Technical Appendix for "Quarantine, Contact Tracing, and Testing: Implications of an Augmented SIR-Model" *

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April 2, 2021

*Any opinions expressed are mine and do not reflect those of the Federal Reserve Bank of Richmond or the Federal Reserve System.

Contents

1 Solving for the transmission rate

For the simulations, we use a discrete time approximation of the continuous time model for which the unit of time is one day. We use the simplified version of the SEIR model, without contact tracing and random testing, to recover the transmission rate.

The system of difference equations is

$$\begin{aligned} I_t^* &= I_{A,t} + \left(1 - \varepsilon^S\right) I_{S,t} \\ \Delta S_{t+1} &= -\alpha_t S_t I_t^* / N \\ \Delta E_{t+1} &= \alpha_t S_t I_t^* / N - \phi E_t \\ \Delta I_{A,t+1} &= \phi E_t - (\beta + \gamma_A) I_{A,t} \\ \Delta I_{S,t+1} &= \beta I_{A,t} - (\delta + \gamma_S) I_{S,t} \\ \Delta R_{t+1} &= \gamma_A I_{A,t} + \gamma_S I_{S,t} \\ \Delta D_{t+1} &= d_{t+1} = \delta I_{S,t}. \end{aligned}$$

We solve the system recursively, working backward, and operating on daily deaths. From the ΔD equation

$$I_{S,t} = d_{t+1}/\delta \text{ and } \Delta I_{S,t+1} = \Delta d_{t+2}/\delta.$$

Sub I_S into the ΔI_S equation

$$\begin{aligned} \beta I_{A,t} &= \Delta I_{S,t+1} + (\delta + \gamma) I_{S,t} \\ &= \Delta d_{t+2}/\delta + (\delta + \gamma)(d_{t+1}/\delta). \end{aligned}$$

Therefore

$$\begin{aligned} I_{A,t} &= [\Delta d_{t+2} + (\delta + \gamma) d_{t+1}] / (\beta\delta) \\ \Delta I_{A,t+1} &= \left[\Delta^2 d_{t+3} + (\delta + \gamma) \Delta d_{t+2}\right] / (\beta\delta). \end{aligned}$$

Sub into the ΔI_A equation

$$\begin{aligned} E_t &= [\Delta I_{A,t+1} + (\beta + \gamma) I_{A,t}] / \phi \\ \Delta E_{t+1} &= \left[\Delta^2 I_{A,t+2} + (\beta + \gamma) \Delta I_{A,t+1}\right] / \phi \\ &= \left[\Delta^2 d_{t+3} + (\delta + \gamma) \Delta d_{t+2}\right] / (\beta\delta) \\ &\quad + (\beta + \gamma) [\Delta d_{t+2} + (\delta + \gamma) d_{t+1}] / (\beta\delta). \end{aligned}$$

Sub into the ΔE equation

$$\begin{aligned} \alpha_t S_t I_t^* / N =& \phi E_t + \Delta E_{t+1} \\ \alpha_t =& \frac{\phi E_t + \Delta E_{t+1}}{S_t I_t^* / N}. \end{aligned}$$

Solve forward with ΔS equation

$$S_{t+1} = S_t + \alpha_t S_t I_t^* / N.$$

2 Calibration

We use the following conventions: (1) $p_{X\to Y}$ for transition probability from X to Y; (2) $T_{X\to Y}$ for the average time to get from state X to Y; and (3) T_{XYZ} for the average time spent in states X, Y, and Z.

For the calculations that follow, it is useful to recall the expected time to exit of a Poisson process with exit rate α,

$$\int_0^t \tau e^{\alpha\tau} d\tau = \frac{1}{\alpha^2}\left[1 + e^{\alpha t}(\alpha t - 1)\right] \text{ and } \lim_{t\to\infty}\int_0^t \tau e^{-\gamma\tau} d\tau = \frac{1}{\gamma^2}. \tag{1}$$

2.1 Transitions from exposed to symptomatic, $p_{E\to S}$ and $T_{E\to S}$

$p_{E\to S}$ is the probability that an exposed individual eventually becomes symptomatic,

$$p_{E\to S} = \int_0^\infty \left\{\int_0^\tau \left[\phi e^{-\phi s}\right]\left[\beta e^{-\beta(\tau-s)}\right]\left[e^{-\gamma_A(\tau-s)}\right] ds\right\} d\tau \tag{2}$$

where the first term is the probability that the individual becomes asymptomatic infectious at s, and then becomes symptomatic at τ, without recovering. Working on this, we get

$$\begin{aligned} p_{E\to S} =& \phi\beta \int_0^\infty \left\{e^{-\phi\tau}\int_0^\tau e^{-\phi(s-\tau)}\left[e^{-\gamma_A(\tau-s)}\right]\left[e^{-\beta(\tau-s)}\right] ds\right\} d\tau \\ =& \phi\beta \int_0^\infty \left\{e^{-\phi\tau}\int_0^\tau e^{-(\gamma_A+\beta-\phi)(\tau-s)} ds\right\} d\tau \\ =& \frac{\phi\beta}{\gamma_A+\beta-\phi}\int_0^\infty \left\{e^{-\phi\tau}\left[1 - e^{-(\gamma_A+\beta-\phi)\tau}\right]\right\} d\tau \\ =& \frac{\phi\beta}{\gamma_A+\beta-\phi}\left[\frac{1}{\phi} - \frac{1}{(\gamma_A+\beta)}\right] \\ =& \frac{\phi\beta}{\gamma_A+\beta-\phi}\left[\frac{\gamma_A+\beta-\phi}{\phi(\gamma_A+\beta)}\right]. \end{aligned} \tag{3}$$

Thus

$$p_{E\to S} = \frac{\beta}{\gamma_A+\beta}. \tag{4}$$

Of course, since an exposed agent always becomes asymptomatic, we could just write

$$p_{E\to S} = p_{A\to S} = \frac{\beta}{\gamma_A+\beta}.$$

$T_{E\to S}$ is the average time to become symptomatic, conditional on eventually becoming symptomatic, that is, the average time to get from state E to S

$$T_{E\to S}=\int_0^\infty \tau\left\{\int_0^\tau \frac{\left[\phi e^{-\phi s}\right]\left[\beta e^{-\beta(\tau-s)}\right]\left[e^{-\gamma_A(\tau-s)}\right]}{p_{E\to S}}ds\right\}d\tau$$

where the term in curly brackets is the probability of becoming symptomatic at τ, conditional on eventually becoming symptomatic. Working on this, we get

$$\begin{aligned}
T_{E\to S}=&\frac{\phi\beta}{p_{E\to S}}\int_0^\infty \tau\left\{e^{-\phi\tau}\int_0^\tau e^{-\phi(s-\tau)}e^{-(\gamma_A+\beta)(\tau-s)}ds\right\}d\tau\\
=&\frac{\phi\beta}{p_{E\to S}}\int_0^\infty \tau\left\{e^{-\phi\tau}\int_0^\tau e^{-(\gamma_A+\beta-\phi)(\tau-s)}ds\right\}d\tau\\
=&\frac{1}{p_{E\to S}}\frac{\phi\beta}{\gamma_A+\beta-\phi}\int_0^\infty \tau\left\{e^{-\phi\tau}\left[1-e^{-(\gamma_A+\beta-\phi)\tau}\right]\right\}d\tau\\
=&\frac{1}{p_{E\to S}}\frac{\phi\beta}{\gamma_A+\beta-\phi}\left[\frac{1}{\phi^2}-\frac{1}{(\gamma_A+\beta)^2}\right].
\end{aligned}$$

Substituting for $p_{E\to S}$ from (3), this simplifies to

$$\begin{aligned}
T_{E\to S}&=\left\{\frac{\phi\beta}{\gamma_A+\beta-\phi}\left[\frac{1}{\phi}-\frac{1}{(\gamma_A+\beta)}\right]\right\}^{-1}\frac{\phi\beta}{\gamma_A+\beta-\phi}\left[\frac{1}{\phi^2}-\frac{1}{(\gamma_A+\beta)^2}\right]\\
&=\left[\frac{1}{\phi^2}-\frac{1}{(\gamma_A+\beta)^2}\right]\left[\frac{1}{\phi}-\frac{1}{(\gamma_A+\beta)}\right]^{-1}\\
&=\left[(\gamma_A+\beta)^2-\phi^2\right]\left[(\gamma_A+\beta)^2\phi-\phi^2(\gamma_A+\beta)\right]^{-1}\\
&=\frac{(\gamma_A+\beta-\phi)(\gamma_A+\beta+\phi)}{(\gamma_A+\beta)\phi(\gamma_A+\beta-\phi)}\\
&=\frac{(\gamma_A+\beta+\phi)}{(\gamma_A+\beta)\phi}.
\end{aligned}$$

Thus

$$T_{E\to S}=\frac{1}{\phi}+\frac{1}{\gamma_A+\beta}. \tag{5}$$

Note that this can be rewritten as

$$T_{E\to S}=\frac{1}{\phi}+\frac{\beta}{\gamma_A+\beta}\frac{1}{\beta}=T_E+p_{A\to S}T_{A\to S}=T_E+T_A$$

where the first term is the average time spent being latent, and the second term is the probability of becoming symptomatic times the average time it takes to become symptomatic from asymptomatic.

2.2 Average duration of infectiousness, T_{AS}

$p_{A\to S}$ and $p_{A\to R}$ are the probabilities that an infected individual becomes symptomatic or recovers before becoming symptomatic, respectively

$$p_{A\to S} = \frac{\beta}{\gamma_A + \beta} \tag{6}$$

$$p_{A\to R} = \frac{\gamma_A}{\gamma_A + \beta}. \tag{7}$$

T_{AS} is the average duration of infectiousness, that is, the average time spent in states A and S

$$\begin{aligned} T_{AS} &= \int_0^\infty \left\{ \tau \left[\gamma_A e^{-\gamma_A \tau}\right] \left[e^{-\beta\tau}\right] + \left(\tau + \frac{1}{\gamma_S + \delta}\right) \left[e^{-\gamma_A \tau}\right] \left[\beta e^{-\beta\tau}\right] \right\} d\tau \\ &= \int_0^\infty \tau \left[\left(\gamma_A + \beta\right) e^{-(\gamma_A+\beta)\tau}\right] d\tau + \left(\frac{1}{\gamma_S + \delta}\right) \frac{\beta}{\beta + \gamma_A} \end{aligned}$$

where the first term in the integral represents being asymptomatic for duration τ, followed by a recovery, and the second term of the integral represents being asymptomatic for duration τ, followed by being symptomatic with an average duration $1/\left(\gamma_S + \delta\right)$. This expression simplifies to

$$T_{AS} = \frac{1}{\beta + \gamma_A} + \frac{\beta}{\beta + \gamma_A} \frac{1}{\gamma_S + \delta} = T_A + p_{A\to S} T_S, \tag{8}$$

which is the average duration of being asymptomatic plus the average duration of being symptomatic times the probability of becoming symptomatic.

2.3 Infection fatality rate (IFR), $p_{E\to D}$

The probability of dying when symptomatic is

$$p_{S\to D} = \int_0^\infty \left(\delta e^{-\delta\tau}\right) e^{-\gamma_S \tau} d\tau = \frac{\delta}{\delta + \gamma_S}.$$

The probability of dying conditional on being infected is equal to the probability of making the transition from asymptomatic to symptomatic times the probability of dying when symptomatic

$$p_{E\to D} = p_{A\to S} p_{S\to D} = \frac{\delta}{\delta + \gamma_S} \frac{\beta}{\beta + \gamma_A}. \tag{9}$$

3 Reproduction rates

We now calculate the average new infections caused by a newly infectious individual. We start with the basic reproduction rate in the SIR model and then calculate the average new infections from an infected individual in an environment with quarantine and contact tracing.

3.1 Basic reproduction rate $\mathcal{R}_0$ for SIR model

The individual is infectious at rate $\alpha S/N$ until recovery or death ($\tilde{\gamma} = \gamma + \delta$). The average number of individuals who are infected by an infectious individual are

$$\begin{aligned}\mathcal{R}_0 = &\int_0^\infty \left[\alpha \frac{S(\tau)}{N}\tau\right]\left[\left(\gamma e^{-\gamma\tau}\right)e^{-\delta\tau} + \left(\delta e^{-\delta\tau}\right)e^{-\tilde{\gamma}\tau}\right] d\tau \\ \approx &\alpha \frac{S(0)}{N}\int_0^\infty (\tau)\left(\tilde{\gamma} e^{-\tilde{\gamma}\tau}\right) d\tau \\ \approx &\alpha\tilde{\gamma}\int_0^\infty \tau e^{-\tilde{\gamma}\tau} d\tau.\end{aligned}$$

For the first approximation, we assume that changes in the measure of susceptible individuals S are small over the duration of an individual infection. For the second approximation, we assume that initially the share of susceptible individuals is close to one. From equation (1), it follows that the basic reproduction rate is

$$\mathcal{R}_0 = \frac{\alpha}{\tilde{\gamma}}. \tag{10}$$

3.2 Reproduction rate with quarantine and contact tracing

- A symptomatic individual who is not quarantined infects on average $\mathcal{R}_S = \alpha (S/N) R_S$ individuals until he recovers or dies, with

$$\begin{aligned}R_S = &\int_0^\infty \tau\left[\left(e^{-\gamma_s\tau}\right)\left(\delta e^{-\delta\tau}\right) + \left(\gamma_S e^{-\gamma_s\tau}\right)\left(e^{-\delta\tau}\right)\right] d\tau \\ = &(\gamma_S + \delta)\int_0^\infty \tau e^{-(\gamma_s+\delta)\tau} d\tau \\ = &\frac{1}{\gamma_S + \delta}.\end{aligned}$$

- An asymptomatic individual who is not quarantined infects on average $\mathcal{R}_A = \alpha (S/N) R_A$ until she recovers or becomes symptomatic, with

$$\begin{aligned}R_A = &\int_0^\infty \sigma\tau\left[\left(e^{-\gamma_A\tau}\right)\left(\beta e^{-\beta\tau}\right) + \left(\gamma_A e^{-\gamma_A\tau}\right)\left(e^{-\beta\tau}\right)\right] d\tau \\ = &\sigma\frac{1}{\gamma_A + \beta}.\end{aligned}$$

- An asymptomatic individual who is quarantined with rates ε_{QA} and ε_{QS} infects on average $\mathcal{R}_{AS} = \alpha\,(S/N)\,R_{AS}$ until he recovers or becomes symptomatic, with

$$\begin{aligned} R_{AS}\left(\varepsilon_{QA},\varepsilon_{QS}\right) &= \int_0^\infty \left[(1-\varepsilon_{QA})\,\sigma\tau + (1-\varepsilon_{QS})\,R_S\right]\left[\left(e^{-\gamma_A\tau}\right)\left(\beta e^{-\beta\tau}\right)\right] d\tau \\ &\quad + \int_0^\infty (1-\varepsilon_{QA})\,\sigma\tau\left[\left(\gamma_A e^{-\gamma_A\tau}\right)\left(e^{-\beta\tau}\right)\right] d\tau \\ &= (1-\varepsilon_{QA})\,R_A + (1-\varepsilon_{QS})\,\frac{\beta}{\gamma_A+\beta}R_S. \end{aligned}$$

- Let $\bar{R}_{AS}\left(\varepsilon_T,\varepsilon_{QA},\varepsilon_{QS}\right)$ denote the expected infection factor for a latent individual (the individual of interest) before knowing whether the individual will be traced

$$\bar{R}_{AS}\left(\varepsilon_T,\varepsilon_{QA},\varepsilon_{QS}\right) = \varepsilon_T R_{AS}\left(\varepsilon_{QA},\varepsilon_{QS}\right) + (1-\varepsilon_T)\,R_{AS}\left(0,\varepsilon_{QS}\right).$$

- Consider an individual who is latent, traced with efficiency ε_T, and quarantined with rates ε_{QA} and ε_{QS}. The individual of interest may have been infected by (1) a symptomatic individual who was not quarantined, there are $(1-\varepsilon_{QS})\,I_S$ of them; (2) an asymptomatic individual who was traced, but not quarantined, there are $(1-\varepsilon_{QA})\,I_{AT}$ of them; and (3) an asymptomatic individual who was not yet traced, there are I_A of them. We want to calculate the average of new infections coming from this individual until she recovers or dies.

  - Case 1 and 2: Since infectious individuals are only traced at the time they become symptomatic, the individual of interest will never be traced. Therefore, the expected number of new infections coming from the infected individual is $\mathcal{R}_{AS}\left(0,\varepsilon_{QS}\right) = \alpha\,(S/N)\,R_{AS}\left(0,\varepsilon_{QS}\right)$;
  - Case 3: The expected number of new infections coming from the infected individual when the infecting individual was asymptomatic is $\mathcal{R}_E = \alpha\,(S/N)\,R_E$, with

R_E

$$= \int_0^\infty \left[\gamma_A e^{-(\gamma_A+\beta)\tau_0}\right] \left[R_{AS}\left(0, \varepsilon_{QS}\right)\right] d\tau_0 \tag{11}$$

$$+ \int_0^\infty \left[\beta e^{-(\gamma_A+\beta)\tau_0}\right] \left[e^{-\phi\tau_0}\right] \bar{R}_{AS}\left(\varepsilon_T, \varepsilon_{QA}, \varepsilon_{QS}\right) d\tau_0 \tag{12}$$

$$+ \int_0^\infty \left[\beta e^{-(\gamma_A+\beta)\tau_0}\right] \left\{\int_0^{\tau_0} \left[\phi e^{-\phi s}\right] \left[e^{-(\gamma_A+\beta)(\tau_0-s)}\right] \left[\sigma\left(\tau_0 - s\right) + \bar{R}_{AS}\left(\varepsilon_T, \varepsilon_{QA}, \varepsilon_{QS}\right)\right] ds\right\} d\tau_0 \tag{13}$$

$$+ \int_0^\infty \left[\beta e^{-(\gamma_A+\beta)\tau_0}\right] \left\{\int_0^{\tau_0} \left[\phi e^{-\phi s}\right] \left[\int_0^{\tau_0-s} \left[\gamma_A e^{-(\gamma_A+\beta)t}\right] \left[\sigma t\right] dt\right] ds\right\} d\tau_0 \tag{14}$$

$$+ \int_0^\infty \left[\beta e^{-(\gamma_A+\beta)\tau_0}\right] \left\{\int_0^{\tau_0} \left[\phi e^{-\phi s}\right] \left[\int_0^{\tau_0-s} \left[\beta e^{-(\gamma_A+\beta)t}\right] \left[\sigma t + \left(1 - \varepsilon_{QS}\right) R_S\right] dt\right] ds\right\} \tag{15}$$

where the five sub-cases are

- ∗ Case 3.1, expression (11): The infecting individual recovers at τ_0, and the infected individual is not traced;
- ∗ Case 3.2-5, expressions (12) through (15): The infecting individual becomes symptomatic at τ_0 and
- ∗ Case 3.2, expression (12): The infected individual never became asymptomatic;
- ∗ Case 3.3, expression (13): The infected individual became asymptomatic at s and stayed so until τ_0;
- ∗ Case 3.4, expression (14): The infected individual became asymptomatic at s and recovered at $s + t$;
- ∗ Case 3.5, expression (15): The infected individual became asymptomatic at s and symptomatic at $s + t$.

- The probabilities are
  for Case 3.1

$$p_{E1} = \int_0^\infty \left[\gamma_A e^{-(\gamma_A+\beta)\tau_0}\right] d\tau_0 = \frac{\gamma_A}{\gamma_A + \beta}, \tag{16}$$

  for Case 3.2

$$p_{E2} = \int_0^\infty \left[\beta e^{-(\gamma_A+\beta+\phi)\tau_0}\right] d\tau_0 = \frac{\beta}{\gamma_A + \beta + \phi}, \tag{17}$$

for Case 3.3

$$
\begin{aligned}
p_{E3} = &\int_0^\infty \left[\beta e^{-(\gamma_A+\beta+\phi)\tau_0}\right] \left\{\int_0^{\tau_0} \left[\phi e^{-\phi(s-\tau_0)}\right] \left[e^{-(\gamma_A+\beta)(\tau_0-s)}\right] ds\right\} d\tau_0 \\
=&\beta\phi \int_0^\infty \left[e^{-(\gamma_A+\beta+\phi)\tau_0}\right] \left\{\int_0^{\tau_0} \left[e^{-(\gamma_A+\beta-\phi)(\tau_0-s)}\right] ds\right\} d\tau_0 \\
=&\beta\phi \int_0^\infty \left[e^{-(\gamma_A+\beta+\phi)\tau_0}\right] \frac{1}{\gamma_A+\beta-\phi} \left[1-e^{-(\gamma_A+\beta-\phi)\tau_0}\right] d\tau_0 \\
=&\frac{\beta\phi}{\gamma_A+\beta-\phi} \int_0^\infty \left[e^{-(\gamma_A+\beta+\phi)\tau_0} - e^{-2(\gamma_A+\beta)\tau_0}\right] d\tau_0 \\
=&\frac{\beta\phi}{\gamma_A+\beta-\phi} \left[\frac{1}{\gamma_A+\beta+\phi} - \frac{1}{2(\gamma_A+\beta)}\right] \\
=&\frac{\beta\phi}{\gamma_A+\beta-\phi} \cdot \frac{2(\gamma_A+\beta)-(\gamma_A+\beta+\phi)}{(\gamma_A+\beta+\phi)\,2(\gamma_A+\beta)} \\
=&\frac{\beta\phi}{\gamma_A+\beta-\phi} \cdot \frac{(\gamma_A+\beta-\phi)}{(\gamma_A+\beta+\phi)\,2(\gamma_A+\beta)} \\
=&\frac{\beta\phi}{2(\gamma_A+\beta+\phi)(\gamma_A+\beta)},
\end{aligned}
\tag{18}
$$

for Case 3.4

$$
\begin{aligned}
p_{E4} = &\int_0^\infty \left[\beta e^{-(\gamma_A+\beta)\tau_0}\right] \left\{\int_0^{\tau_0} \left[\phi e^{-\phi s}\right] \left[\int_0^{\tau_0-s} \left[\gamma_A e^{-(\gamma_A+\beta)t}\right] dt\right] ds\right\} d\tau_0 \\
=&\beta\phi\gamma_A \int_0^\infty \left[e^{-(\gamma_A+\beta)\tau_0}\right] \left\{\int_0^{\tau_0} \left[e^{-\phi s}\right] \frac{1}{\gamma_A+\beta} \left[1-e^{-(\gamma_A+\beta)(\tau_0-s)}\right] ds\right\} d\tau_0 \\
=&\frac{\beta\phi\gamma_A}{\gamma_A+\beta} \int_0^\infty \left[e^{-(\gamma_A+\beta+\phi)\tau_0}\right] \left\{\int_0^{\tau_0} \left[e^{-\phi(s-\tau_0)}\right] \left[1-e^{-(\gamma_A+\beta)(\tau_0-s)}\right] ds\right\} d\tau_0 \\
=&\frac{\beta\phi\gamma_A}{\gamma_A+\beta} \int_0^\infty \left[e^{-(\gamma_A+\beta+\phi)\tau_0}\right] \left\{\frac{1}{\phi}\left[e^{\phi\tau_0}-1\right] - \frac{1}{\gamma_A+\beta-\phi}\left[1-e^{-(\gamma_A+\beta-\phi)\tau_0}\right]\right\} d\tau_0 \\
=&\frac{\beta\phi\gamma_A}{\gamma_A+\beta} \left\{\frac{1}{\phi}\left[\frac{1}{(\gamma_A+\beta)} - \frac{1}{\gamma_A+\beta+\phi}\right] - \frac{1}{\gamma_A+\beta-\phi}\left[\frac{1}{(\gamma_A+\beta+\phi)} - \frac{1}{2(\gamma_A+\beta)}\right]\right\} \\
=&\frac{\beta\phi\gamma_A}{\gamma_A+\beta} \left\{\frac{1}{\phi}\left[\frac{\phi}{(\gamma_A+\beta+\phi)(\gamma_A+\beta)}\right] - \frac{1}{\gamma_A+\beta-\phi}\left[\frac{\gamma_A+\beta-\phi}{(\gamma_A+\beta+\phi)\,2(\gamma_A+\beta)}\right]\right\} \\
=&\frac{\beta\phi\gamma_A}{\gamma_A+\beta} \left\{\frac{1}{(\gamma_A+\beta+\phi)(\gamma_A+\beta)} - \frac{1}{(\gamma_A+\beta+\phi)\,2(\gamma_A+\beta)}\right\} \\
=&\frac{1}{2}\frac{\beta\phi\gamma_A}{(\gamma_A+\beta)^2} \cdot \frac{1}{\gamma_A+\beta+\phi},
\end{aligned}
\tag{19}
$$

for Case 3.5

$$
\begin{aligned}
p_{E5} =& \int_0^\infty \left[\beta e^{-(\gamma_A+\beta)\tau_0}\right] \left\{ \int_0^{\tau_0} \left[\phi e^{-\phi s}\right] \left[\int_0^{\tau_0 - s} \left[\beta e^{-(\gamma_A+\beta)t}\right] dt \right] ds \right\} d\tau_0 \\
=&\beta\phi\beta \int_0^\infty \left[e^{-(\gamma_A+\beta)\tau_0}\right] \left\{ \int_0^{\tau_0} \left[e^{-\phi s}\right] \left[\frac{1}{\gamma_A+\beta} \left[1 - e^{-(\gamma_A+\beta)(\tau_0-s)}\right] \right] ds \right\} d\tau_0 \\
=&\frac{\phi\beta^2}{\gamma_A+\beta} \int_0^\infty \left[e^{-(\gamma_A+\beta+\phi)\tau_0}\right] \left\{ \int_0^{\tau_0} \left[e^{-\phi(s-\tau_0)}\right] \left[1 - e^{-(\gamma_A+\beta)(\tau_0-s)}\right] ds \right\} d\tau_0 \\
=&\frac{\phi\beta^2}{\gamma_A+\beta} \int_0^\infty \left[e^{-(\gamma_A+\beta+\phi)\tau_0}\right] \left\{ \frac{1}{\phi} \left[e^{\phi\tau_0} - 1\right] - \frac{1}{\gamma_A+\beta-\phi} \left[1 - e^{-(\gamma_A+\beta-\phi)\tau_0}\right] \right\} d\tau_0 \\
=&\frac{\phi\beta^2}{\gamma_A+\beta} \left\{ \frac{1}{\phi} \left[\frac{1}{\gamma_A+\beta} - \frac{1}{\gamma_A+\beta+\phi} \right] - \frac{1}{\gamma_A+\beta-\phi} \left[\frac{1}{\gamma_A+\beta+\phi} - \frac{1}{2(\gamma_A+\beta)} \right] \right\} \\
=&\frac{\phi\beta^2}{\gamma_A+\beta} \left\{ \frac{1}{\phi} \left[\frac{\phi}{(\gamma_A+\beta)(\gamma_A+\beta+\phi)} \right] - \frac{1}{\gamma_A+\beta-\phi} \left[\frac{\gamma_A+\beta-\phi}{(\gamma_A+\beta+\phi)\, 2(\gamma_A+\beta)} \right] \right\} \\
=&\frac{\phi\beta^2}{\gamma_A+\beta} \cdot \frac{1}{(\gamma_A+\beta+\phi)} \left\{ \frac{1}{(\gamma_A+\beta)} - \frac{1}{2(\gamma_A+\beta)} \right\} \\
=&\frac{1}{2} \frac{\phi\beta^2}{(\gamma_A+\beta)^2} \cdot \frac{1}{(\gamma_A+\beta+\phi)}.
\end{aligned}
\tag{20}
$$

Note that

$$
p_{E5} = \frac{\beta}{\gamma_A} p_{E4}.
$$

The sum of the probabilities is one

$$
\begin{aligned}
& \frac{\gamma_A}{\gamma_A+\beta} + \frac{\beta}{\gamma_A+\beta+\phi} + \frac{\beta\phi}{2(\gamma_A+\beta+\phi)(\gamma_A+\beta)} + \frac{\beta\phi\gamma_A}{2(\gamma_A+\beta+\phi)(\gamma_A+\beta)^2} \\
& + \frac{\phi\beta^2}{2(\gamma_A+\beta+\phi)(\gamma_A+\beta)^2} \\
=& \frac{\gamma_A}{\gamma_A+\beta} + \frac{\beta}{\gamma_A+\beta+\phi} + \frac{\beta\phi}{2(\gamma_A+\beta+\phi)(\gamma_A+\beta)} \left[1 + \frac{\gamma_A+\beta}{(\gamma_A+\beta)}\right] \\
=& \frac{\gamma_A}{\gamma_A+\beta} + \frac{\beta}{\gamma_A+\beta+\phi} + \frac{\beta\phi}{(\gamma_A+\beta+\phi)(\gamma_A+\beta)} \\
=& \frac{\gamma_A}{\gamma_A+\beta} + \frac{\beta}{\gamma_A+\beta+\phi} \left[1 + \frac{\phi}{\gamma_A+\beta}\right] \\
=& \frac{\gamma_A}{\gamma_A+\beta} + \frac{\beta}{\gamma_A+\beta} = 1.
\end{aligned}
$$

- Before we proceed, note that the following is true

$$
\int_0^\infty e^{-\alpha\tau_0} \left\{ \int_0^{\tau_0} s e^{-\beta s} ds \right\} d\tau_0 = \frac{1}{\alpha(\alpha+\beta)^2}. \tag{21}
$$

We use equation (1)

$$
\begin{aligned}
&\int_0^\infty e^{-\alpha\tau_0}\left\{\int_0^{\tau_0} se^{-\beta s}ds\right\}d\tau_0\\
=&\int_0^\infty e^{-\alpha\tau_0}\left\{\frac{1}{\beta^2}\left[1-(\beta\tau_0+1)\,e^{-\beta\tau_0}\right]\right\}d\tau_0\\
=&\frac{1}{\beta^2}\left\{\int_0^\infty e^{-\alpha\tau_0}d\tau_0-\left[\beta\int_0^\infty \tau_0e^{-(\alpha+\beta)\tau_0}d\tau_0+\int_0^\infty e^{-(\alpha+\beta)\tau_0}d\tau_0\right]\right\}\\
=&\frac{1}{\beta^2}\left\{\frac{1}{\alpha}-\left[\frac{\beta}{(\alpha+\beta)^2}+\frac{1}{\alpha+\beta}\right]\right\}\\
=&\frac{1}{\beta^2}\left\{\frac{(\alpha+\beta)^2-\alpha\beta-\alpha\,(\alpha+\beta)}{\alpha\,(\alpha+\beta)^2}\right\}\\
=&\frac{1}{\beta^2}\left\{\frac{(\alpha+\beta)\,\beta-\alpha\beta}{\alpha\,(\alpha+\beta)^2}\right\}\\
=&\frac{1}{\alpha\,(\alpha+\beta)^2}.
\end{aligned}
$$

- The expected infections are
  for Case 3.1

$$
\begin{aligned}
R_{E1}=&R_{AS}\,(0,\varepsilon_{QS})\int_0^\infty\left[\gamma_Ae^{-(\gamma_A+\beta)\tau_0}\right]d\tau_0=p_{E1}R_{AS}\,(0,\varepsilon_{QS})\\
=&\frac{\gamma_A}{\gamma_A+\beta}R_{AS}\,(0,\varepsilon_{QS})\,,
\end{aligned}
\tag{22}
$$

  for Case 3.2

$$
\begin{aligned}
R_{E2}=&\bar{R}_{AS}\,(\varepsilon_T,\varepsilon_{QA},\varepsilon_{QS})\int_0^\infty\left[\beta e^{-(\gamma_A+\beta)\tau_0}\right]\left[e^{-\phi\tau_0}\right]d\tau_0=p_{E2}\bar{R}_{AS}\,(\varepsilon_T,\varepsilon_{QA},\varepsilon_{QS})\\
=&\frac{\beta}{\gamma_A+\beta+\phi}\bar{R}_{AS}\,(\varepsilon_T,\varepsilon_{QA},\varepsilon_{QS})\,,
\end{aligned}
\tag{23}
$$

for Case 3.3 [where you use equation (21) at the fourth step]

$$\begin{aligned}
R_{E3} =& \int_0^\infty \left[\beta e^{-(\gamma_A+\beta)\tau_0}\right] \left\{ \int_0^{\tau_0} \left[\phi e^{-\phi s}\right] \left[e^{-(\gamma_A+\beta)(\tau_0-s)}\right] \left[\sigma\left(\tau_0 - s\right) + \bar{R}_{AS}\left(\varepsilon_T, \varepsilon_{QA}, \varepsilon_{QS}\right)\right] ds \right\} d\tau_0 \\
=& \int_0^\infty \left[\beta e^{-(\gamma_A+\beta)\tau_0}\right] \left\{ \int_0^{\tau_0} \left[\phi e^{-\phi s}\right] \left[e^{-(\gamma_A+\beta)(\tau_0-s)}\right] \left[\sigma\left(\tau_0 - s\right)\right] ds \right\} d\tau_0 \\
&+ p_{E3}\bar{R}_{AS}\left(\varepsilon_T, \varepsilon_{QA}, \varepsilon_{QS}\right) \\
=& \beta\phi\sigma \int_0^\infty \left[e^{-(\gamma_A+\beta+\phi)\tau_0}\right] \left\{ \int_0^{\tau_0} \left[e^{-\phi(s-\tau_0)}\right] \left[e^{-(\gamma_A+\beta)(\tau_0-s)}\right] \left[\left(\tau_0 - s\right)\right] ds \right\} d\tau_0 \\
&+ p_{E3}\bar{R}_{AS}\left(\varepsilon_T, \varepsilon_{QA}, \varepsilon_{QS}\right) \\
=& \beta\phi\sigma \int_0^\infty \left[e^{-(\gamma_A+\beta+\phi)\tau_0}\right] \left\{ \int_0^{\tau_0} \left[e^{-(\gamma_A+\beta-\phi)(\tau_0-s)}\right] \left[\left(\tau_0 - s\right)\right] ds \right\} d\tau_0 \\
&+ p_{E3}\bar{R}_{AS}\left(\varepsilon_T, \varepsilon_{QA}, \varepsilon_{QS}\right) \\
=& \beta\phi\sigma \frac{1}{\left(\gamma_A+\beta+\phi\right)\left[\left(\gamma_A+\beta+\phi\right)+\left(\gamma_A+\beta-\phi\right)\right]^2} + p_{E3}\bar{R}_{AS}\left(\varepsilon_T, \varepsilon_{QA}, \varepsilon_{QS}\right) \\
=& \sigma \frac{\beta\phi}{\left(\gamma_A+\beta+\phi\right)\left[2\left(\gamma_A+\beta\right)\right]^2} + p_{E3}\bar{R}_{AS}\left(\varepsilon_T, \varepsilon_{QA}, \varepsilon_{QS}\right) \\
=& p_{E3}\left[\frac{\sigma}{2\left(\gamma_A+\beta\right)} + \bar{R}_{AS}\left(\varepsilon_T, \varepsilon_{QA}, \varepsilon_{QS}\right)\right],
\end{aligned} \tag{24}$$

for Case 3.4

$$
\begin{aligned}
R_{E4} =& \int_0^\infty \left[\beta e^{-(\gamma_A+\beta)\tau_0}\right] \left\{ \int_0^{\tau_0} \left[\phi e^{-\phi s}\right] \left[\int_0^{\tau_0 - s} \left[\gamma_A e^{-(\gamma_A+\beta)t}\right] [\sigma t]\, dt \right] ds \right\} d\tau_0 \\
=& \beta\phi\gamma_A\sigma \int_0^\infty e^{-(\gamma_A+\beta+\phi)\tau_0} \left\{ \int_0^{\tau_0} e^{-\phi(s-\tau_0)} \left[\int_0^{\tau_0-s} t e^{-(\gamma_A+\beta)t} dt \right] ds \right\} d\tau_0 \\
=& \beta\phi\gamma_A\sigma \int_0^\infty e^{-(\gamma_A+\beta+\phi)\tau_0} \left\{ \int_0^{\tau_0} \frac{e^{-\phi(s-\tau_0)}}{(\gamma_A+\beta)^2} \right. \\
& \left. \times \left[1 - \left[(\gamma_A+\beta)(\tau_0 - s) + 1\right] e^{-(\gamma_A+\beta)(\tau_0-s)}\right] ds \right\} d\tau_0 \\
=& \frac{\beta\phi\gamma_A\sigma}{(\gamma_A+\beta)^2} \int_0^\infty e^{-(\gamma_A+\beta+\phi)\tau_0} \left\{ \int_0^{\tau_0} e^{\phi(\tau_0-s)} \right. \\
& \left. \times \left[1 - \left[(\gamma_A+\beta)(\tau_0 - s) + 1\right] e^{-(\gamma_A+\beta)(\tau_0-s)}\right] ds \right\} d\tau_0 \\
=& \frac{\beta\phi\gamma_A\sigma}{(\gamma_A+\beta)^2} \int_0^\infty e^{-(\gamma_A+\beta+\phi)\tau_0} \left\{ \int_0^{\tau_0} e^{\phi s} \left[1 - \left[(\gamma_A+\beta) s + 1\right] e^{-(\gamma_A+\beta)s}\right] ds \right\} d\tau_0 \\
=& \frac{\beta\phi\gamma_A\sigma}{(\gamma_A+\beta)^2} \int_0^\infty e^{-(\gamma_A+\beta+\phi)\tau_0} \left\{ \int_0^{\tau_0} \left[e^{\phi s} - e^{-(\gamma_A+\beta-\phi)s} - (\gamma_A+\beta) s e^{-(\gamma_A+\beta-\phi)s}\right] ds \right\} d\tau_0 \\
=& \frac{\beta\phi\gamma_A\sigma}{(\gamma_A+\beta)^2} \left\{ \int_0^\infty e^{-(\gamma_A+\beta+\phi)\tau_0} \left\{ \int_0^{\tau_0} \left[e^{\phi s} - e^{-(\gamma_A+\beta-\phi)s}\right] ds \right\} d\tau_0 \right. \\
& \left. - \int_0^\infty e^{-(\gamma_A+\beta+\phi)\tau_0} \int_0^{\tau_0} (\gamma_A+\beta) s e^{-(\gamma_A+\beta-\phi)s} ds d\tau_0 \right\} \\
=& \frac{\beta\phi\gamma_A\sigma}{(\gamma_A+\beta)^2} \left\{ \int_0^\infty e^{-(\gamma_A+\beta+\phi)\tau_0} \left\{ \frac{1}{\phi}\left[e^{\phi\tau_0} - 1\right] - \frac{1}{\gamma_A+\beta-\phi}\left[1 - e^{-(\gamma_A+\beta-\phi)\tau_0}\right] \right\} d\tau_0 \right. \\
& \left. - (\gamma_A+\beta) \frac{1}{(\gamma_A+\beta+\phi)\left[(\gamma_A+\beta+\phi) + (\gamma_A+\beta-\phi)\right]^2} \right\} \\
=& \frac{\beta\phi\gamma_A\sigma}{(\gamma_A+\beta)^2} \left\{ \frac{1}{\phi}\left[\frac{1}{\gamma_A+\beta} - \frac{1}{\gamma_A+\beta+\phi}\right] - \frac{1}{\gamma_A+\beta-\phi}\left[\frac{1}{\gamma_A+\beta+\phi} - \frac{1}{2(\gamma_A+\beta)}\right] \right. \\
& \left. - (\gamma_A+\beta) \frac{1}{(\gamma_A+\beta+\phi)\left[2(\gamma_A+\beta)\right]^2} \right\} \\
=& \frac{\beta\phi\gamma_A\sigma}{(\gamma_A+\beta)^2} \left\{ \frac{1}{\phi}\left[\frac{\phi}{(\gamma_A+\beta)(\gamma_A+\beta+\phi)}\right] - \frac{1}{\gamma_A+\beta-\phi}\left[\frac{\gamma_A+\beta-\phi}{(\gamma_A+\beta+\phi)\, 2(\gamma_A+\beta)}\right] \right. \\
& \left. - (\gamma_A+\beta) \frac{1}{(\gamma_A+\beta+\phi)\left[2(\gamma_A+\beta)\right]^2} \right\} \\
=& \frac{\beta\phi\gamma_A\sigma}{(\gamma_A+\beta)^2} \left\{ \frac{1}{(\gamma_A+\beta)(\gamma_A+\beta+\phi)} - \frac{1}{(\gamma_A+\beta+\phi)\, 2(\gamma_A+\beta)} \right. \\
& \left. - \frac{1}{(\gamma_A+\beta+\phi)\, 4(\gamma_A+\beta)} \right\} \\
=& \frac{\beta\phi\gamma_A\sigma}{4(\gamma_A+\beta+\phi)(\gamma_A+\beta)^3},
\end{aligned} \tag{25}
$$

for Case 3.5

$$\begin{aligned}
R_{E5} =& \int_0^\infty \left[\beta e^{-(\gamma_A+\beta)\tau_0}\right] \left\{ \int_0^{\tau_0} \left[\phi e^{-\phi s}\right] \left[\int_0^{\tau_0 - s} \left[\beta e^{-(\gamma_A+\beta)t}\right] \left[\sigma t + (1-\varepsilon_{QS}) R_S\right] dt \right] ds \right\} d\tau_0 \\
=& \beta^2 \phi \sigma \int_0^\infty e^{-(\gamma_A+\beta)\tau_0} \left\{ \int_0^{\tau_0} e^{-\phi s} \left[\int_0^{\tau_0 - s} t e^{-(\gamma_A+\beta)t} dt \right] ds \right\} d\tau_0 \\
&+ (1-\varepsilon_{QS}) R_S \beta^2 \phi \int_0^\infty e^{-(\gamma_A+\beta)\tau_0} \left\{ \int_0^{\tau_0} e^{-\phi s} \left[\int_0^{\tau_0 - s} e^{-(\gamma_A+\beta)t} ds \right] dt \right\} d\tau_0 \\
=& \frac{\beta}{\gamma_A} \beta \gamma_A \phi \sigma \int_0^\infty e^{-(\gamma_A+\beta)\tau_0} \left\{ \int_0^{\tau_0} e^{-\phi s} \left[\int_0^{\tau_0 - s} t e^{-(\gamma_A+\beta)t} dt \right] ds \right\} d\tau_0 \\
&+ (1-\varepsilon_{QS}) R_S p_{E5} \\
=& \frac{\beta}{\gamma_A} R_{E4} + (1-\varepsilon_{QS}) R_S \frac{\beta}{\gamma_A} p_{E4} \\
=& \frac{\beta}{\gamma_A} \left[R_{E4} + p_{E4} (1-\varepsilon_{QS}) R_S\right],
\end{aligned} \tag{26}$$

and

$$R_E = \sum_{i=1}^{5} R_{E,i}.$$

- A newly infected individual then infects on average $\mathcal{R} = \alpha (S/N) R$ individuals where

$$R = \frac{\left[(1-\varepsilon_{QA}) I_{AT} + (1-\varepsilon_{QS}) I_S\right] R_{AS}(0, \varepsilon_{QS}) + I_A R_E}{(1-\varepsilon_{QA}) I_{AT} + (1-\varepsilon_{QS}) I_S + I_A}.$$

3.3 Traceable individuals

This time we consider an asymptomatic infectious individual who is quarantined once he becomes symptomatic. For this case, we calculate the average number of exposed and infectious asymptomatic individuals who this individual has created.

By the time an asymptomatic individual becomes symptomatic, on average who individual has infected $\alpha(t) S(t)/N(t) R_{AQ}$ other individuals, where

$$\begin{aligned}
R_{AQ} =& \int_0^\infty (\sigma\tau) \left(\beta e^{-\beta\tau}\right) \left(e^{-\gamma_A \tau}\right) d\tau \\
=& \sigma\beta \int_0^\infty \tau e^{-(\beta+\gamma_A)\tau} d\tau \\
=& \sigma \frac{\beta}{(\beta+\gamma_A)^2}.
\end{aligned}$$

The average number of individuals who the infectious individual has infected and who are not yet infectious at the time the individual becomes symptomatic is $\alpha(t) S(t)/N(t) R_{AQE}$ where

$$R_{AQE} = \int_0^\infty \sigma p_{EE}(\tau) \left[\beta e^{-(\beta+\gamma_A)\tau}\right] d\tau.$$

where the term in brackets is the probability that the infectious individual has been asymptomatic for duration τ, and $p_{EE}(\tau)$ denotes the fraction of individuals who were infected by the infectious individual over the interval τ and who have not yet become infectious at the time the individual becomes symptomatic. The probability that an individual who was infected time s ago and has not yet become infectious is $e^{-\phi s}$. Thus

$$p_{EE}(\tau) = \int_0^\tau e^{-\phi s} ds = \frac{1}{\phi}\left(1 - e^{-\phi\tau}\right)$$

and

$$\begin{aligned} R_{AQE} =& \beta\sigma \int_0^\infty \left[\frac{1}{\phi}\left(1 - e^{-\phi\tau}\right)\right] e^{-(\beta+\gamma_A)\tau} d\tau \\ =& \frac{\beta\sigma}{\phi}\left[\frac{1}{\beta+\gamma_A} - \frac{1}{\beta+\gamma_A+\phi}\right] \\ =& \frac{\beta\sigma}{\phi}\frac{\phi}{(\beta+\gamma_A)(\beta+\gamma_A+\phi)}. \end{aligned}$$

Therefore

$$R_{AQE} = \sigma\frac{\beta}{(\beta+\gamma_A)(\beta+\gamma_A+\phi)}. \quad (27)$$

The average number of individuals who an infectious individual has infected and who are infectious but asymptomatic at the time the individual becomes symptomatic is $\alpha(t)\, S(t)/N(t) R_{AQA}$ where

$$R_{AQA} = \int_0^\infty \left[\sigma \int_0^\tau p_{EA}(s)\, ds\right]\left[\beta e^{-(\beta+\gamma_A)\tau}\right] d\tau$$

and $p_{EA}(s)$ denotes the fraction of individuals who were infected time s ago, have become infectious in the meantime, but have not yet recovered or become symptomatic. Thus

$$\begin{aligned} p_{EA}(s) =& \int_0^s \phi e^{-\phi v} e^{-(\gamma_A+\beta)(s-v)} dv \\ =& \phi e^{-(\gamma_A+\beta)s} \int_0^s e^{-(\phi-\gamma_A-\beta)v} dv \\ =& e^{-(\gamma_A+\beta)s} \frac{\phi}{\phi-\gamma_A-\beta}\left[1 - e^{-(\phi-\gamma_A-\beta)s}\right], \\ \lim_{s\to\infty} p_{EA}(s) =& 0. \end{aligned}$$

Substituting in the expression for R_{AQA}, we get

$$\begin{aligned}
R_{AQA} =& \sigma\beta\frac{\phi}{\phi-\gamma_A-\beta}\int_0^\infty\left[\int_0^\tau e^{-(\gamma_A+\beta)s}\left[1-e^{-(\phi-\gamma_A-\beta)s}\right]ds\right]e^{-(\beta+\gamma_A)\tau}d\tau \\
=& \frac{\sigma\phi\beta}{\phi-\gamma_A-\beta}\int_0^\infty\left[\frac{1}{\gamma_A+\beta}\left[1-e^{-(\gamma_A+\beta)\tau}\right]-\frac{1}{\phi}\left(1-e^{-\phi\tau}\right)\right]e^{-(\beta+\gamma_A)\tau}d\tau \\
=& \frac{\sigma\phi\beta}{\phi-\gamma_A-\beta}\left\{\frac{1}{\gamma_A+\beta}\left[\frac{1}{\gamma_A+\beta}-\frac{1}{2(\gamma_A+\beta)}\right]-\frac{1}{\phi}\left[\frac{1}{\beta+\gamma_A}-\frac{1}{\beta+\gamma_A+\phi}\right]\right\} \\
=& \frac{\sigma\phi\beta}{\phi-\gamma_A-\beta}\left\{\frac{1}{2}\frac{1}{(\gamma_A+\beta)^2}-\frac{1}{\phi}\left[\frac{\phi}{(\beta+\gamma_A)(\beta+\gamma_A+\phi)}\right]\right\} \\
=& \frac{\sigma\phi\beta}{(\phi-\gamma_A-\beta)(\beta+\gamma_A)}\left\{\frac{1}{2}\frac{1}{\beta+\gamma_A}-\frac{1}{\beta+\gamma_A+\phi}\right\} \\
=& \frac{\sigma\phi\beta}{(\phi-\gamma_A-\beta)(\beta+\gamma_A)}\left\{\frac{(\beta+\gamma_A+\phi)-2(\beta+\gamma_A)}{2(\beta+\gamma_A)(\beta+\gamma_A+\phi)}\right\} \\
=& \frac{\sigma\phi\beta}{(\phi-\gamma_A-\beta)(\beta+\gamma_A)}\frac{\phi-\beta-\gamma_A}{2(\beta+\gamma_A)(\beta+\gamma_A+\phi)}.
\end{aligned}$$

Therefore

$$R_{AQA} = \sigma\frac{\phi\beta}{2(\beta+\gamma_A)^2(\beta+\gamma_A+\phi)}. \tag{28}$$

The average number of individuals who the infectious individual has infected and who have become symptomatic or recovered at the time the individual becomes symptomatic is $\alpha(t)\,S(t)/N(t)R_{AQR}$ where

$$R_{AQR} = \int_0^\infty \sigma\int_0^\tau p_{ER}(s)\,d\tau\left[\beta e^{-(\beta+\gamma_A)\tau}\right]d\tau$$

and $p_{ER}(s)$ denotes the fraction of individuals who were infected time s ago and who have recovered or become symptomatic,

$$p_{ER}(s) = \int_0^s \phi e^{-\phi v}\left[1-e^{-(\gamma_A+\beta)(s-v)}\right]dv.$$

Note that

$$\begin{aligned}
& p_{EE}(\tau)+\int_0^\tau\left[p_{EA}(s)+p_{ER}(s)\right]ds \\
=& \int_0^\tau e^{-\phi s}ds+\int_0^\tau\left[\int_0^s \phi e^{-\phi v}dv\right]ds \\
=& \frac{1}{\phi}\left(1-e^{-\phi\tau}\right)+\phi\int_0^\tau\left[\frac{1}{\phi}\left(1-e^{-\phi s}\right)\right]ds \\
=& \frac{1}{\phi}\left(1-e^{-\phi\tau}\right)+\tau-\frac{1}{\phi}\left(1-e^{-\phi\tau}\right) \\
=& \tau,
\end{aligned}$$

which are all the individuals who have been infected over the interval τ. Therefore

$$\begin{aligned}
R_{AQR} = & \int_0^\infty \sigma \left[\tau - p_{EE} - \int_0^\tau p_{EA}\left(s\right) d\tau\right] \left[\beta e^{-(\beta+\gamma_A)\tau}\right] d\tau \\
= & \sigma \int_0^\infty \tau \left[\beta e^{-(\beta+\gamma_A)\tau}\right] d\tau - \sigma \int_0^\infty p_{EE} \left[\beta e^{-(\beta+\gamma_A)\tau}\right] d\tau \\
& - \sigma \int_0^\infty \left[\int_0^\tau p_{EA}\left(s\right) d\tau\right] \left[\beta e^{-(\beta+\gamma_A)\tau}\right] d\tau \\
= & R_{AQ} - R_{AQE} - R_{AQA} \\
= & \frac{\sigma\beta}{\left(\beta+\gamma_A\right)^2} - \frac{\sigma\beta}{\left(\beta+\gamma_A\right)\left(\beta+\gamma_A+\phi\right)} - \frac{\sigma\beta\phi}{2\left(\beta+\gamma_A\right)^2\left(\beta+\gamma_A+\phi\right)} \\
= & \frac{\sigma\beta}{\beta+\gamma_A}\left[\frac{1}{\beta+\gamma_A} - \frac{1}{\left(\beta+\gamma_A+\phi\right)} - \frac{\phi}{2\left(\beta+\gamma_A\right)\left(\beta+\gamma_A+\phi\right)}\right] \\
= & \frac{\sigma\beta}{\beta+\gamma_A}\left[\frac{2\left(\beta+\gamma_A+\phi\right) - 2\left(\beta+\gamma_A\right) - \phi}{2\left(\beta+\gamma_A\right)\left(\beta+\gamma_A+\phi\right)}\right] \\
= & \frac{\sigma\beta}{\beta+\gamma_A}\left[\frac{\phi}{2\left(\beta+\gamma_A\right)\left(\beta+\gamma_A+\phi\right)}\right].
\end{aligned}$$

Thus

$$R_{AQR} = \frac{\sigma\beta\phi}{2\left(\beta+\gamma_A\right)^2\left(\beta+\gamma_A+\phi\right)} = R_{AQA}.$$

What about the relative magnitudes?

$$\begin{aligned}
\frac{R_{AQE}}{R_{AQ}} = & \frac{\sigma\beta/\left[\left(\beta+\gamma_A\right)\left(\beta+\gamma_A+\phi\right)\right]}{\sigma\beta/\left(\beta+\gamma_A\right)^2} = \frac{\beta+\gamma_A}{\beta+\gamma_A+\phi} \\
\frac{R_{AQA}}{R_{AQ}} = & \frac{\sigma\beta\phi/\left[2\left(\beta+\gamma_A\right)^2\left(\beta+\gamma_A+\phi\right)\right]}{\sigma\beta/\left(\beta+\gamma_A\right)^2} = \frac{\phi}{2\left(\beta+\gamma_A+\phi\right)} \\
\frac{R_{AQR}}{R_{AQ}} = & \frac{\sigma\beta\phi/\left[2\left(\beta+\gamma_A\right)^2\left(\beta+\gamma_A+\phi\right)\right]}{\sigma\beta/\left(\beta+\gamma_A\right)^2} = \frac{\phi}{2\left(\beta+\gamma_A+\phi\right)}.
\end{aligned}$$