = f_j\ (y_1^{(o)}, y_2^{(o)}, \ldots, y_{j-1}^{(o)}, y_{j+1}^{(o)}, \ldots, y_N^{(o)} | X, Z) \qquad j=1,\ldots,N$$

Using the values of y_j obtained in the first trial solution, solve the system again to obtain a second trial solution. Iterate in this manner such that:

$$y_j^{(r)} = f_j(y_1^{(r-1)}, y_2^{(r-1)}, \ldots, y_{j-1}^{(r-1)}, y_{j+1}^{(r-1)}, \ldots, y_N^{(r-1)} | X, Z) \qquad j=1,\ldots,N$$

until:

$$\left| \frac{y_j^{(r)} - y_j^{(r-1)}}{y_j^{(r-1)}} \right| < \text{tolerance}$$

While the above presentation indicates the basic Gauss-Seidel routine, several additional features may be used to improve its solution characteristics. First, convergence may be enhanced by using the values $y_i^{(r)}$, $i < j$, already

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[4]This exposition derives from that of Evans, Computer Simulations of Non-Linear Econometric Models, pp. 6-7.

calculated for the rth iteration to determine $y_j^{(r)}$. That is:

$$y_j^{(r)} = f_j(y_1^{(r)}, y_2^{(r)}, \ldots, y_{j-1}^{(r)}, y_{j+1}^{(r-1)}, \ldots, y_N^{(r-1)} | X,Z) \qquad j=1,\ldots,N$$

While this method may speed convergence, it may also lead to divergence if the equations are not properly ordered.[5]

Secondly, as an alternative to the straightforward substitution of current iteration values as initial estimates of the succeeding solution, more control over the solution may be obtained if the following up-date routine is used:

$$y_j^{(r+1)} = y_j^{(r)} + \underline{a} \cdot s_j (y_j^{(r)} - y_j^{(r-1)})$$

where $\underline{a}$ is a dampening factor and s_j is a sign factor.[6] The dampening factor $\underline{a}$ allows the starting estimate of y_j for a particular iteration to be approximated as the weighted average of the previous and current estimates. Increasing $\underline{a}$ may speed convergence, but at the risk of inducing divergence. A value of $\underline{a}$ less than 1.0 will enhance convergence, but at a slow pace.

The sign factor is needed to indicate the direction of change necessary to reduce the residual error; that is, to move toward convergence. It is calculated for the initial trial solution, and is positive or negative, depending on the sign of the partial derivative of the function f_j with respect to the endogenous variable y_j. That is:

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[5]Evans, Computer Simulations of Non-Linear Econometric Models, p. 7. See Also, Michael J. Hartley, "Instructions for the Use of the Econometric Model Solution Program," (Durham, North Carolina: Duke University, 1969, mimeographed).

[6]Holt *et al.*, Program Simulate II, Section 9.5b.

$$s_j = \begin{bmatrix} +1 \\ -1 \end{bmatrix} \text{ if } \left. \frac{\partial f_j(y^{(r)})}{\partial y_j} \right|_{y^{(o)}} \begin{bmatrix} \geq 0 \\ < 0 \end{bmatrix}$$

where $y^{(o)}$ is the vector of initial trial solutions for the rth iteration.[7] Changing the sign factor for one or more errant variables may turn a divergent system into a convergent one.

In addition to these factors, several other changes in a system of equations may enhance convergence. These include the possibility of reordering the equations. In some cases a recursive ordering may be found, and convergence is automatic. In more complex systems some simultaneity is present, and the correct ordering of the equations may be a question of trial and error. Secondly, the equations themselves must be normalized on their "dominant" variable.[8] Finally, if all else fails, the tolerance may be relaxed to allow a less precise solution. Reasoned and diligent use of these various controls, together with familiarity with the logic of the equation system being solved, should result in convergence of any soluable system.

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[7]<u>Idem</u>.

[8]<u>Ibid</u>., p. 9.5c.