Working Paper Series

# Debt Limits and Credit Bubbles in General Equilibrium

WP 19-19

V. Filipe Martins-da-Rocha
Sao Paulo School of Economics-FGV and CNRS

Toan Phan
Federal Reserve Bank of Richmond

Yiannis Vailakis
University of Glasgow



FEDERAL RESERVE BANK OF RICHMOND®
Richmond ▪ Baltimore ▪ Charlotte

# Debt Limits and Credit Bubbles in General Equilibrium*

V. Filipe Martins-da-Rocha[†] Toan Phan[‡] Yiannis Vailakis[§]

October 2, 2019

Working Paper No. 19-19

**Abstract**

We provide a novel characterization of self-enforcing debt limits in a general equilibrium framework of risk sharing with limited commitment, where defaulters are subject to recourse (a fractional loss of current and future endowments) and exclusion from future credit. We show that debt limits are exactly equal to the present value of recourse plus a credit bubble component. We provide applications to models of sovereign debt, private collateralized debt, and domestic public debt. Implications include an original equivalence mapping among distinct institutional arrangements, thereby clarifying the relationship between different enforcement mechanisms and the connection between asset and credit bubbles.

**Keywords:** Limited commitment; general equilibrium; rational credit bubbles.

JEL codes: E00; E10; F00.

# 1 Introduction

Consider an environment where economic agents borrow and save in order to smooth their consumption against fluctuating endowments but agents cannot commit to repaying debt.

*We thank Kartik Athreya, Bernardo Guimaraes, Guido Lorenzoni, Pietro Peretto, Nico Trachter, Jan Werner and various seminar/conference participants at the Federal Reserve Bank of Richmond, Virginia Tech, Duke University Triangle Dynamic Macro workshop, University of Tokyo, University of California Santa Cruz, Santa Clara University, University of Virginia, the Economics School of Louvain, the Sao Paulo School of Economics, the Canon Institute for Global Studies, and the 71st European Meeting of the Econometric Society for useful comments and suggestions. The views expressed herein are those of the authors and not those of the Federal Reserve Bank of Richmond or the Federal Reserve System.

[†]Sao Paulo School of Economics–FGV and CNRS; filipe.econ@gmail.com.

[‡]The Federal Reserve Bank of Richmond; toanvphan@gmail.com.

[§]The University of Glasgow; yiannis.vailakis@glasgow.ac.uk.

Default induces both direct sanctions (loss of a fraction of current and future endowments) and a reputation loss (no access to future credit). How much debt can these agents issue in this setting?

This is an important and classic question with many relevant implications. When agents are sovereign governments, the question maps to the sustainability of sovereign debt, which is the topic of a large and growing body of research (Wright 2013, Aguiar and Amador 2014).[1] When agents are individuals or corporations, the question maps to the sustainability of consumer debt (Livshits 2015) or corporate debt (Azariadis et al. 2015). Since one agent's liability is another agent's asset in general equilibrium, the question also maps to the availability of assets or liquidity. It is well-known that a scarcity of assets facilitates bubbles (Tirole 1985; Farhi and Tirole 2012) and a growing body of papers has argued both empirically and theoretically that asset and credit bubbles play an important role in the observed phenomena of long recessions, low interest rates, liquidity traps, and global imbalances (Caballero et al. 2008; Jordà et al. 2015; Caballero and Farhi 2017; Barlevy 2018; Biswas et al. 2018; Ikeda and Phan 2019).

In this paper, we revisit these issues with an emphasis on general equilibrium implications. We consider an endowment economy where agents can issue and trade debt contracts. Agents cannot commit to honor their liabilities, but should they default, agents will face two forms of punishment. First, they will lose a nonnegative fraction of their current and future endowments. As mentioned, this loss can be mapped to output loss in the case of a sovereign default or to recourse and seized collateral in the case of consumer and corporate default. Second, agents will be excluded from future borrowing but they can still save. As in Alvarez and Jermann (2000), agents face endogenous "not-too-tight" debt limits, which in equilibrium are set at the largest possible levels such that repayment is always individually rational. These debt limits determine the borrowing capacity and thus the availability of assets in equilibrium and are the focus of this paper.

We provide an intuitive but powerful result that characterizes equilibrium debt limits: for each agent, the debt limit in any contingency is exactly the sum of a "secured" component, which is equal to the present value of the current and future endowment losses (discounted at the endogenous equilibrium interest rates), and an "unsecured" credit bubble component. In other words, the secured component of the debt limits is supported by the direct endowment loss. The remaining component, supported by the threat of credit exclusion, is necessarily

[1]Also related is the sustainability of public debt (D'Erasmo et al. 2016).

bubbly in the sense that it satisfies an exact rollover condition.

Despite its intuitive nature, proving this property is rather challenging. This is because in our environment, unlike in Eaton and Gersovitz (1981) or Alvarez and Jermann (2000), defaulting agents can still save, and hence the punishment value for defaulting involves equilibrium prices. Our proof strategy exploits a novel intermediate finding about the equilibrium interest rates that has no analogue in the absence of output losses. Specifically, we show that, in any equilibrium with self-enforcing debt, the present value of endowment losses upon default must be finite, or, equivalently, the implied interest rates must be higher than the growth rates of aggregate endowment losses. A standard argument then reveals that the process of present values of endowment losses is itself not too tight. The result then follows from the fact that the difference between two not-too-tight processes is necessarily a bubble.

We also show that a necessary condition for the emergence of credit bubbles is that the economy's total resources have an infinite present value. A necessary condition for this is that the aggregate endowment losses are negligible with respect to the economy's total resources. Note that in the special case where the endowment losses are set to zero, our characterization result implies as a corollary that the competitive equilibrium either features no borrowing/lending or features trades where borrowers purely roll over their debt as credit bubbles. Thus, our paper nests the well-known theorems in Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2009).

Our result has a broad set of applications and implications. First, it simplifies substantially the computation of equilibria, ruling out complications related to the fixed-point determination of not-too-tight debt limits. This is important since in our setting the value of default depends on prices, and thus equilibrium allocations cannot be obtained by solving a planner's problem as is the case when the default option is autarky (Alvarez and Jermann 2001). We illustrate this by looking at a simple deterministic economy with constant endowment loss and endowment growth. We fully characterize equilibrium outcomes and prove the existence of competitive equilibria where a positive fundamental component in debt limits coexists with a positive bubble component.

Another implication of our characterization theorem is an interesting equivalence result: a competitive equilibrium with inside liquidity (i.e., tradable debt securities issued by private agents) is isomorphic to a competitive equilibrium with outside liquidity in the form of public debt (Holmström and Tirole 2011). Outstanding debt in both environments reflects the present value of overall resources used to back up liabilities plus a bubble component. However, at the self-enforcing equilibrium the privilege of issuing the credit bubble is attributed to agents, whereas at the public debt equilibrium this privilege is entitled to the government. A technical implication of this equivalence result is that it allows us to prove the existence of a competitive equilibrium with limited commitment in a simpler way. It is well-known that standard proofs based on truncation arguments are generally difficult to apply to models with limited commitment due to the presence of the self-enforcing conditions. The equivalence allows us to employ a truncation technique to tackle an alternative problem that is free of the not-too-tight restrictions. Nevertheless, an additional source of difficulty remains as the supply of government debt is endogenous in the definition of the equilibrium with public debt. We suggest a way to deal with this issue that can be of independent interest.

We subsequently argue, via a second equivalence result, that equilibria with not-too-tight debt constraints can be mapped to equilibria with collateral constraints (Chien and Lustig 2010; Gottardi and Kubler 2015) and vice versa. In the latter environment, in addition to the issuance of private debt, each agent is free to issue and trade a durable and collateralizable asset (a Lucas tree) whose dividend process amounts to a fraction of her income. Upon default, the borrower's equity is confiscated and passed to the hands of creditors (as opposed to the previous environment where endowment losses are not passed to creditors). Apart from the seizure of equity holdings, there is no additional punishment for default: the agent can trade on financial markets. The default option is endogenous and reflects not only the loss of future investment income (the asset's dividends), but also the loss of using the asset as a source of liquidity. In equilibrium, equity prices reflect assets' fundamental value and a bubble component. Thus, the credit bubble component in the previous setting with self-enforcing debt can be mapped to the bubble component in asset prices in this setting with collateral and vice versa.

The equivalence result challenges a common view that models with collateral constraints are more in line with data at business cycle frequencies. For instance, Chien and Lustig (2010) found that their model produces more volatile equity risk-premiums than the limited commitment model of Alvarez and Jermann (2001). We show that, when debt enforcement relies on recourse and one-sided exclusion, the self-enforcing mechanism is observationally equivalent to the collateral mechanism.

Finally, it is well-known that Markovian (with respect to a simple state space) collateral equilibria have a tractable representation, where Negishi's method can be applied to prove existence and even compute them in a fairly simple way (e.g., Gottardi and Kubler 2015).

Although this representation is often cited as a reason to prefer these types of models for equilibrium analysis, our equivalence result shows that equilibria with self-enforcing debt can also be cast in terms of solutions to a programming problem, thereby suggesting that their computation is not of a different complexity. We prove existence and uniqueness of a Markov equilibrium, thereby complementing the analysis in Gottardi and Kubler (2015) along this line.

**Related literature.** The paper is related to the general equilibrium literature on risk sharing with limited commitment and endogenous borrowing constraints. A nonexhaustive list of contributions includes Kehoe and Levine (1993), Alvarez and Jermann (2000, 2001), Kehoe and Perri (2002), Kehoe and Perri (2004), Ábrahám and Cárceles-Poveda (2006, 2010), Bloise and Reichlin (2011), Bloise et al. (2013), Werner (2014), Martins-da-Rocha and Vailakis (2015), Bidian (2016), where default induces full exclusion from financial markets, and Bulow and Rogoff (1989), Gul and Pesendorfer (2004), Hellwig and Lorenzoni (2009), Werner (2014), Bidian and Bejan (2015) and Martins-da-Rocha and Vailakis (2017a,b), where defaulters are only excluded from future credit. To the best of our knowledge, our paper is the first that introduces the empirically relevant recourse feature of endowment losses from default into this general equilibrium environment.

Our paper is also related to Woodford (1990), Holmström and Tirole (1998, 2011), and Werner (2014) in exploring the relationship between private liquidity and public liquidity in environments with financial frictions that lead to scarce collateral. The work is also related to the rational asset price bubbles literature, which dates back to Samuelson (1958), Diamond (1965), and Tirole (1985). Recent papers in this literature include Farhi and Tirole (2012), Martin and Ventura (2012), Hirano and Yanagawa (2016), Miao and Wang (2018), and Bengui and Phan (2018). To the best of our knowledge, our paper is the first to formalize a mapping between the credit bubble component of debt limits and the bubble component of asset prices.

The plan for the rest of the paper is as follows. Section 2 sets up the model. Section 3 provides the main result. Section 4 provides applications and implications. Section 5 concludes.

# 2 Model

## 2.1 Fundamentals

Consider an infinite-horizon endowment economy with a single nonstorable consumption good at each date. Time and uncertainty are both discrete. We use an event tree $\Sigma$ to describe the revelation of information over an infinite horizon. There is a unique initial date-0 event $s^0 \in \Sigma$ and for each date $t \in \{0, 1, 2, \ldots\}$ there is a finite set $S^t \subseteq \Sigma$ of date-$t$ events $s^t$. Each $s^t$ has a unique predecessor $\sigma(s^t)$ in $S^{t-1}$ and a finite number of successors $s^{t+1}$ in $S^{t+1}$ for which $\sigma(s^{t+1}) = s^t$. The notation $s^{t+1} \succ s^t$ specifies that $s^{t+1}$ is a successor of $s^t$. The event $s^{t+\tau}$ is said to follow event $s^t$, also denoted $s^{t+\tau} \succ s^t$, if $\sigma^{(\tau)}(s^{t+\tau}) = s^t$. The set $S^{t+\tau}(s^t) := \{s^{t+\tau} \in S^{t+\tau} \ : \ s^{t+\tau} \succ s^t\}$ denotes the collection of all date-$(t+\tau)$ events following $s^t$. Abusing notation, we let $S^t(s^t) := \{s^t\}$. The subtree starting at event $s^t$ is then given by

$$\Sigma(s^t) := \bigcup_{\tau \geqslant 0} S^{t+\tau}(s^t).$$

We use the notation $s^\tau \succeq s^t$ when $s^\tau \succ s^t$ or $s^\tau = s^t$. In particular, we have $\Sigma(s^t) = \{s^\tau \in \Sigma \ : \ s^\tau \succeq s^t\}$.

There is a finite set $I$ of household types, each consisting of a unit measure of identical, infinitely lived agents who consume the single perishable good. Agents cannot commit to future actions: at any event $s^t$, they can refuse to honor past promises and default.

Preferences over (nonnegative) consumption processes $c = (c(s^t))_{s^t \succeq s^0}$ are represented by the lifetime expected and discounted utility

$$U(c) := \sum_{t \geqslant 0} \beta^t \sum_{s^t \in S^t} \pi(s^t) u(c(s^t))$$

where $\beta \in (0, 1)$ is the discount factor, $\pi(s^t)$ is the unconditional probability of $s^t$, and $u : [0, \infty) \to \mathbb{R}$ is a utility function that is strictly increasing, concave, continuous on $[0, \infty)$, differentiable on $(0, \infty)$, and satisfies Inada's condition $\lim_{\varepsilon \to 0}[u(\varepsilon) - u(0)]/\varepsilon = \infty$.[2] As in Hellwig and Lorenzoni (2009), we assume that $u$ is bounded. The only role of this assumption is to guarantee that the lifetime utility $U$ is continuous and the demand set is non-empty. The analysis can be extended and our results continue to hold even when $u$ is unbounded,

[2]For simplicity, we assume that agents' preferences are homogeneous. All arguments can be adapted to handle the case where the preferences differ among agents.

in particular, when $u$ belongs to the class of constant relative risk aversion utility functions $u = c^{1-\alpha}/(1-\alpha)$ with $\alpha > 0$. The example in Section 4.1 illustrates this point.[3]

Given an event $s^t$, we denote by $U(c|s^t)$ the lifetime continuation utility conditional to $s^t$ defined by

$$U(c|s^t) := u(c(s^t)) + \sum_{\tau \geqslant 1} \beta^\tau \sum_{s^{t+\tau} \succ s^t} \pi(s^{t+\tau}|s^t) u(c(s^{t+\tau}))$$

where $\pi(s^{t+\tau}|s^t) := \pi(s^{t+\tau})/\pi(s^t)$ is the conditional probability of $s^{t+\tau}$ given $s^t$.

Agents' endowments are subject to random shocks. We denote by $y^i = (y^i(s^t))_{s^t \succeq s^0}$ the process of positive endowments $y^i(s^t) > 0$ of a representative agent of type $i$. A collection $(c^i)_{i \in I}$ of consumption processes is called a consumption allocation. It is said to be resource feasible if $\sum_{i \in I} c^i = \sum_{i \in I} y^i$. We also fix an allocation $(a^i(s^0))_{i \in I}$ of initial financial claims $a^i(s^0) \in \mathbb{R}$ that satisfies the usual market-clearing condition: $\sum_{i \in I} a^i(s^0) = 0$.

## 2.2 Markets

At any event $s^t$, agents can issue and trade a complete set of one-period contingent bonds, each one promising to pay one unit of the consumption good contingent on the realization of any successor event $s^{t+1} \succ s^t$. Let $q(s^{t+1}) > 0$ denote the price at event $s^t$ of the $s^{t+1}$-contingent bond. Agent $i$'s bond holdings are $a^i = (a^i(s^t))_{s^t \succeq s^0}$, where $a^i(s^t) \leqslant 0$ denotes a liability and $a^i(s^t) \geqslant 0$ denotes a claim. Debt is observable and subject to state-contingent (nonnegative and finite) debt limits $D^i = (D^i(s^t))_{s^t \succeq s^0}$. Given the initial financial claim $a^i(s^0)$, we denote by $B^i(D^i, a^i(s^0)|s^0)$ the budget set of an agent who never defaults. It consists of all pairs $(c^i, a^i)$ of consumption and bond holdings satisfying the following budget and solvency constraints: for any event $s^t \succeq s^0$,

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a^i(s^{t+1}) \leqslant y^i(s^t) + a^i(s^t) \tag{2.1}$$

and

$$a^i(s^{t+1}) \geqslant -D^i(s^{t+1}), \quad \text{for all } s^{t+1} \succ s^t. \tag{2.2}$$

[3]A general treatment of unbounded utility functions requires some additional technical assumptions on endowment processes together with a suitable modification of the utility function $u$ outside a specific interval such that the equilibrium outcomes remain unaffected. For a detailed discussion, see Martins-da-Rocha and Santos (2019).

Fix an event $s^t$ and some initial claim $a \in \mathbb{R}$. We denote by $V^i(D^i, a|s^t)$ the value function defined by

$$V^i(D^i, a|s^t) := \sup\{U(c^i|s^t) \ : \ (c^i, a^i) \in B^i(D^i, a|s^t)\}$$

where $B^i(D^i, a|s^t)$ is the set of all plans $(c^i, a^i)$ satisfying $a^i(s^t) = a$ together with restrictions (2.1) and (2.2) at every successor node $s^\tau \succeq s^t$.

Without any loss of generality, we restrict attention to debt limits $(D^i(s^t))_{s^t \succeq s^0}$ that are *consistent*, meaning that at every event $s^t$, the maximal debt can be repaid out of the current resources and the largest possible debt contingent on future events, i.e.,

$$D^i(s^t) \leqslant y^i(s^t) + \sum_{s^{t+1} \succeq s^t} q(s^{t+1}) D^i(s^{t+1}), \quad \text{for all } s^t \succeq s^0. \tag{2.3}$$

This condition is necessary for the budget set $B^i(D^i, -D^i(s^t)|s^t)$ to be nonempty.

## 2.3 Default Costs

Agents might not honor their debt obligations and default if it is optimal for them.[4] Following Bulow and Rogoff (1989), we assume that, upon default, debtors start with neither assets nor liabilities, are excluded from future credit, but retain the ability to purchase bonds. In addition, debt repudiation leads to a loss $\ell^i(s^t) \in [0, y^i(s^t)]$ of the endowment, where $\ell^i(s^t)$ may vary across agents and events.[5] Formally, agent $i$'s default option at event $s^t$ is the largest continuation utility when starting with zero liabilities, cannot borrow, and her income contracts by the amount $\ell^i(s^\tau)$ at every $s^\tau \succeq s^t$:

$$V^i_{\ell^i}(0, 0|s^t) := \sup\{U(c^i|s^t) \ : \ (c^i, a^i) \in B^i_{\ell^i}(0, 0|s^t)\}, \tag{2.4}$$

where $B^i_{\ell^i}(0, 0|s^t)$ denotes the budget set corresponding to $B^i(0, 0|s^t)$ when the endowment $y^i(s^\tau)$ is replaced by $y^i(s^\tau) - \ell^i(s^\tau)$ at any event $s^\tau \succeq s^t$.

[4] Since the default punishment is independent of the default level, there will be no partial default in equilibrium: agents either repay or default fully on their promises.

[5] In their modeling of unsecured consumer credit, Chatterjee et al. (2007) assume that a defaulting household cannot borrow and incurs a small reduction in its earning capability. Disruption of international trade and of the domestic financial system can lead to a sovereign's drop of if trade and/or credit are essential for production. Among others, Mendoza and Yue (2012), Gennaioli et al. (2014), and Phan (2017) model explicitly how sovereign default may lead to efficiency losses in production. We follow the tradition in the sovereign debt literature (see for instance Cohen and Sachs 1986, Cole and Kehoe 2000, Aguiar and Gopinath 2006, Arellano 2008, Ábrahám and Cárceles-Poveda 2010, and Bai and Zhang 2010, 2012) and model the negative implications on output as a loss of an exogenous fraction of income.

## 2.4 Not-Too-Tight Debt Limits

Since agents trade contingent bonds, potential lenders have no reason to provide credit if they anticipate that debtors will default at an event $s^t$. Debt limits should reflect this property. We say that debt limits are *self enforcing* if for every individual $i$, at every event $s^t$,

$$V^i(D^i, -D^i(s^t)|s^t) \geqslant V^i_{\ell^i}(0, 0|s^t).$$

The left-hand side is the value of market participation beginning with the maximum sustainable debt, whereas the right-hand side is the value of default.

Competition among lenders naturally leads to consider the maximum self-enforcing debt limit compatible with repayment. Following Alvarez and Jermann (2000) and Hellwig and Lorenzoni (2009), we say that debt limits are *not too tight* if the individual is indifferent between repaying and defaulting, i.e.,

$$V^i(D^i, -D^i(s^t)|s^t) = V^i_{\ell^i}(0, 0|s^t). \tag{2.5}$$

## 2.5 Competitive Equilibrium

A competitive equilibrium with self-enforcing private debt is defined as follows.

**Definition 2.1.** Given initial asset holdings $(a^i(s^0))_{i\in I}$ such that $\sum_{i\in I} a^i(s^0) = 0$, a *competitive equilibrium with self-enforcing debt* $(q, (c^i, a^i, D^i)_{i\in I})$ is a collection of state-contingent bond prices $q$, a resource feasible consumption allocation $(c^i)_{i\in I}$, a market clearing allocation of bond holdings $(a^i)_{i\in I}$, and a family of consistent, nonnegative, and finite debt limits $(D^i)_{i\in I}$ such that:

(a) for every agent $i \in I$, taking prices and the debt limits as given, the plan $(c^i, a^i)$ is optimal among budget feasible plans in $B^i(D^i, a^i(s^0)|s^0)$;

(b) for every agent $i \in I$, the debt limits $D^i$ are not too tight, i.e., equation (2.5) is satisfied at any event.

Throughout the paper, we restrict attention to competitive equilibria with self-enforcing debt such that initial asset holdings are consistent with repayment incentives, i.e., $a^i(s^0) \geqslant -D^i(s^0)$ for each $i$.[6]

[6] This is in particular the case when $a^i(s^0) = 0$ for each agent $i$.

We subsequently define some additional objects that are useful for the rest of the analysis. Given bond prices $q = (q(s^t))_{s^t \succ s^0}$, we denote by $p(s^t)$ the associated date-0 price of consumption at $s^t$ defined recursively by $p(s^0) = 1$ and $p(s^{t+1}) = q(s^{t+1})p(s^t)$ for all $s^{t+1} \succ s^t$. Given date-0 prices, we define the *present value* of a process $x$ conditional to an event $s^t$ as

$$\mathrm{PV}(x|s^t) := \frac{1}{p(s^t)} \sum_{s^\tau \succeq s^t} p(s^\tau)x(s^\tau).$$

The *wealth* of an agent at event $s^t$ is defined as the present value of her endowments:

$$W^i(s^t) := \mathrm{PV}(y^i|s^t).$$

Finally, a process $(M^i(s^t))_{s^t \succeq s^0}$ allows for *exact rollover* if it satisfies the following property:

$$M^i(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1})M^i(s^{t+1}), \quad \text{for all } s^t \succeq s^0. \tag{2.6}$$

Slightly abusing language, we will refer to exact rollover processes as *discounted martingales* or *bubbles*.

*Remark* 2.1. If $\ell^i(s^t) = y^i(s^t)$ at every event $s^t$, then agent $i$ has no incentive to ever default. In this case, the agent can effectively commit to her financial promises because the value of the default option $V^i_{y^i}(0, 0|s^t)$ equals $U(0|s^t)$, and, therefore, any process of consistent debt limits is self-enforcing. Our specification of output costs thus encompasses a *mixed* environment where some agents can perfectly commit to financial contracts while others have limited commitment. The textbook model with full commitment obtains when $\ell^i(s^t) = y^i(s^t)$ for any agent $i$ and every event $s^t$. The model nests both Bulow and Rogoff (1989) (unilateral lack of commitment) and Hellwig and Lorenzoni (2009) (multilateral lack of commitment) settings where $\ell \equiv 0$.

# 3 Main Result

At the core of the analysis is a novel characterization of equilibrium debt limits when default entails both credit exclusion and a loss of a fraction of the private endowment. Not-too-tight debt limits are decomposed into a "fundamental" component, $\mathrm{PV}(\ell^i)$, associated to direct default costs, and a "credit bubble" component, $M^i$, that captures the possibility of rolling over a fraction of debt indefinitely. That is, credit beyond the fundamental component is sustainable only if debtors can exactly refinance past liabilities by issuing new debt. The following theorem provides the formal statement.

**Theorem 3.1.** *Not-too-tight debt limits equal the sum of the present value of endowment losses upon default and a bubble component, i.e.,*

$$D^i = \mathrm{PV}(\ell^i) + M^i,$$

*where $M^i$ is a nonnegative discounted martingale process.*

Before we proceed to the proof of Theorem 3.1, we present two intermediate results. The first and crucial observation, that has no analogue in the absence of output contraction, is to show that the present value of foregone endowment imposes a lower bound on not-too-tight debt limits. A direct implication of this property is that the process $\mathrm{PV}(\ell^i)$ is finite. This is summarized in the following lemma.

**Lemma 3.1.** *Not-too-tight debt limits are at least as large as the present value of endowment losses: for each agent $i$, $D^i(s^t) \geqslant \mathrm{PV}(\ell^i|s^t)$ at any event $s^t$.*

A natural approach to prove this result is to show that $D^i(s^t) \geqslant \ell^i(s^t) + \widetilde{D}^i(s^t)$, where $\widetilde{D}^i(s^t) := \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1})$ is the present value of next period's debt limits, and then use a standard iteration argument. Because, in equilibrium, debt limits are not too tight, this is equivalent to proving that agent $i$ does not have an incentive to default when her net asset position is $\ell^i(s^t) + \widetilde{D}^i(s^t)$, i.e.,

$$V^i(D^i, -\ell^i(s^t) - \widetilde{D}^i(s^t)|s^t) \geqslant V^i_{\ell^i}(0, 0|s^t). \tag{3.1}$$

By definition, the default value function $V^i_{\ell^i}$ satisfies:

$$V^i_{\ell^i}(0, 0|s^t) \geqslant u(y^i(s^t) - \ell^i(s^t)) + \beta \sum_{s^{t+1} \succ s^t} \pi(s^{t+1}|s^t) V^i_{\ell^i}(0, 0|s^{t+1}). \tag{3.2}$$

If we had an equality in (3.2), then inequality (3.1) would be straightforward. Indeed, consuming $y^i(s^t) - \ell^i(s^t)$ and borrowing up to each debt limit $D^i(s^{t+1})$ at event $s^t$ leads to the right-hand side continuation utility in (3.2) and satisfies the solvency constraint at event $s^t$ in the budget set defining the left-hand side of (3.1). However, in our environment where an agent can save upon default, (3.2) need not be satisfied with an equality.[7] Overcoming this problem is the technical challenge in the proof of Lemma 3.1. The formal argument is presented in Appendix A.1.

[7] In the simpler environment where saving is not possible after default (as it is the case in Alvarez and Jermann 2000) we always have an equality in (3.2).

A second observation is that the process $\mathrm{PV}(\ell^i)$ of present values of endowment losses, when it is finite, is itself not too tight. The following lemma provides the formal statement. The proof follows from a simple translation invariance of the flow budget constraints presented in Appendix A.2.

**Lemma 3.2.** *If* $\mathrm{PV}(\ell^i|s^0)$ *is finite, then the process* $\mathrm{PV}(\ell^i)$ *is not too tight, i.e.,*

$$V^i(\mathrm{PV}(\ell^i), -\mathrm{PV}(\ell^i|s^t)|s^t) = V^i_{\ell^i}(0,0|s^t), \quad \text{for all } s^t \succeq s^0.$$

Equipped with Lemma 3.1 and Lemma 3.2, we can provide a simple proof of Theorem 3.1.

*Proof of Theorem 3.1.* Fix a process $D^i$ of not-too-tight debt limits. Lemma 3.1 implies that $\mathrm{PV}(\ell^i|s^0)$ is finite. From Lemma 3.2 we also deduce that the process $\underline{D}^i := \mathrm{PV}(\ell^i)$ is not too tight. Martins-da-Rocha and Santos (2019) proved that the difference between two processes of not-too-tight debt limits must be a discounted martingale. Therefore, there exists a process $M^i$ satisfying (2.6) such that $D^i = \underline{D}^i + M^i$. By Lemma 3.1, $D^i \geqslant \underline{D}^i$, in which case the process $M^i$ must be nonnegative. $\square$

We next show that a necessary condition for the emergence of credit bubbles is that the direct sanctions against defaulters become negligible (relative to aggregate resources) in the distant future. Indeed, as argued below, when the fraction of endowment losses to aggregate endowment is uniformly bounded away from zero, the bubble component disappears and debt can never exceed the present value of direct default costs.

Specifically, we say that endowment losses are *nonnegligible* if there exists $\varepsilon > 0$ such that:

$$\sum_{i \in I} \ell^i(s^t) \geqslant \varepsilon \sum_{i \in I} y^i(s^t), \quad \text{for all } s^t \succeq s^0. \tag{3.3}$$

The condition implies that, in equilibrium, interest rates should be sufficiently high, in the sense that the aggregate wealth of the economy is finite. A direct consequence is that the determination of not-too-tight debt limits is akin to the pricing of long-lived assets in a competitive equilibrium with full commitment. On the one hand, high interest rates rule out the possibility of rolling over debt indefinitely and forces equilibrium debt to not exceed the present value of endowment losses exactly the same way asset bubbles are ruled out in Santos and Woodford (1997). On the other hand, debtors would always choose to honor any debt that is at least as large as the present value of the endowment losses (see Lemma 3.1). Together, those forces imply that equilibrium debt limits should exactly reflect the present value of foregone endowment upon default. All this is summarized in the following corollary.

**Corollary 3.1.** *If the endowment losses are nonnegligible, then the not-too-tight debt limits equal exactly the present value of foregone endowment upon default, i.e.,*

$$D^i(s^t) = \mathrm{PV}(\ell^i|s^t), \quad \textit{for all } s^t \succeq s^0. \tag{3.4}$$

*Proof.* Let $(q, (c^i, a^i, D^i)_{i\in I})$ be a competitive equilibrium with self-enforcing debt. Since endowment losses are nonnegligible, we must have

$$\sum_{i\in I} W^i(s^0) = \sum_{i\in I} \mathrm{PV}(y^i|s^0) \leqslant \frac{1}{\varepsilon}\sum_{i\in I} \mathrm{PV}(\ell^i|s^0).$$

Lemma 3.1 then implies that the aggregate wealth of the economy must be finite. Since consumption markets clear, we obtain that the present value of optimal consumption is finite for all agents. In addition, due to the Inada's condition, the optimal consumption is strictly positive.[8] Lemma A.1 in Martins-da-Rocha and Vailakis (2017a) then implies that the following market transversality condition holds true:[9]

$$\lim_{t\to\infty} \sum_{s^t\in S^t} p(s^t)[a^i(s^t) + D^i(s^t)] = 0. \tag{3.5}$$

We know from Theorem 3.1 that for each $i$, there exists a nonnegative discounted martingale process $M^i$ satisfying $D^i = \mathrm{PV}(\ell^i) + M^i$. The market transversality condition can then be written as follows

$$\lim_{t\to\infty} \sum_{s^t\in S^t} p(s^t)a^i(s^t) = -p(s^0)M^i(s^0).$$

Since bond markets clear, we deduce that $\sum_{i\in I} M^i(s^0) = 0$ and we get the desired result: $M^i = 0$ for each $i$. □

Corollary 3.1 provides a general equilibrium foundation for Bulow and Rogoff (1989)'s no-trade result. Indeed, in their framework, upon default, the sovereign can take the saved repayments to Swiss bankers who are committed to honoring contracts made with governments. This can be translated in our setting by assuming that a subset $I^c \subset I$ of agents (the Swiss bankers) face the maximum endowment loss, i.e., $\ell^i = y^i$ for all $i \in I^c$ while the remaining agents (the debtors) face no endowment loss, i.e., $\ell^i = 0$ for all $i \in I^{nc} := I \setminus I^c$.

[8] See the Supplemental Material of Martins-da-Rocha and Santos (2019) for a detailed proof.

[9] The market transversality condition differs from the individual transversality condition. Indeed, due to the lack of commitment, agent $i$'s debt limits may bind, in which case we do not necessarily have that $p(s^t) = \beta^t\pi(s^t)u'(c^i(s^t))/u'(c^i(s^0))$.

As long as the endowment of committed agents is large relative to the one of noncommitted agents, i.e, there exists $\alpha > 0$ such that

$$\sum_{i\in I^c} y^i(s^t) \geqslant \alpha \sum_{i\in I^{nc}} y^i(s^t), \tag{3.6}$$

then condition (3.3) holds true (for $\varepsilon \leqslant \alpha/(1+\alpha)$) and Corollary 3.1 applies: $D^i(s^t) = 0$ for any $i \in I^{nc}$ and all $s^t \succeq s^0$. Instead of assuming that the endogenously determined interest rates are high enough (as in Bulow and Rogoff 1989 or Proposition 3.1 in Hellwig and Lorenzoni 2009), we show this is always true when endowments satisfy condition (3.6).

# 4 Applications

## 4.1 Equilibrium Computation: An Example

We illustrate the applicability of Theorem 3.1 by analyzing a simple deterministic economy with identical agents whose endowments switch between a high and low value and default entails the loss of a fixed amount $\ell > 0$. Our characterization result simplifies substantially the computation of equilibria. It also enables us to overcome the complications that arise in constructing and characterizing bubbly equilibria where a positive fundamental component coexists with a positive bubble component.[10]

There are two agents $I := \{i_1, i_2\}$ with a constant relative risk aversion utility function

$$u(c) = \frac{c^{1-\alpha}}{1-\alpha}, \quad \alpha > 0.$$

At every date $t$, each agent $i$'s income $y^i_t$ alternates between a high value $y_{\mathrm{H},t}$ and a low value $y_{\mathrm{L},t}$. Agent $i_1$ starts with the high income. Incomes grow at a constant gross rate $\rho > 1$:

$$(y_{\mathrm{H},t}, y_{\mathrm{L},t}) = \rho^t (y_{\mathrm{H}}, y_{\mathrm{L}}), \quad \text{with} \quad y_{\mathrm{H}} > y_{\mathrm{L}} > \ell > 0.$$

We focus on symmetric equilibria and denote by $x_t$ the not-too-tight debt limit at date $t$, i.e., $D^i_t = x_t$ for each $i$.[11] It follows from our general characterization result that there exists $M_0 \geqslant 0$ such that

$$x_t = \frac{1}{p_t}[\ell(p_t + p_{t+1} + \ldots) + M_0],$$

[10] For details of the difficulty in establishing the coexistence of the two components in asset price bubbles, see Miao and Wang (2018).

[11] Symmetry means that the debt limit is the same for both agents.

where $p_t := q_1 \cdots q_t$. We restrict attention to equilibria where the high-income agent at date $t$ purchases the amount $x_{t+1}$ of the one-period bond and the low-income agent issues the largest debt $x_{t+1}$. To support this allocation as a competitive equilibrium, we assume an initial positive transfer $x_0$ for the low-income agent and an initial debt level $x_0$ for the high-income agent. Then, for all $t$, the consumption of the high-income agent is

$$c_{\text{H},t} = y_{\text{H},t} - (x_t + q_{t+1} x_{t+1})$$

while the consumption of the low-income agent is

$$c_{\text{L},t} = y_{\text{L},t} + (x_t + q_{t+1} x_{t+1}).$$

Let $z_t$ denote the net trade position, i.e.,

$$z_t := x_t + q_{t+1} x_{t+1}.$$

Since there is growth, we let $\hat{z}_t := \rho^{-t} z_t$ represent the detrended trade position at period $t$. Observe that

$$\hat{z}_t := \frac{1}{\rho^t p_t}[\ell(p_t + 2p_{t+1} + 2p_{t+2} + \ldots) + 2M_0]. \tag{4.1}$$

The Euler equation associated to the high-income agent's saving decision is

$$q_{t+1} = \beta \frac{u'(c_{\text{L},t+1})}{u'(c_{\text{H},t})} = \beta \left( \frac{y_{\text{H}} - \hat{z}_t}{\rho(y_{\text{L}} + \hat{z}_{t+1})} \right)^{\alpha}, \tag{4.2}$$

while the Euler equation associated to the low-income agent's borrowing decision is

$$q_{t+1} \geqslant \beta \frac{u'(c_{\text{H},t+1})}{u'(c_{\text{L},t})} = \beta \left( \frac{y_{\text{L}} + \hat{z}_t}{\rho(y_{\text{H}} - \hat{z}_{t+1})} \right)^{\alpha}.$$

The last inequality follows from (4.2) when $\hat{z}_t \leqslant (y_{\text{H}} - y_{\text{L}})/2$. Therefore, to get the existence of a competitive equilibrium it is sufficient to find a sequence $(q_{t+1})_{t \geqslant 0}$ of positive bond prices and a sequence $(\hat{z}_t)_{t \geqslant 0}$ of detrended net trades such that

$$q_{t+1} = \beta \left( \frac{y_{\text{H}} - \hat{z}_t}{\rho(y_{\text{L}} + \hat{z}_{t+1})} \right)^{\alpha} \tag{4.3}$$

and

$$\hat{z}_t = \frac{\ell}{\rho^t} + \rho q_{t+1} \left( \hat{z}_{t+1} + \frac{\ell}{\rho^{t+1}} \right), \tag{4.4}$$

where each $\hat{z}_t$ belongs to $[0, (y_{\rm H} - y_{\rm L})/2]$.

From now on, we impose the following restrictions on primitives:

$$1 > \underbrace{\beta\left(\frac{y_{\rm H}}{\rho y_{\rm L}}\right)^{\alpha}}_{=:\,\bar{q}} > \frac{1}{\rho} > \beta\left(\frac{1}{\rho}\right)^{\alpha}. \tag{4.5}$$

This condition implies that there exists a unique $M_0^\star$ such that

$$2M_0^\star < (y_{\rm H} - y_{\rm L})/2 \quad \text{and} \quad \beta\left(\frac{y_{\rm H} - 2M_0^\star}{\rho(y_{\rm L} + 2M_0^\star)}\right)^{\alpha} = \frac{1}{\rho}. \tag{4.6}$$

It is straightforward to check that if $\ell = 0$, then there exists a competitive equilibrium with time-invariant interest rate characterized by

$$p_t = \frac{1}{\rho^t} \quad \text{and} \quad x_t = \rho^t M_0^\star.$$

When the endowment loss $\ell$ is positive but small enough, in the sense that $\ell/(1-\bar{q}) < M_0^\star$, we show in the online Supplemental Material of this paper that for any arbitrary value $M_0(\ell)$ satisfying

$$\frac{\ell}{1-\bar{q}} + M_0(\ell) \leqslant M_0^\star,$$

there exists a competitive equilibrium $(q_{t+1}(\ell), \hat{z}_t(\ell))_{t\geqslant 0}$ such that $p_t(\ell) \geqslant \rho^{-t}$.[12] Note that there is an indeterminacy of equilibria parametrized by the size of the initial bubble $M_0(\ell)$. We also show in the online Supplemental Material of this paper that the detrended net trade vanishes (i.e., $\lim_{t\to\infty} \hat{z}_t(\ell) = 0$). This property follows from the fact that the initial net trade satisfies $\hat{z}_0(\ell) < 2M_0^\star$.

Nevertheless, because of the difference equation (4.4), we may look for other type of equilibria with an initial net trade satisfying $\hat{z}_0 > 2M_0^\star$. In fact, depending on the choice of the initial value $\hat{z}_0$, an equilibrium can display very different properties. To illustrate this, we analyze numerically two types of possible equilibrium outcomes for a given endowment loss $\ell$: one where the initial trade is fixed and equal to $2M_0^\star$ (Figure 1) and another one where the asymptotic detrended trade is fixed and equal to $2M_0^\star$ (Figure 2). We use the following parameters: $\alpha = 1$ (that is, the utility function is $u(c) = \ln(c)$), $y_L = 75$, $y_H = 140$, $\beta = 0.55$, $\rho = 1.05$, and set various values of $\ell$.

[12] Recall that $p_t(\ell) := q_1(\ell)\ldots q_t(\ell)$ is the Arrow–Debreu price at $t = 0$ of one unit of consumption at date $t$.

In Figure 1, setting $\hat{z}_0 = 2M_0^\star$, we solve equations (4.3) and (4.4) for $\ell$ equal to: $\ell_0 = 0$, $\ell_1 = 0.01$, $\ell_2 = 0.015$, and $\ell_3 = 0.020$. We appeal to Theorem 3.1 to get that $D_t(\ell) = (z_t(\ell) + \ell)/2$. The initial value $M_0(\ell)$ of the bubble component can then be identified by computing $\lim_{t\to\infty} p_t(\ell)D_t(\ell)$.[13] Bond prices increase and net trade decreases with the endowment loss. Since the initial trade is fixed and satisfies

$$2M_0^\star = \hat{z}_0 = \ell(1 + 2p_1(\ell) + 2p_2(\ell) + \ldots) + 2M_0(\ell),$$

the initial size of the bubble must decrease with the endowment loss. Moreover, since bond prices converge to $\rho_0^{-1} := \bar{q} > \rho^{-1}$, the (detrended) debt limit converges to zero.

In Figure 2, instead of fixing the initial value of the detrended trade, we look for a solution involving the largest $\hat{z}_0$ compatible with equilibrium for each of the following values for $\ell$: $\ell_0 = 0$, $\ell_1 = 0.05$, $\ell_2 = 0.12$, and $\ell_3 = 0.40$.[14] Observe that, given $\ell > 0$, along any equilibrium path we now have $p_t(\ell) \leqslant \rho^{-t}$ for every $t \geqslant 0$ (as opposed to $p_t(\ell) \geqslant \rho^{-t}$ in Figure 1). We still have that the level of initial bubble $M_0(\ell)$ decreases with the endowment loss, but now trade opportunities increase as $\ell$ increases. Moreover, debt does not vanish asymptotically but rather converges to the pure bubble debt level $M_0^\star$ (that is independent of the level of endowment loss), so trade persists in the limit. Bond prices also have a different behavior. The larger the endowment loss $\ell$, the higher the implied interest rate, the lower the asset price.

[13] The values for $\ell_1$, $\ell_2$, and $\ell_3$ have been chosen to get a nice distribution of $M_0(\ell)$ between 0 and $M_0^\star$.

[14] We refer to the online Supplemental Material of this paper where we justify the existence of such a solution. We have modified the values for $\ell_1$, $\ell_2$, and $\ell_3$ to keep a nice distribution of $M_0(\ell)$ between 0 and $M_0^\star$.

Figure 1: $\hat{z}_0 = 2M_0^\star$

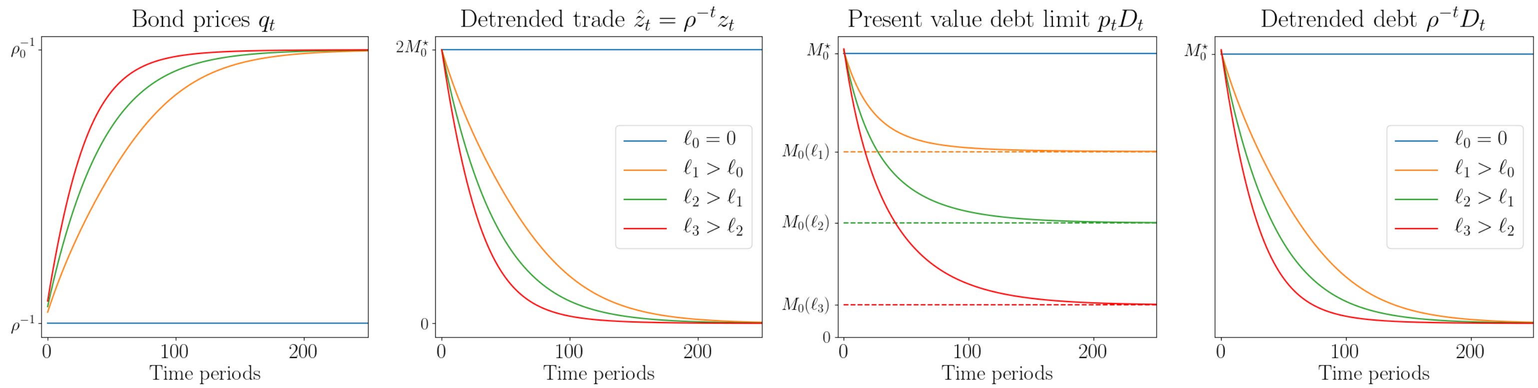



Figure 2: $\lim_{t\to\infty} \hat{z}_t = 2M_0^\star$

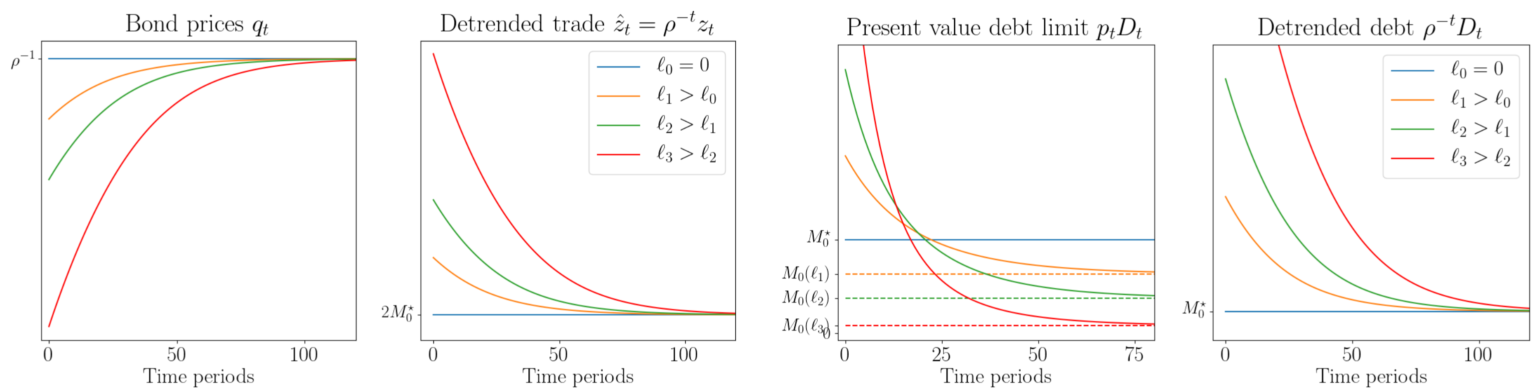
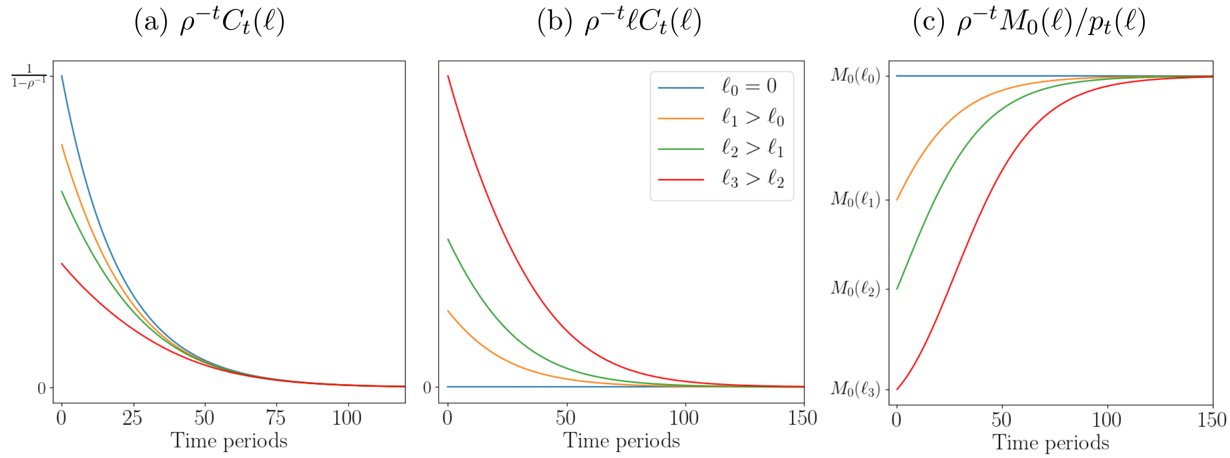


Figure 3: Races defining the debt level




We now demonstrate how our characterization result can help disentangle the forces that drive the pattern of the sustained debt shown in Figure 2. Theorem 3.1 implies the following decomposition:

$$D_t(\ell) = \ell \underbrace{(1 + q_{t+1}(\ell) + q_{t+1}q_{t+2}(\ell) + \dots)}_{C_t(\ell)} + \frac{M_0(\ell)}{p_t(\ell)}$$

where $\ell C_t(\ell)$ is the fundamental component with $C_t(\ell)$ be the (cum-dividend) price of a console at date $t$ and $M_0(\ell)/p_t(\ell)$ is the bubble component. There is a first race between the endowment loss $\ell$ and the price of the console $C_t(\ell)$ since the later decreases when the endowment loss increases (Figure 3(a)). The first effect dominates, so a larger $\ell$ implies a larger fundamental component (Figure 3(b)). There is also a second race that dictates the way the bubble component changes as $\ell$ increases. We have seen that the initial value of the bubble $M_0(\ell)$ decreases with $\ell$ while the long-term gross return $p_t(\ell)$ increases with $\ell$ (Figure 2). The first effect is stronger, so a larger $\ell$ implies a lower bubble component $M_0(\ell)/p_t(\ell)$ (Figure 3(c)). The overall effect of a change of the endowment loss to the debt limits, and therefore to liquidity, is therefore determined by the competition between the rise in the fundamental component and the decrease in the bubble component. For the chosen specification of the economy's fundamentals and the type of equilibria analyzed in Figure 2, the rise in the fundamental component outweighs the drop in the bubble component, so risk sharing is enhanced as direct sanctions upon default become more severe.

## 4.2 Public Debt

This section shows that the consumption allocation of an equilibrium with self-enforcing debt can be implemented as an equilibrium allocation of an economy in which debt is only issued by a government, and vice versa. In that respect, we establish a one-to-one mapping between an environment with private liquidity (debt issued by private agents) and public liquidity (debt issued by the government).

This result might come as a surprise as it is generally thought that public debt provides a superior instrument for allocating risk efficiently: using its tax power, a government can enlarge agents' insurance opportunities by transferring additional resources to the debt holders who are hit by adverse shocks (see, for instance, Holmström and Tirole 2011). Furthermore, there are differences that relate to institutional arrangements. Specifically, in an equilibrium with backed public debt, the tax revenue is seized along the equilibrium path and transferred to the government. Implicitly, it is assumed that the government is endowed with an enforcement technology. At the self-enforcing equilibrium, instead, the loss of resources only occurs on out-of-equilibrium paths and nothing is transferred to creditors upon default (i.e., they represent dead-weight losses).

We argue that these differences are immaterial in a setting with complete markets and linear pricing. The proof exploits the characterization of equilibrium private debt (Theorem 3.1) together with a straightforward decomposition of equilibrium public debt (Proposition 4.1 below). By securing a fraction of debt, tax revenue is a source of liquidity as it is the case with endowment losses in a setting with self-enforcing debt. Any amount of public debt in excess of backed resources is valued in the market as a speculative bubble, exactly as private debt (in excess of the present value of income losses) is valued at the competitive equilibrium with self-enforcing debt.

### 4.2.1 Institutional Arrangements

The environment is as before, but now we assume that individual agents can no longer issue debt, i.e., $D^i(s^t) = 0$ for every agent $i$ and any event $s^t$. Instead, debt is only issued by a government, which backs its liabilities by taxing income according to a tax schedule $(\tau^i(s^t))_{s^t \succeq s^0}$ for each type $i \in I$.

Given an initial asset position $\theta^i(s^0) \geqslant 0$, let $\widehat{B}^i(\theta^i(s^0)|s^0)$ denote the budget set of agent $i \in I$ in this economy. It contains all pairs $(c^i, \theta^i)$ of consumption $c^i = (c^i(s^t))_{s^t \succeq s^0}$ and public debt holdings $\theta^i = (\theta^i(s^t))_{s^t \succ s^0}$ satisfying the (after-tax) budget constraint at any event $s^t$, i.e.,

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\theta^i(s^{t+1}) \leqslant \underbrace{(1-\tau^i(s^t))y^i(s^t)}_{\text{after-tax endowment}} + \theta^i(s^t), \tag{4.7}$$

together with the no-borrowing restrictions

$$\theta^i(s^{t+1}) \geqslant 0 \quad \text{at all successors } s^{t+1} \succ s^t. \tag{4.8}$$

At any event $s^t$, the government issues public debt $d(s^{t+1}) \geqslant 0$ contingent to every successor event $s^{t+1}$. The outstanding debt $d(s^t)$ at an event $s^t$ is financed partially by tax revenues while the rest is rolled over across next period's contingencies. The government's budget restriction is then

$$d(s^t) \leqslant \sum_{i \in I} \tau^i(s^t)y^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})d(s^{t+1}). \tag{4.9}$$

**Definition 4.1.** Given a tax schedule $(\tau^i(s^t))_{s^t \succeq s^0}$ for each agent $i \in I$ and an allocation $(\theta^i(s^0))_{i \in I}$ of initial asset positions, a *competitive equilibrium with public debt* $(q, d, (c^i, \theta^i)_{i \in I})$ consists of state-contingent bond prices $q$, a resource feasible consumption allocation $(c^i)_{i \in I}$, an allocation of government bond holdings $(\theta^i)_{i \in I}$, and the government's net liability positions $d$ such that:

(i) for each agent $i \in I$, taking prices as given, the plan $(c^i, \theta^i)$ is optimal among budget feasible plans in $\widehat{B}^i(\theta^i(s^0)|s^0)$;

(ii) the debt market clears

$$\sum_{i \in I} \theta^i(s^t) = d(s^t), \quad \text{for all } s^t \succeq s^0; \tag{4.10}$$

(iii) the government's budget constraint (4.9) is satisfied at all contingencies $s^t \succeq s^0$.

### 4.2.2 Characterization of Public Debt

We show that, given the individual tax schedules, equilibrium public debt is decomposed to a fundamental component reflecting the present value of total tax revenue and a speculative bubble. As opposed to an equilibrium with self-enforcing debt, the privilege of issuing the speculative bubble is now attributed to the government that rolls over part of its debt forever.

**Proposition 4.1.** *At any competitive equilibrium with public debt,*

$$d(s^t) = \sum_{i\in I} \mathrm{PV}(\tau^i y^i | s^t) + M(s^t), \quad \textit{for all } s^t \succeq s^0,$$

*where $M$ is a nonnegative discounted martingale process. Equivalently, the government's debt level at any contingency is decomposed into the present value of total tax revenue and a bubble component.*

*Proof.* Fix a competitive equilibrium $(q, d, (c^i, \theta^i)_{i\in I})$ with public debt. To simplify the presentation, let $\delta := \sum_{i\in I} \tau^i y^i$ represent the taxed aggregate income process. Combining resource feasibility of $(c^i)_{i\in I}$, the government's budget constraint, and the debt market clearing condition, we deduce that the government's budget constraints is satisfied with equality, i.e.,

$$d(s^t) = \delta(s^t) + \sum_{s^{t+1}\succ s^t} q(s^{t+1}) d(s^{t+1}).$$

Consolidating the above equations from date 0 to any arbitrary date $T > 0$, we get

$$p(s^0)d(s^0) = \sum_{t=0}^{T} \sum_{s^t\in S^t} p(s^t)\delta(s^t) + \sum_{s^{T+1}\in S^{T+1}} p(s^{T+1})d(s^{T+1}).$$

Since $d(s^{T+1}) \geqslant 0$, we deduce that the process $\delta$ has finite present value, i.e., $\mathrm{PV}(\delta|s^0) < \infty$. In particular, consolidating the government's budget constraints along the sub-tree $\Sigma(s^t)$, we get that

$$d(s^t) = \mathrm{PV}(\delta|s^t) + \underbrace{\frac{1}{p(s^t)} \lim_{T\to\infty} \sum_{s^T\in S^T} p(s^T)d(s^T)}_{=:\, M(s^t)}.$$

The desired result follows from the fact that $M$ satisfies exact rollover. □

*Remark* 4.1. Notice that characterizing public debt (i.e., proving Proposition 4.1) is far simpler than characterizing private debt (i.e., proving Theorem 3.1). In fact, showing that the total tax revenue has finite present value is straightforward, as the government's budget restriction (4.9) and market clearing imply the recursive property

$$d(s^t) = \delta(s^t) + \sum_{s^{t+1}\succ s^t} q(s^{t+1}) d(s^{t+1}).$$

Proving the same claim for the private endowment losses is more involved, as Lemma 3.1 reveals. This is because we do not know *a priori* whether the not-too-tight debt limits satisfy the recursive property

$$D^i(s^t) = \ell^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D^i(s^{t+1}),$$

so this recursive equation is obtained after applying a fixed-point theorem to a suitably defined operator.

#### 4.2.3 Equivalence Result

We now have all elements to establish an equivalence mapping between a competitive equilibrium with public debt and a self-enforcing equilibrium. The implication is that public liquidity is akin to private liquidity. Who is issuing the debt does not affect the economy's risk-sharing opportunities. Formally, using the characterization results established in Theorem 3.1 and Proposition 4.1, we obtain the following equivalence result.[15]

**Theorem 4.1.** *A consumption allocation is the outcome of a competitive equilibrium with public debt backed by taxes $(\tau^i)_{i\in I}$ if, and only if, it is the outcome of a competitive equilibrium with self-enforcing debt and endowment losses $(\ell^i)_{i\in I}$ such that $\ell^i = \tau^i y^i$ for any $i \in I$. Formally, we have the following properties:*

*(a) Let $(\tau^i)_{i\in I}$ be a family of tax schedules. If $(q, d, (c^i, \theta^i)_{i\in I})$ is a competitive equilibrium with public debt where $M$ is the bubble component of the government debt, i.e.,*

$$d = \sum_{i\in I} \mathrm{PV}(\tau^i y^i) + M,$$

*then the family of income loss processes $(\ell^i)_{i\in I}$ satisfying $\ell^i = \tau^i y^i$ supports $(q, (c^i, a^i, D^i)_{i\in I})$ as a competitive equilibrium with self-enforcing debt with*

$$D^i := \mathrm{PV}(\ell^i) + M^i \quad \text{and} \quad a^i := \theta^i - D^i,$$

*where $(M^i)_{i\in I}$ is any family of nonnegative discounted martingale processes satisfying*

$$\sum_{i\in I} M^i = M.$$

[15] Note that by setting $\ell = \tau \equiv 0$, Theorem 4.1 nests the equivalence result (Theorem 2) in Hellwig and Lorenzoni (2009).

(b) *Reciprocally, if* $(q,(c^i,a^i,D^i)_{i\in I})$ *is a competitive equilibrium with self-enforcing debt associated with a family* $(\ell^i)_{i\in I}$ *of income losses and* $M^i$ *is the bubble component of agent* $i$*'s debt limits, i.e.,*

$$D^i = \mathrm{PV}(\ell^i) + M^i,$$

*then the family of tax schedules* $(\tau^i)_{i\in I}$ *satisfying* $\ell^i = \tau^i y^i$ *supports* $(q,d,(c^i,\theta^i)_{i\in I})$ *as a competitive equilibrium with public debt with*

$$d := \sum_{i\in I} D^i \quad \text{and} \quad \theta^i := a^i + D^i.$$

*Proof.* We only provide the proof of Part (a). Part (b) follows in the same spirit. The following claim formalizes an observation that is a consequence of a translation invariance property of the flow budget constraints.

**Claim 4.1.** *If the plan* $(c^i,a^i)$ *belongs to the budget set* $B^i(D^i,a^i(s^0)|s^0)$*, then the plan* $(c^i,\theta^i)$ *belongs to the budget set* $\widehat{B}^i(\theta^i(s^0)|s^0)$ *where* $\theta^i := a^i + D^i$*. Reciprocally, if the plan* $(c^i,\theta^i)$ *belongs to the budget set* $\widehat{B}^i(\theta^i(s^0)|s^0)$*, then the plan* $(c^i,a^i)$ *belongs to the budget set* $B^i(D^i,a^i(s^0)|s^0)$ *where* $a^i := \theta^i - D^i$*.*

Fix a competitive equilibrium $(q,d,(c^i,\theta^i)_{i\in I})$ with public debt backed by taxes $(\tau^i)_{i\in I}$. It follows from Proposition 4.1 that $\mathrm{PV}(\tau^i y^i|s^0)$ is finite for each $i$. Moreover, there exists a nonnegative process $M$ satisfying exact roll-over such that

$$d = M + \sum_{i\in I} \mathrm{PV}(\tau^i y^i).$$

Fix an arbitrary family $(M^i)_{i\in I}$ of nonnegative processes satisfying exact roll-over and such that

$$\sum_{i\in I} M^i = M.$$

Pose $D^i := \mathrm{PV}(\ell^i) + M^i$, $a^i := \theta^i - D^i$ and $\ell^i := \tau^i y^i$. We claim that $(q,(c^i,a^i,D^i)_{i\in I})$ constitutes a competitive equilibrium with self-enforcing debt and endowment losses $(\ell^i)_{i\in I}$. Indeed, we first observe that the market clearing conditions are satisfied:

$$\begin{aligned}\sum_{i\in I} a^i &= \sum_{i\in I}\theta^i - \sum_{i\in I} D^i\\ &= d - \sum_{i\in I}\mathrm{PV}(\ell^i) - M\\ &= 0.\end{aligned}$$

It follows from Theorem 3.1 that debt limits are not too tight. The debt constraints are satisfied since $\theta^i \geqslant 0$ for each $i$. To conclude the proof we only have to prove that $(c^i, a^i)$ is optimal in the budget set $B^i(D^i, a^i(s^0)|s^0)$. This is, however, true given Claim 4.1. $\square$

### 4.3 Collateralized Debt

We next focus on economies in which all borrowing and lending operations are fully secured by collateral and default carries no other consequences than the loss of the collateral. Following Chien and Lustig (2010), we assume that agents back their promises by means of issuing and trading a long-lived asset whose dividend process accrues a fraction of their income. The default option is endogenous and reflects the continuation value associated with losing the privilege of the stream of liquidity services collateral provides to its owner. In contrast to the economy with self-enforcing debt, there is no reputational effect: collateral secures the full face value of the debt contract as opposed to endowment losses that have no value to creditors.

We show that despite the differences in institutional arrangements, an economy with self-enforcing debt can support as much risk-sharing as an economy with collateralized debt. The established equivalence unravels an interesting link between credit and asset bubbles. By securing debt, collateral is a source of liquidity in two ways. First, collateralized debt reflects the assets' fundamental value (i.e., the investment income due to dividend payments). This is equivalent to endowment losses in economies with self-enforcing debt. Second, any debt level in excess of the fundamental value reflects a bubble in the asset price, the same way credit beyond the present value of income losses reflects a speculative bubble at the self-enforcing equilibrium.

The finding has potential implications for the assessment of models with market exclusion. Chien and Lustig (2010) find that their model supports more volatile risk-premiums and interest rates compared to the setting in Alvarez and Jermann (2001), where autarky is enforced upon default. Our equivalence result suggests that, for a different default option, economies with self-enforcing debt might produce the same variation in equity risk-premiums as in collateralized economies.

### 4.3.1 Institutional Arrangements

Assume that each agent $i$ can pledge at the initial event $s^0$ a part $\ell^i(s^t)$ of the endowment $y^i(s^t)$ at any event $s^t \succ s^0$. This is the equivalent of issuing a long-lived asset (Lucas tree) whose dividend process is $(\ell^i(s^t))_{s^t \succ s^0}$. We now refer to the process $\ell^i$ as the *collateralizable income*.

Shares of Lucas trees can be traded at every event $s^t$. We denote by $P^j(s^t)$ the ex-dividend price of agent $j$'s tree, and we let $\alpha^i_j(s^t) \geqslant 0$ represent agent $i$'s share on agent $j$'s equity at event $s^t$.

Agents can also trade at every event $s^t$ one-period-ahead contingent claims $b^i(s^{t+1}) \in \mathbb{R}$ for each successor event $s^{t+1}$ at a price $q(s^{t+1})$ (expressed in units of $s^t$-consumption). Agents can default and file for bankruptcy. In this case, all assets (equity holdings) and current period dividends are seized and transferred to lenders to redeem their debt. However, the nonpledged endowment cannot be seized and agents still maintain access to financial markets. This specification of the default punishment leads to the following debt constraints:

$$b^i(s^t) \geqslant -\sum_{j\in I} \alpha^i_j(\sigma(s^t))\big[P^j(s^t) + \ell^j(s^t)\big], \quad \text{for all } s^t \succ s^0. \tag{4.11}$$

Constraint (4.11) states that no agent can promise to deliver more than the value of her equity holdings in any state. Following Chien and Lustig (2010), we refer to *liquidity risk* as the risk associated with the constraint (4.11) be binding.

For each agent $i$, given an initial financial claim $b^i(s^0)$, we let $\widetilde{B}^i(b^i(s^0)|s^0)$ denote the budget set consisting of all triples $(c^i, \alpha^i, b^i)$ of consumption processes $(c^i(s^t))_{s^t \succeq s^0}$, equity holdings $(\alpha^i(s^t))_{s^t \succ s^0}$, and contingent claims $(b^i(s^t))_{s^t \succ s^0}$ such that:

$$c^i(s^0) + \sum_{j\in I} P^j(s^0)\alpha^i_j(s^0) + \sum_{s^1 \succ s^0} q(s^1)b^i(s^1) \leqslant y^i(s^0) + b^i(s^0) + P^i(s^0), \tag{4.12}$$

and at any event $s^t \succ s^0$,

$$c^i(s^t) + \sum_{j\in I} P^j(s^t)\alpha^i_j(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})b^i(s^{t+1}) \leqslant \\ y^i(s^t) - \ell^i(s^t) + b^i(s^t) + \sum_{j\in I} \alpha^i_j(\sigma(s^t))\big[\ell^j(s^t) + P^j(s^t)\big], \tag{4.13}$$

where $b(s^t)$ is subject to the liquidity constraint (4.11). Notice that at any event $s^t \succ s^0$, agent $i$'s available income is only $y^i(s^t) - \ell^i(s^t)$ since the rest has been sold at $t = 0$ in the equity markets.

**Definition 4.2.** Given initial financial claims $(b^i(s^0))_{i\in I}$ such that $\sum_{i\in I} b^i(s^0) = 0$, a *competitive equilibrium with collateralized debt* $(q, (P^i)_{i\in I}, (c^i, \alpha^i, b^i)_{i\in I})$ consists of state-contingent bond prices $q$, equity prices $(P^i)_{i\in I}$, a resource feasible consumption $(c^i)_{i\in I}$, an allocation of nonnegative equity holdings $(\alpha^i)_{i\in I}$, and an allocation of contingent claims $(b^i)_{i\in I}$ such that:

(a) for each agent $i \in I$, taking prices as given, the plan $(c^i, \alpha^i, b^i)$ is optimal among budget feasible plans in $\widetilde{B}^i(b^i(s^0)|s^0)$;

(b) all equity markets clear:

$$\forall i \in I, \quad \sum_{j\in I} \alpha_i^j(s^t) = 1, \quad \text{for all } s^t \succeq s^0; \tag{4.14}$$

(c) the bond market clears:

$$\sum_{i\in I} b^i(s^t) = 0, \quad \text{for all } s^t \succ s^0. \tag{4.15}$$

### 4.3.2 Characterization of Asset Prices

We derive a well-understood asset pricing equation that turns out to be useful for our equivalence result. At any contingency, equity prices are decomposed into a fundamental component related to dividend payments and a bubble component. Similarly to the equilibrium with self-enforcing debt, asset price bubbles emerge when dividends are small relative to aggregate resources (see Santos and Woodford 1997).

**Proposition 4.2.** *At any competitive equilibrium with collateralizable wealth, the cum-dividend price of agent j's equity is given by:*

$$\ell^j + P^j = \mathrm{PV}(\ell^j) + M^j \tag{4.16}$$

*for some nonnegative exact roll-over process $M^j$.*

*Proof.* Fix an agent $j$ and an event $s^t \succeq s^0$. Market clearing implies that there exists at least one agent $i \in I$ who is holding a positive amount $\alpha_j^i(s^t) > 0$ of the agent's $j$ equity. Fix $\varepsilon \in \mathbb{R}$ such that $\varepsilon \geqslant -\alpha_j^i(s^t)$. The following changes in contingent claims and equity $j$'s holding are admissible

$$\tilde{\alpha}_j^i(s^t) := \alpha_j^i(s^t) + \varepsilon \quad \text{and} \quad \tilde{b}^i(s^{t+1}) := b^i(s^{t+1}) - \varepsilon[P^j(s^{t+1}) + \ell^j(s^{t+1})].$$

Since agent's $j$ welfare cannot improve after these changes, we must have

$$P^j(s^t) = \sum_{s^{t+1} \succ s^t} q(s^{t+1})[P^j(s^{t+1}) + \ell^j(s^{t+1})]. \tag{4.17}$$

Given this recursive equation, it follows that $\mathrm{PV}(\tau^j y^j | s^0)$ is finite. Moreover, for every event $s^t$, the following limit

$$M^j(s^t) = \lim_{\tau \to \infty} \frac{1}{p(s^t)} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau) P^j(s^\tau)$$

is well-defined, so we obtain equation (4.16). □

### 4.3.3 Equivalence Result

We now have all elements to establish an equivalence mapping between a competitive equilibrium with collateralized debt and a competitive equilibrium with public debt. Given Theorem 4.1, the coincidence extends to a competitive equilibrium with self-enforcing debt. More precisely, starting from an equilibrium in an economy where debt is self-enforcing, we can always support the same consumption allocation in an economy that is subject to liquidity risks for some appropriately chosen levels of pledgeable resources. In that respect, there is no difference in terms of trade between equilibria with self-enforcing debt and equilibria with debt secured by collateral.

**Theorem 4.2.** *Given a family $(\ell^i)_{i \in I}$ representing individual pledgeable resources, a consumption allocation is the outcome of a competitive equilibrium with collaterized debt if, and only if, it is the outcome of a competitive equilibrium with public debt backed by taxes $(\tau^i)_{i \in I}$ satisfying $\ell^i = \tau^i y^i$ (or, equivalently, a competitive equilibrium with self-enforcing debt and endowment losses $(\ell^i)_{i \in I}$). Formally, we have the following properties:*

*(a) Let $(q, (P^i)_{i \in I}, (c^i, \alpha^i, b^i)_{i \in I})$ be a competitive equilibrium with collateralizable income represented by $(\ell^i)_{i \in I}$ and denote by $M^i$ the bubble component of agent $i$'s Lucas tree. Then, $(q, d, (c^i, \theta^i)_{i \in I})$ is a competitive equilibrium with public debt backed by the family of taxes $(\tau^i)_{i \in I}$ satisfying $\ell^i = \tau^i y^i$, where*

$$\theta^i(s^t) := b^i(s^t) + \sum_{j \in I} \alpha^i_j(\sigma(s^t))[P^j(s^t) + \tau^j(s^t) y^j(s^t)] \tag{4.18}$$

*and*

$$d := \sum_{i \in I} \mathrm{PV}(\tau^i y^i) + M^i. \tag{4.19}$$

*(b) Let $(q, d, (c^i, \theta^i)_{i\in I})$ be a competitive equilibrium with public debt backed by the family of taxes $(\tau^i)_{i\in I}$. Denote by $M$ the nonnegative process satisfying exact roll-over such that $d = \sum_{i\in I} \mathrm{PV}(\tau^i y^i) + M$. Fix any family $(M^i)_{i\in I}$ of nonnegative processes satisfying exact roll-over and such that $\sum_{i\in I} M^i = M$. Fix also a family $(\alpha^i)_{i\in I}$ of shares satisfying market clearing.*[16] *Then, $(q, (P^i)_{i\in I}, (c^i, \alpha^i, b^i)_{i\in I})$ constitutes a competitive equilibrium with collateralized debt (associated to individual pledgeable resources $\ell^i = \tau^i y^i$) where*

$$P^i := \mathrm{PV}(\ell^i) - \ell^i + M^i \tag{4.20}$$

*and*

$$b^i(s^t) := \theta^i(s^t) - \sum_{j\in I} \alpha^i_j(\sigma(s^t))[P^j(s^t) + \ell^j(s^t)]. \tag{4.21}$$

*Proof.* We only prove part (a). Part (b) follows in the same spirit. The debt constraints (4.11) imply that $\theta^i(s^t)$, as defined by (4.18), is nonnegative. Combining the market clearing conditions (4.14) and (4.15), we get that

$$\sum_{i\in I} \theta^i(s^t) = \sum_{j\in I}[P^j(s^t) + \tau^j(s^t)y^j(s^t)] = \sum_{i\in I} \mathrm{PV}(\tau^i y^i) + M^i,$$

where the last equality follows form the asset pricing equation (4.16). The market clearing condition (4.10) then follows from our choice of the government debt. Combining the flow budget constraints (4.12) and (4.13) together with the asset pricing equation (4.16), we get that the pair $(c^i, \theta^i)$ belongs to the budget set $\widehat{B}^i(\theta^i(s^0)|s^0)$. Reciprocally, let $(\tilde{c}^i, \tilde{\theta}^i)$ be a plan in the budget set $\widehat{B}^i(\theta^i(s^0)|s^0)$. Let $\tilde{b}^i$ be the process of bond holdings defined by

$$\tilde{b}^i(s^t) := \tilde{\theta}^i(s^t) - \sum_{j\in I} \alpha^i_j(\sigma(s^t))[P^j(s^t) + \tau^j(s^t)y^j(s^t)].$$

Using the asset pricing equation (4.16) and the fact that $\tilde{b}^i(s^0) = b^i(s^0)$, we can show that the plan $(\tilde{c}^i, \alpha^i, \tilde{b}^i)$ belongs to the budget set $\widetilde{B}(b^i(s^0)|s^0)$. Since $(c^i, \alpha^i, b^i)$ is optimal in $\widetilde{B}(b^i(s^0)|s^0)$, we must have $U(\tilde{c}^i|s^0) \leqslant U(c^i|s^0)$. This proves that the plan $(c^i, \theta^i)$ is optimal in $\widehat{B}^i(\theta^i(s^0)|s^0)$. We can conclude that $(q, d, (c^i, \theta^i)_{i\in I})$ is a competitive equilibrium with public debt backed by the family of taxes $(\tau^i)_{i\in I}$. □

*Remark* 4.2. The following observation is a direct consequence of our equivalence result. Assume, as in Chien and Lustig (2010), that there exists $\ell > 0$ such that $\ell^i(s^t) = \ell$ for all $i$

[16] In the sense that $\sum_{j\in I} \alpha^i_j(s^t) = 1$ for all $i \in I$ and all $s^t \succeq s^0$.

and $s^t$ and that there is no endowment growth. If $\ell > 0$, then Corollary 3.1 implies that the present value of pledgeable resources is finite and assets are priced at their fundamental value, so they are bubble-free. One may think that for $\ell = 0$ (i.e., assets pay no dividends), asset prices must equal zero, so autarky is the only equilibrium outcome. However, this claim is true provided that the aggregate wealth is still finite, or equivalently, when the implied interest rates are higher than growth rates. Nevertheless, as documented by Hellwig and Lorenzoni (2009), when $\ell = 0$, equilibrium interest rates can be sufficiently low so that the economy's aggregate wealth is infinite. The implication for the collateral equilibrium, is that, even if the trees pay no dividend, assets may be priced as a speculative bubble. Indeed, it is sufficient to appeal to Theorem 4.1 and Theorem 4.2 and translate the bubbly equilibrium of Hellwig and Lorenzoni (2009) in the environment of Chien and Lustig (2010).

The intuition for this discrepancy relies on the dual role of collateral as a source of liquidity. As dividends become negligible (i.e., $\ell$ approaches zero), the value of the asset increases to compensate for the decreased investment value. In the limit, the value of the collateral asset is still positive, reflecting purely a bubble, even though there is no collateral in the market anymore.

## 4.4 Existence of a Competitive Equilibrium

It is well-known that proving the existence of a competitive equilibrium with self-enforcing debt is difficult. The presence of self-enforcing conditions does not permit a direct adaptation of a standard truncation technique, as is the case in models with exogenous debt limits. We show below how to bypass these complications by proving the existence of a competitive equilibrium with public debt and then appealing to our equivalence result (Theorem 4.1). Our insight derives from the observation that applying a truncation argument to the economy with public debt, though it requires taking into account the endogeneity of the supply side (condition 4.9), is far more simple than dealing directly with the endogeneity of individual debt limits in the economy with self-enforcing debt.

Formally, the idea of proof strategy goes as follows. We can apply standard arguments based on continuity, convexity, and compactness to prove the existence of a competitive equilibrium $(q_T, d_T, (c^i_T, \theta^i_T)_{i\in I})$ with public debt in a truncated economy with finite horizon $T$. Passing to a subsequence if necessary, we can assume that the sequence of consumption allocation $(c^i_T(s^t))_{T\geqslant 1}$ converges to a feasible allocation $(c^i(s^t))_{i\in I}$. Convergence of bond prices $(q_T(s^t))_{T\geqslant 1}$ to some price $q(s^t)$ follows then from the Euler equation. The flow budget constraints remain valid in the limit. The only difficult part is to show that bond holdings converge (or admit a converging subsequence. Since markets clear, it is sufficient to show that outstanding public debt is bounded. Intuitively, if the sequence $(d_T(s^0))_{T\geqslant 1}$ of initial public debt is unbounded, then there must be at least one agent with an arbitrary large amount $\theta^i_T(s^0)$ of initial resources as $T$ goes to infinite. This agent could support an arbitrary large consumption plan that would necessarily violate optimallity.

**Theorem 4.3.** *Fix a family of tax schedules $(\tau^i)_{i\in I}$ and consider an arbitrary decomposition of tax revenues*

$$\sum_{i\in I}\tau^i y^i = \sum_{i\in I}\delta^i$$

*where $\delta^i$ is a nonnegative process. If aggregate tax revenues are nonzero at any contingency, i.e.,*

$$\sum_{i\in I}\tau^i(s^t)y^i(s^t) > 0, \quad \textit{for all event } s^t,$$

*then there exists a competitive equilibrium with backed public debt $(q, d, (c^i, \theta^i)_{i\in I})$ where the allocation of initial bond holdings satisfies $\theta^i(s^0) \geqslant \mathrm{PV}(\delta^i|s^0)$ for each agent $i$.*

A formal proof is presented in Appendix A.3.[17] Existence of a competitive equilibrium with self-enforcing debt is then a direct corollary of the last theorem and Theorem 4.1.

**Corollary 4.1.** *Fix a family of income loss processes $(\ell^i)_{i\in I}$ and consider an arbitrary decomposition*

$$\sum_{i\in I}\ell^i = \sum_{i\in I}\delta^i$$

*where $\delta^i$ is a nonnegative process. There exists a competitive equilibrium with self-enforcing debt $(q, (c^i, a^i, D^i)_{i\in I})$ where the allocation of initial bond holdings satisfies $a^i(s^0) = \mathrm{PV}(\delta^i - \ell^i|s^0)$ for each agent $i$.*

*Proof.* Fix tax schedules $(\tau^i)_{i\in I}$ satisfying $\ell^i = \tau^i y^i$ for each $i$. From Theorem 4.3 there exists a competitive equilibrium with backed public debt $(q, d, (c^i, \theta^i)_{i\in I})$ where the allocation of initial bond holdings satisfies $\theta^i(s^0) \geqslant \mathrm{PV}(\delta^i|s^0)$ for each agent $i$. We can apply

[17] Assuming that aggregate tax revenues are nonzero at any contingency facilitates proving that the limiting bond prices are nonzero. This requirement could be relaxed, but the argument becomes more involved.

Proposition 4.1 to get the existence of a nonnegative discounted martingale process $M$ such that

$$d = M + \sum_{i \in I} \mathrm{PV}(\tau^i y^i).$$

We pose $M^i(s^0) := \theta^i(s^0) - \mathrm{PV}(\delta^i | s^0)$, for each $i$. Observe that $M^i(s^0) \geqslant 0$ and

$$\sum_{i \in I} M^i(s^0) = d(s^0) - \sum_{i \in I} \mathrm{PV}(\tau^i y^i | s^0) = M(s^0).$$

We can then extend the definition of $M^i(s^0)$ to the whole tree $\Sigma$ such that each $M^i$ is a nonnegative discounted martingale and $M = \sum_{i \in I} M^i$.[18] We can now apply Theorem 4.1 to deduce that $(q, (c^i, a^i, D^i)_{i \in I})$ is a competitive equilibrium with self-enforcing debt where $D^i := \mathrm{PV}(\ell^i) + M^i$ and $a^i := \theta^i - D^i$ for each $i \in I$. The way we define each $M^i(s^0)$, implies that $a^i(s^0) = \mathrm{PV}(\delta^i - \ell^i | s^0)$. □

*Remark* 4.3. If we choose the decomposition such that $\delta^i = \ell^i$ for each $i$, we then get the existence of a competitive equilibrium with zero initial financial endowments ($a^i(s^0) = 0$). Existence is also guaranteed for many other allocations of initial financial endowments. For instance, we may have an agent $j$ starting with the initial financial endowment $\sum_{i \in I} \mathrm{PV}(\ell^i | s^0)$ while all other agents $i \neq j$ start with (self-enforcing) debt $\mathrm{PV}(\ell^i | s^0)$.

Existence of a competitive equilibrium with self-enforcing debt as presented in Corollary 4.1 is an important result. However, equilibria are only tractable if they are Markov for some simple endogenous state variable. For the rest of the section we assume that uncertainty is governed by a Markov process on a finite space $S$ with strictly positive transitions, i.e., there exists a vector

$$(\pi(s'|s))_{(s,s') \in S^2} \quad \text{with } \pi(s'|s) > 0$$

such that

(a) $s^0 =: s_0 \in S$ and for every $t \geqslant 1$, an event $s^t$ is a history $(s_0, s_1, \ldots, s_t)$ where each $s_r \in S$;

(b) conditional probabilities satisfy $\pi(s^{t+1} | s^t) = \pi(s_{t+1} | s_t)$ for all $s^{t+1} \succ s^t$.

[18] A possible way (among infinitely many) to construct the family $(M^i)_{i \in I}$ is as follows: if $M(s^0) = 0$, then $M^i(s^0) = 0$ for all $i$, and we can pose $M^i(s^t) := 0$ for every $s^t \succ s^0$. If $M(s^0) > 0$, then let $\alpha^i := M^i(s^0)/M(s^0)$ and pose $M^i(s^t) := \alpha^i M(s^t)$ for every $i$ and every $s^t \succ s^0$.

Individual endowments $y^i$ and income losses $\ell^i$ oscillate according to this Markov process and, hence, all fundamentals are measurable with respect to the finite Markov state space $S$.[19] This, in particular, implies that the economy cannot grow or decline over time, that is, the aggregate endowment is bounded.

In general, at a competitive equilibrium, prices and debt limits would be affected by the distribution of wealth, as well as possibly by future expectations, and would not be measurable with respect to the Markov state space $S$. Following the recent contributions of Chien and Lustig (2010) and Gottardi and Kubler (2015), we consider the *instantaneous Negishi weights* as a candidate for the endogenous state variable. Indeed, fix an arbitrary event $s^t$ and observe that an allocation $(c^i(s^t))_{i\in I}$ of strictly positive consumption satisfies $\sum_{i\in I} c^i(s^t) = \bar{y}(s_t)$ where $\bar{y} := \sum_{i\in I} y^i$ if, and only if, there exists a unique (up to positive scaling) family $\lambda(s^t) = (\lambda^i(s^t))_{i\in I} \in \mathbb{R}^I_{++}$ of strictly positive instantaneous Negishi weights such that

$$(c^i(s^t))_{i\in I} \in \operatorname{argmax}\left\{\sum_{i\in I} \lambda^i(s^t)u(x^i) \ : \ (x^i)_{i\in I} \in \mathbb{R}^I_{++} \quad \text{and} \quad \sum_{i\in I} x^i = \bar{y}(s_t)\right\}. \tag{4.22}$$

The unique feasible allocation that solves the maximization problem (4.22) can be expressed as a continuous function $(\lambda(s^t), s_t) \longmapsto \tilde{c}(\lambda(s^t), s_t) = (\tilde{c}^i(\lambda(s^t), s_t))_{i\in I}$ from $\mathbb{R}^I_{++} \times S$ to $\mathbb{R}^I_{+}$. It is then natural to introduce the following definition.

**Definition 4.3.** A stationary Markov equilibrium with instantaneous Negishi weigths is a family of policy functions $((\tilde{q}(s')_{s'\in S}), (\tilde{c}^i, \tilde{a}^i, \tilde{D}^i)_{i\in I})$ defined on $\mathbb{R}^I_{++} \times S$ and a transition function $L : \mathbb{R}^I_{++} \times S \to \mathbb{R}^I_{++}$ satisfying the following properties.

(1) For each state $s$, the set $\Lambda(s) \subseteq \mathbb{R}^I_{++}$ of instantaneous Negishi weights $\lambda$ such that $\tilde{a}^i(\lambda, s) \geqslant -\tilde{D}^i(\lambda, s)$ for each $i$ is nonempty and $L(\lambda, s) \subseteq \Lambda(s)$ for all instantaneous Negishi weights $\lambda \in \mathbb{R}^I_{++}$.

(2) For any initial $\lambda_0 \in \Lambda(s_0)$, the family $(q, (c^i, a^i, D^i)_{i\in I})$ defined by

$$(c^i(s^t), a^i(s^t), D^i(s^t)) := (\tilde{c}^i(\lambda(s^t), s_t), \tilde{a}^i(\lambda(s^t), s_t), \tilde{D}^i(\lambda(s^t), s_t)) \tag{4.23}$$

and for any $s^{t+1} \succ s^t$

$$q(s^{t+1}) := \tilde{q}(\lambda(s^t), s_t)(s_{t+1}) \tag{4.24}$$

[19] In other words, $y^i(s^t)$ and $\ell^i(s^t)$ only depend on the current shock $s_t$. We abuse notations by writing $y^i(s_t)$ and $\ell^i(s_t)$.

is a competitive equilibrium with self-enforcing debt where the process $(\lambda(s^t))_{s^t \succeq s^0}$ is defined recursively by

$$\lambda(s^0) := \lambda_0 \quad \text{and} \quad \lambda(s^t) := L(\lambda(\sigma(s^t)), s_t), \quad \text{for all } s^t \succ s^0. \tag{4.25}$$

We can now combine Theorem 5 in Gottardi and Kubler (2015) with our equivalence results to prove the existence and uniqueness of a stationary Markov equilibrium under specific conditions on primitives.

**Proposition 4.3.** *Assume that endowment losses are nonnegligible and that the map $c \mapsto u'(c)c$ is increasing over $\mathbb{R}_{++}$. Then there exists a unique stationary Markov equilibrium with self-enforcing debt.*

The proof of this result is presented in Appendix A.4. In the spirit of Theorem 6 in Gottardi and Kubler (2015), if there are only two agents, we can identify further conditions on primitives under which the unique stationary Markov equilibrium has finite support. This is important since such an equilibrium can be characterized by a finite system of equations and can typically be computed easily. One can also conduct local comparative statics using the implicit function theorem.

# 5 Conclusion

Since the global financial crisis, there has been a revival in the literature that aims to understand credit and asset bubbles. In this paper, we have shown that credit bubbles can naturally arise in various general equilibrium environments with limited commitment, in a similar manner to how asset price bubbles can naturally arise in environments with limited asset supply. We also provided several applications of our theory to the context of sovereign debt, domestic public debt, and consumer debt. While we have focused on equilibria where bubbles are safe, it is possible to extend our framework to incorporate sunspot shocks (as in Calvo 1988, Cole and Kehoe 2000, Azariadis et al. 2015) to allow for the possibility of stochastically bursting credit bubbles. It would be interesting for future research to study the general equilibrium effects of the booms and busts of credit bubbles in the various economic environments that we have studied.

# References

Ábrahám, A. and Cárceles-Poveda, E. (2006). Endogenous incomplete markets, enforcement constraints, and intermediation. Theoretical Economics, 1(4):439–459.

Ábrahám, A. and Cárceles-Poveda, E. (2010). Endogenous trading constraints with incomplete asset markets. Journal of Economic Theory, 145(3):974–1004.

Aguiar, M. and Amador, M. (2014). Sovereign debt. In Gita Gopinath, E. H. and Rogoff, K., editors, Handbook of International Economics, volume 4, chapter 11, pages 647–687. Elsevier.

Aguiar, M. and Gopinath, G. (2006). Defaultable debt, interest rates and the current account. Journal of International Economics, 69(1):64–83.

Alvarez, F. and Jermann, U. J. (2000). Efficiency, equilibrium, and asset pricing with risk of default. Econometrica, 68(4):775–797.

Alvarez, F. and Jermann, U. J. (2001). Quantitative asset pricing implications of endogenous solvency constraints. Review of Financial Studies, 14(4):1117–1151.

Arellano, C. (2008). Default risk and income fluctuations in emerging economies. American Economic Review, 98(3):690–712.

Azariadis, C., Kaas, L., and Wen, Y. (2015). Self-fulfilling credit cycles. Review of Economic Studies, 83(4):1364–1405.

Bai, Y. and Zhang, J. (2010). Solving the Feldstein–Horioka puzzle with financial frictions. Econometrica, 78(2):603–632.

Bai, Y. and Zhang, J. (2012). Financial integration and international risk sharing. Journal of International Economics, 86(1):17–32.

Barlevy, G. (2018). Bridging between policymakers' and economists' views on bubbles. Economic Perspectives, 42(4):1–21.

Bengui, J. and Phan, T. (2018). Asset pledgeability and endogenously leveraged bubbles. Journal of Economic Theory, 177:280–314.

Bidian, F. (2016). Robust bubbles with mild penalties for default. Journal of Mathematical Economics, 65:141–153.

Bidian, F. and Bejan, C. (2015). Martingale properties of self-enforcing debt. Economic Theory, 60(1):35–57.

Biswas, S., Hanson, A., and Phan, T. (2018). Bubbly recessions. Richmond Fed Working Paper.

Bloise, G. and Reichlin, P. (2011). Asset prices, debt constraints and inefficiency. Journal of Economic Theory, 146(4):1520–1546.

Bloise, G., Reichlin, P., and Tirelli, M. (2013). Fragility of competitive equilibrium with risk of default. Review of Economic Dynamics, 16(2):271–295.

Bulow, J. and Rogoff, K. (1989). Sovereign debt: Is to forgive to forget? American Economic Review, 79(1):43–50.

Caballero, R. and Farhi, E. (2017). The safety trap. Review of Economic Studies, 85(1):223–274.

Caballero, R. J., Farhi, E., and Gourinchas, P.-O. (2008). An equilibrium model of global imbalances and low interest rates. American Economic Review, 98(1):358.

Calvo, G. A. (1988). Servicing the public debt: The role of expectations. American Economic Review, pages 647–661.

Chatterjee, S., Corbae, D., Nakajima, M., and Ríos-Rull, J.-V. (2007). A quantitative theory of unsecured consumer credit with risk of default. Econometrica, 75(6):1525–1589.

Chien, Y. and Lustig, H. (2010). The market price of aggregate risk and the wealth distribution. Review of Financial Studies, 23(4):1596–1650.

Cohen, D. and Sachs, J. (1986). Growth and external debt under risk of debt repudiation. European Economic Review, 30(3):529–560.

Cole, H. and Kehoe, T. J. (2000). Self-fulfilling debt crises. Review of Economic Studies, 67(1):91–116.

D'Erasmo, P., Mendoza, E., and Zhang, J. (2016). What is a sustainable public debt? In Taylor, J. B. and Uhlig, H., editors, Handbook of Macroeconomics, volume 2, chapter 32, pages 2493–2597. Elsevier.

Diamond, P. A. (1965). National debt in a neoclassical growth model. American Economic Review, 55(5):1126–1150.

Eaton, J. and Gersovitz, M. (1981). Debt with potential repudiation: Theoretical and empirical analysis. Review of Economic Studies, 48(2):289–309.

Farhi, E. and Tirole, J. (2012). Bubbly liquidity. Review of Economic Studies, 79(2):678–706.

Gennaioli, N., Martin, A., and Rossi, S. (2014). Sovereign default, domestic banks, and financial institutions. Journal of Finance, 69(2):819–866.

Gottardi, P. and Kubler, F. (2015). Dynamic competitive equilibrium with complete markets and collateral constraints. Review of Economic Studies, 82(3):1119–1153.

Gul, F. and Pesendorfer, W. (2004). Self-control and the theory of consumption. Econometrica, 72(1):119–158.

Hellwig, C. and Lorenzoni, G. (2009). Bubbles and self-enforcing debt. Econometrica, 77(4):1137–1164.

Hirano, T. and Yanagawa, N. (2016). Asset bubbles, endogenous growth, and financial frictions. Review of Economic Studies, 84(1):406–443.

Holmström, B. and Tirole, J. (1998). Private and public supply of liquidity. Journal of political Economy, 106(1):1–40.

Holmström, B. and Tirole, J. (2011). Inside and outside liquidity. MIT press.

Ikeda, D. and Phan, T. (2019). Asset bubbles and global imbalances. American Economic Journal: Macroeconomics, 11(3):209–51.

Jordà, Ò., Schularick, M., and Taylor, A. M. (2015). Leveraged bubbles. Journal of Monetary Economics, 76:S1–S20.

Kehoe, P. and Perri, F. (2002). International business cycles with endogenous incomplete markets. Econometrica, 70(3):907–928.

Kehoe, P. and Perri, F. (2004). Competitive equilibria with limited enforcement. Journal of Economic Theory, 119(1):184–206.

Kehoe, T. J. and Levine, D. K. (1993). Debt-constrained asset markets. Review of Economic Studies, 60(4):865–888.

Livshits, I. (2015). Recent developments in consumer credit and default literature. Journal of Economic Surveys, 29(4):594–613.

Martin, A. and Ventura, J. (2012). Economic growth with bubbles. American Economic Review, 102(6):3033–3058.

Martins-da-Rocha, V. F. and Santos, M. (2019). Self-enforcing debt and rational bubbles. Available at SSRN: https://ssrn.com/abstract=3169229 or http://dx.doi.org/10.2139/ssrn.3169229.

Martins-da-Rocha, V. F. and Vailakis, Y. (2015). Constrained efficiency without commitment. Journal of Mathematical Economics, 61:276–286.

Martins-da-Rocha, V. F. and Vailakis, Y. (2017a). Borrowing in excess of natural ability to repay. Review of Economic Dynamics, 23:42–59.

Martins-da-Rocha, V. F. and Vailakis, Y. (2017b). On the sovereign debt paradox. Economic Theory, 64(4):825–846.

Mendoza, E. G. and Yue, V. Z. (2012). A general equilibrium model of sovereign default and business cycles. Quarterly Journal of Economics, 127(2):889–946.

Miao, J. and Wang, P. (2018). Asset bubbles and credit constraints. American Economic Review, 108(9):2590–2628.

Phan, T. (2017). Sovereign debt signals. Journal of International Economics, 104:157–165.

Samuelson, P. A. (1958). An exact consumption-loan model of interest with or without the social contrivance of money. Journal of Political Economy, 66(6):467–482.

Santos, M. and Woodford, M. (1997). Rational asset pricing bubbles. Econometrica, 65(1):19–57.

Tirole, J. (1985). Asset bubbles and overlapping generations. Econometrica, pages 1071–1100.

Werner, J. (2014). Rational asset pricing bubbles and debt constraints. Journal of Mathematical Economics, 53:145–152.

Woodford, M. (1990). Public debt as private liquidity. American Economic Review, 80(2):382–388.

Wright, M. L. (2013). Theory of sovereign debt and default. In Caprio, G., Bacchetta, P., Barth, J. R., Hoshi, T., Lane, P. R., Mayes, D. G., Mian, A. R., and Taylor, M., editors, Handbook of Safeguarding Global Financial Stability, chapter 20, pages 187–193. Academic Press, San Diego.

# A Appendix: Omitted Proofs

## A.1 Proof of Lemma 3.1

Since we are exclusively concerned with the single-agent problem, we simplify notation throughout this section by dropping the superscript $i$.

Let $D$ be a process of not-too-tight bounds. We first show that there exists a nonnegative process $\underline{D}$ satisfying

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), \underline{D}(s^{t+1})\}, \quad \text{for all } s^t \succeq s^0. \tag{A.1}$$

Indeed, let $\Phi$ be the mapping $B \in \mathbb{R}^\Sigma \longmapsto \Phi B \in \mathbb{R}^\Sigma$ defined by

$$(\Phi B)(s^t) := \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \min\{D(s^{t+1}), B(s^{t+1})\}, \quad \text{for all } s^t \succeq s^0.$$

Denote by $[0, \bar{D}]$ the set of all processes $B \in \mathbb{R}^\Sigma$ satisfying $0 \leqslant B \leqslant \bar{D}$ where

$$\bar{D}(s^t) := \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) D(s^{t+1}), \quad \text{for all } s^t \succeq s^0.$$

The mapping $\Phi$ is continuous (for the product topology), and we have $\Phi[0, \bar{D}] \subseteq [0, \bar{D}]$. Since $[0, \bar{D}]$ is convex and compact (for the product topology), it follows that $\Phi$ admits a fixed point $\underline{D}$ in $[0, \bar{D}]$.

**Claim A.1.** *The process $\underline{D}$ is tighter than the process $D$, i.e., $\underline{D} \leqslant D$.*

*Proof of Claim A.1.* Fix a node $s^t$. Since $V_\ell(0,0|s^t) = V(D, -D(s^t)|s^t)$ and $V(D,\cdot|s^t)$ is strictly increasing, it is sufficient to show that $V(D, -\underline{D}(s^t)|s^t) \geqslant V_\ell(0,0|s^t)$. Denote by $(\tilde{c}, \tilde{a})$ the optimal consumption and bond holdings associated with the default option at $s^t$, i.e., $(\tilde{c}, \tilde{a}) \in d_\ell(0,0|s^t)$.[20] We let $\widehat{D}$ be the process defined by $\widehat{D}(s^t) := \min\{D(s^t), \underline{D}(s^t)\}$ for all $s^t$. Observe that

$$\begin{aligned} y(s^t) - \underline{D}(s^t) &= y(s^t) - \ell(s^t) - \sum_{s^{t+1} \succ s^t} q(s^{t+1}) \widehat{D}(s^{t+1}) \\ &= \tilde{c}(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})[\tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})] \\ &= \tilde{c}(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1}) a(s^{t+1}) \end{aligned}$$

where $a(s^{t+1}) := \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1})$. Since $\widehat{D} \leqslant D$, we have $a(s^{t+1}) \geqslant -D(s^{t+1})$. At any successor event $s^{t+1} \succ s^t$, we have

$$\begin{aligned} y(s^{t+1}) + a(s^{t+1}) &= y(s^{t+1}) + \tilde{a}(s^{t+1}) - \widehat{D}(s^{t+1}) \\ &\geqslant y(s^{t+1}) + \tilde{a}(s^{t+1}) - \underline{D}(s^{t+1}) \\ &\geqslant y(s^{t+1}) - \ell(s^{t+1}) + \tilde{a}(s^{t+1}) - \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) \widehat{D}(s^{t+2}) \\ &\geqslant \tilde{c}(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2})[\tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})] \\ &\geqslant \tilde{c}(s^{t+2}) + \sum_{s^{t+2} \succ s^{t+1}} q(s^{t+2}) a(s^{t+2}) \end{aligned}$$

where $a(s^{t+2}) := \tilde{a}(s^{t+2}) - \widehat{D}(s^{t+2})$.[21] Observe that $a(s^{t+2}) \geqslant -D(s^{t+2})$ (since $\widehat{D} \leqslant D$).

Defining $a(s^\tau) := \tilde{a}(s^\tau) - \widehat{D}(s^\tau)$ for any successor $s^\tau \succ s^t$ and iterating the above argument, we can show that $(\tilde{c}, a)$ belongs to the budget set $B(D, -\underline{D}(s^t)|s^t)$. It follows that

$$V(D, -\underline{D}(s^t)|s^t) \geqslant U(\tilde{c}|s^t) = V_\ell(0,0|s^t)$$

implying the desired result: $\underline{D}(s^t) \leqslant D(s^t)$. □

[20] Equivalently, $(\tilde{c}, \tilde{a})$ satisfies $U(\tilde{c}|s^t) := V_\ell(0,0|s^t)$ and belongs to $B_\ell(0,0|s^t)$.

[21] To get the second weak inequality, we use equation (A.1).

It follows from Claim A.1 that $\underline{D}$ satisfies

$$\underline{D}(s^t) = \ell(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\underline{D}(s^{t+1}), \quad \text{for all } s^t \succeq s^0. \tag{A.2}$$

Applying equation (A.2) recursively, we get

$$p(s^t)\underline{D}(s^t) = p(s^t)\ell(s^t) + \sum_{s^{t+1} \in S^{t+1}(s^t)} p(s^{t+1})\ell(s^{t+1}) + \dots$$
$$\dots + \sum_{s^T \in S^T(s^t)} p(s^T)\ell(s^T) + \sum_{s^{T+1} \in S^{T+1}(s^t)} p(s^{T+1})\underline{D}(s^{T+1})$$

for any $T > t$. Since $\underline{D}$ is nonnegative, it follows that

$$p(s^t)\underline{D}(s^t) \geqslant \sum_{\tau=t}^{T} \sum_{s^\tau \in S^\tau(s^t)} p(s^\tau)\ell(s^\tau).$$

Passing to the limit when $T$ goes to infinity, we get that $\mathrm{PV}(\ell|s^t)$ is finite for any event $s^t$ (in particular for $s^0$). Recalling that $D \geqslant \underline{D}$, we also get that $D(s^t) \geqslant \mathrm{PV}(\ell|s^t)$.

## A.2 Proof of Lemma 3.2

Denote by $(\underline{c}^i, \underline{a}^i)$ the optimal consumption and bond holdings in the budget set $B^i_{\ell^i}(0, 0|s^\tau)$ for some arbitrary event $s^\tau$. We pose $\underline{D}^i := \mathrm{PV}(\ell^i)$ and observe that

$$\underline{D}^i(s^t) = \ell^i(s^t) + \sum_{s^{t+1} \succ s^t} q(s^{t+1})\underline{D}^i(s^{t+1}).$$

We can easily show that $(\underline{c}^i, a^i)$ is optimal in the budget set $B^i(\underline{D}^i, -\underline{D}^i(s^0)|s^\tau)$ where $a^i := \underline{a}^i - \underline{D}^i$. We then deduce that $V^i(\underline{D}^i, -\underline{D}^i(s^\tau)|s^\tau) = V^i_{\ell^i}(0, 0|s^\tau)$.

## A.3 Proof of Theorem 4.3

To simplify the presentation, we let $\delta := \sum_{i \in I} \tau^i y^i$. Fix an arbitrary decomposition of $\delta$ as follows:

$$\delta = \sum_{i \in I} \delta^i \quad \text{where } \delta^i \geqslant 0 \text{ for each } i.$$

We claim that there exists a competitive equilibrium $(q, d, (c^i, \theta^i)_{i \in I})$ with public debt backed by the tax schedules $(\tau^i)_{i \in I}$ and such that the initial bond holdings satisfy

$$\theta^i(s^0) \geqslant \mathrm{PV}(\delta^i | s^0), \quad \text{for each } i \in I.$$

To prove existence, we follow the standard truncation approach. Given an arbitrary date $\xi \geqslant 1$, for every event $s^t$ with $t \leqslant \xi$, the $\xi$-truncated present value $\mathrm{PV}_\xi(x|s^t; q)$ of some process $x$ under the price process $q$ is defined by

$$\mathrm{PV}_\xi(x|s^t; q) = \frac{1}{p(s^t)} \sum_{r=t}^{\xi} \sum_{s^r \in S^r(s^t)} p(s^r) x(s^r)$$

where $p$ is the process of Arrow–Debreu prices associated with $q$.

Standard arguments (continuity, compacity, and convexity) can be applied to prove the existence of a $\xi$-truncated equilibrium with public debt defined as a family $(q_\xi, d_\xi, (c^i_\xi, \theta^i_\xi)_{i \in I})$ where

(i) for each agent $i \in I$, taking prices as given, the plan $(c^i_\xi, \theta^i_\xi)$ maximizes the utility $U(c)$ among all budget feasible plans $(c^i, \theta^i)$ satisfying the initial condition

$$\theta^i(s^0) = \mathrm{PV}_\xi(\delta^i | s^0; q_\xi),$$

for every $t \leqslant \xi$ and every event $s^t \in S^t$,

$$c^i(s^t) + \sum_{s^{t+1} \succ s^t} q_\xi(s^{t+1}) \theta^i(s^{t+1}) = (1 - \tau^i(s^t)) y^i(s^t) + \theta^i(s^t) \quad \text{and} \quad \theta^i(s^t) \geqslant 0$$

and for every $t > \xi$ and every $s^t \in S^t$

$$c^i(s^t) = y^i(s^t) \quad \text{and} \quad \theta^i(s^t) = 0;$$

(ii) the government debt market clears

$$\sum_{i \in I} \theta^i_\xi(s^t) = d_\xi(s^t), \quad \text{for all } t \leqslant \xi \text{ and } s^t \in S^t;$$

(iii) the government budget constraint is satisfied

$$d_\xi(s^t) = \mathrm{PV}_\xi(\delta | s^t; q_\xi), \quad \text{for all } t \leqslant \xi \text{ and } s^t \in S^t.$$

For each $\xi$, the consumption allocation $(c^i_\xi)_{i\in I}$ satisfies market clearing. Passing to a subsequence if necessary, we can assume that there exists a consumption allocation $(c^i)_{i\in I}$, also satisfying market clearing, such that, for each $i$, the sequence $(c^i_\xi)_{\xi\geqslant 1}$ converges (for the product topology) to $c^i$. We can also assume that for each event $s^t$, the sequence of bond prices $(q_\xi(s^t))_{\xi\geqslant 1}$ converges to some number $q(s^t) \in [0,\infty]$.

**Claim A.2.** *For every event $s^t$, we have $q(s^t) < \infty$.*

*Proof.* Fix an event $s^t$ and assume, by way of contradiction, that the sequence $(q_\xi(s^t))_{\xi\geqslant 1}$ converges to $\infty$. Fix an arbitrary agent $i$. If there exists some large enough $\bar{\xi}$ such that $\theta^i_\xi(s^t) = 0$ for every $\xi \geqslant \bar{\xi}$, then we have

$$c^i_\xi(s^t) \leqslant (1-\tau^i(s^t))y^i(s^t)$$

and passing to the limit, we get that $c^i(s^t) \leqslant (1-\tau^i(s^t))y^i(s^t)$. Assume now that there exists a strictly increasing function $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\theta^i_{\varphi(\xi)}(s^t) > 0$ for every $\xi$. From the first-order condition associated with agent $i$'s maximization problem, we have that

$$q_{\varphi(\xi)}(s^t) = \beta\pi(s^t|\sigma(s^t))\frac{u'(c^i_{\varphi(\xi)}(s^t))}{u'(c^i_{\varphi(\xi)}(\sigma(s^t)))}.$$

Passing to the limit when $\xi$ approaches infinity, we deduce that $c^i(s^t) = 0$.[22] We have then proved that either $c^i(s^t) \leqslant (1-\tau^i(s^t))y^i(s^t)$ or $c^i(s^t) = 0$. This contradicts the fact that the allocation $(c^i)_{i\in I}$ satisfies market clearing since we assumed that $\sum_{i\in I}\tau^i(s^t)y^i(s^t) > 0$. $\square$

We claim that the sequence $(d_\xi(s^0))_{\xi\geqslant 1}$ is bounded. Indeed, first observe that there exists a date $T$ such that $U_T(2\overline{y}|s^0) > U(\overline{y}|s^0)$ where

$$U_T(c|s^0) = \sum_{t=0}^{T}\beta^t\sum_{s^t\in S^t}\pi(s^t)u(c(s^t)).$$

Since the sequence $(q_\xi)_{\xi\geqslant 1}$ of bond prices converges to $q$, there exists $\xi_0$ large enough such that

$$\mathrm{PV}_T(2\overline{y}|s^0;q_\xi) \leqslant \mathrm{PV}_T(2\overline{y}|s^0;q) + 1, \quad \text{for all } \xi \geqslant \xi_0.$$

Now, assume by way of contradiction that $(d_\xi(s^0))_{\xi\geqslant 1}$ is unbounded. There must exist $\xi$, say $\xi \geqslant \xi_0$, such that $d_\xi(s^0) \geqslant (\#I)(\mathrm{PV}_T(2\overline{y}|s^0;q)+1)$. In particular, there must exist an

[22]Recall that $c^i_{\varphi(\xi)}(\sigma(s^t))$ is bounded from above by aggregate endowments.

agent $i$ such that $\theta^i_\xi(s^0) \geqslant \mathrm{PV}_T(2\overline{y}|s^0; q) + 1$. With this initial bond holding, agent $i$ can finance the consumption $2\overline{y}(s^t)$ for any $t \leqslant T$. By optimality, we deduce that

$$U(c^i_\xi|s^0) \geqslant U_T(2\overline{y}|s^0) > U(\overline{y}|s^0),$$

which contradicts the feasibility of the allocation $(c^i_\xi)_{i\in I}$.

We have thus proved that the sequence $(d_\xi(s^0))_{\xi\geqslant 1}$ is bounded. This implies that for each event $s^t$, the sequence $(d_\xi(s^t))_{\xi\geqslant 1}$ is bounded. Since the markets for bonds clear, we deduce that for each $i$ and each event $s^t$, the sequence $(\theta^i_\xi(s^t))_{\xi\geqslant 1}$ is bounded. Passing to a subsequence if necessary, we can assume that for each $i$, there exists a process of bond holdings $\theta^i$ such that the sequence $(\theta^i_\xi)_{\xi\geqslant 1}$ converges for the product topology to $\theta^i$. It follows from the flow budget constraints satisfied by $(c^i_\xi, \theta^i_\xi)$ that

$$c^i(s^t) + \sum_{s^{t+1}\succ s^t} q(s^{t+1})\theta^i(s^{t+1}) \leqslant (1 - \tau^i(s^t))y^i(s^t) + \theta^i(s^t).$$

In other words, the plan $(c^i, \theta^i)$ is a budget feasible plan in $\widehat{B}^i(\theta^i(s^0)|s^0)$. We omit the standard arguments to show that the plan $(c^i, a^i)$ is optimal among budget feasible plans in $\widehat{B}^i(\theta^i(s^0)|s^0)$.

Fix an event $s^t$. Since the government debt market clears for every $\xi$-truncated equilibrium, the sequence $(d_\xi(s^t))_{\xi\geqslant 1}$ of government debts converges to some $d(s^t)$ satisfying

$$d(s^t) = \sum_{i\in I} \theta^i(s^t).$$

We still have to prove that the government budget restriction is satisfied. Recall that for any $\xi$-truncated economy (with $\xi > t$), the government budget constraint is satisfied

$$d_\xi(s^t) = \delta(s^t) + \sum_{s^{t+1}\succ s^t} q_\xi(s^{t+1})d_\xi(s^{t+1}).$$

Passing to the limit, we get that

$$d(s^t) = \delta(s^t) + \sum_{s^{t+1}\succ s^t} q(s^{t+1})d(s^{t+1}).$$

We have thus proved that $(q, d, (c^i, \theta^i)_{i\in I})$ is a competitive equilibrium with public debt backed by the tax schedule $(\tau^i)_{i\in I}$.

Fix now an agent $i$ and an arbitrary $T > 0$. Recall that for any $\xi \geqslant T$, we have

$$\theta^i_\xi(s^0) = \mathrm{PV}_\xi(\delta^i|s^0; q_\xi) \geqslant \mathrm{PV}_T(\delta^i|s^0; q_\xi).$$

Passing to the limit when $\xi$ approaches infinity, we get that[23]

$$\theta^i(s^0) \geqslant \mathrm{PV}_T(\delta^i|s^0; q) = \mathrm{PV}_T(\delta^i|s^0).$$

Since the sequence $(\mathrm{PV}_T(\delta^i|s^0))_{T>0}$ is increasing, we deduce that it converges and satisfies

$$\theta^i(s^0) \geqslant \lim_{T\to\infty} \mathrm{PV}_T(\delta^i|s^0) = \mathrm{PV}(\delta^i|s^t).$$

## A.4 Proof of Proposition 4.3

To simplify the presentation, we let $e^i(s) := y^i(s) - \ell^i(s)$ denote agent $i$'s endowment after default. It follows from Theorem 5 in Gottardi and Kubler (2015) that there exists a family $(V(\cdot, s), L(\cdot, s))_{s\in S}$ of functions

$$V(\cdot, s) : \mathbb{R}^I_{++} \longrightarrow \mathbb{R}^I \quad \text{and} \quad L(\cdot, s) : \mathbb{R}^I_{++} \longrightarrow \mathbb{R}^I_{++}$$

satisfying the following properties: for every $\lambda \in \mathbb{R}^I_{++}$, for every $s \in S$, and for every $\alpha > 0$,

(a) $L(\lambda, s) \geqslant \lambda$ and $L(\alpha\lambda, s) = \alpha L(\lambda, s)$;

(b) $V(L(\lambda, s), s) \geqslant 0$ and $V(\alpha\lambda, s) = V(\lambda, s)$;

(c) for each $i$,

$$V^i(L(\lambda, s), s) > 0 \Longrightarrow L^i(\lambda, s) = \lambda^i;$$

(d) for each $i$,

$$V^i(\lambda, s) = u'(\tilde{c}^i(\lambda, s))[\tilde{c}^i(\lambda, s) - e^i(s)] + \beta \sum_{s'\in S} \pi(s'|s) V^i(L(\lambda, s'), s');$$

where we recall that $\tilde{c}(\lambda, s) = (\tilde{c}^i(\lambda, s))_{i\in I}$ is the unique solution of the following maximization problem

$$\max\left\{\sum_{i\in I} \lambda^i u(x^i) \ : \ (x^i)_{i\in I} \in \mathbb{R}^I_{++} \quad \text{and} \quad \sum_{i\in I} x^i = \bar{y}(s)\right\}.$$

[23]When the present value is computed with the limit price process $q$, we omit to specify the price in the definition.

We let $\tilde{\theta}^i(\cdot, s) : \mathbb{R}^I_{++} \to \mathbb{R}$ be defined by

$$\tilde{\theta}^i(\lambda, s) := \frac{V^i(\lambda, s)}{u'(\tilde{c}^i(\lambda, s))}, \quad \text{for all } \lambda \in \mathbb{R}^I_{++}$$

and for each current state $s$ and contingent to each possible future state $s'$, we let $\tilde{q}(\cdot, s)(s')$ be defined by

$$\tilde{q}(\lambda, s)(s') := \beta\pi(s'|s) \max_{i \in I} \frac{u'(\tilde{c}^i(L(\lambda, s'), s'))}{u'(\tilde{c}^i(\lambda, s))}, \quad \text{for all } \lambda \in \mathbb{R}^I_{++}.$$

Combining conditions (a), (c), and (d), we get that for each $i$,

$$\tilde{c}^i(\lambda, s) + \sum_{s' \in S} q(\lambda, s)(s')\tilde{\theta}^i(L(\lambda, s'), s') = e^i(s) + \tilde{\theta}^i(\lambda, s).$$

Condition (b) implies that

$$\tilde{\theta}(L(\lambda, s'), s') \geqslant 0, \quad \text{for all } s'.$$

If $\lambda \in \Lambda(s)$, we also have that $\tilde{\theta}(\lambda, s) \geqslant 0$.

Fix an arbitrary $\lambda_0 \in \Lambda(s_0)$. Define the process $(\lambda(s^t))_{s^t \succeq s^0}$ according to (4.25), the process $(q(s^t))_{s^t \succ s^0}$ according to (4.24), the process $(c^i(s^t))_{s^t \succeq s^0}$ according to (4.23) and let $\theta^i(s^t) := \tilde{\theta}^i(\lambda(s^t), s_t)$. We have proved that $(c^i, \theta^i) \in \widehat{B}^i(\theta^i(s^0)|s^0)$. Moreover, following the arguments in Gottardi and Kubler (2015), the plan $(c^i, \theta^i)$ is actually optimal in $\widehat{B}^i(\theta^i(s^0)|s^0)$.

Fix an arbitrary date $T$. Consolidating the budget restrictions from date 0 to date $T$ and summing over the agents, we get

$$\sum_{t=0}^{T} \sum_{s^t \in S^t} p(s^t)\delta(s^t) + \sum_{s^{T+1} \in S^{T+1}} p(s^{T+1}) \sum_{i \in I} \theta^i(s^{T+1}) = \sum_{i \in I} \theta^i(s^0)$$

where $\delta := \sum_{i \in I} \ell^i$. The above inequality implies that $\mathrm{PV}(\delta|s^0)$ is finite. Since endowment losses are nonnegligible, we deduce that $\mathrm{PV}(\bar{y}|s^0)$ is finite. In particular, for each $i$ the consumption plan $c^i$ has finite present value. We can then apply Lemma B.1 in Martins-da-Rocha and Vailakis (2017a) to deduce the following market transversality condition

$$\lim_{t \to \infty} \sum_{s^t \in S^t} p(s^t)\theta^i(s^t) = 0.$$

The above condition implies that

$$\sum_{i \in I} \theta^i(s^0) = \mathrm{PV}(\delta|s^0).$$

This is exactly the budget restriction of the government at the initial event $s^0$. Reproducing the above argument at any event $s^t$, we can show that

$$\sum_{i \in I} \theta^i(s^t) = \mathrm{PV}(\delta | s^t).$$

We have thus proved that $(q, d, (c^i, \theta^i)_{i \in I})$ is a competitive equilibrium with public debt where $d := \mathrm{PV}(\delta)$.

Observe that for any arbitrary $s_0 \in S$ and any $\lambda_0 \in \Lambda(s_0)$, we have constructed a competitive equilibrium with public debt

$$(q_{\lambda_0,s_0}, d_{\lambda_0,s_0}, (c^i_{\lambda_0,s_0}, \theta^i_{\lambda_0,s_0})_{i \in I})$$

such that $\mathrm{PV}(\ell^i | s_0; q_{\lambda_0,s_0}) < \infty$. For any $s \in S$ and any $\lambda \in \Lambda(s)$, we pose

$$\tilde{D}^i(\lambda, s) := \mathrm{PV}(\ell^i | s; q_{\lambda,s}).$$

By construction, we have

$$\tilde{D}^i(\lambda, s) = \ell^i(s) + \sum_{s' \in S} \tilde{q}(\lambda, s)(s') \tilde{D}^i(L(\lambda, s'), s').$$

Since $L(\lambda, s') \in \Lambda(s')$ for any instantaneous Negishi weights $\lambda \in \mathbb{R}^I_{++}$, we can use the above recursive formula to extend the definition of $\tilde{D}^i(\cdot, s)$ to the whole set $\mathbb{R}^I_{++}$.

# B Online Supplemental Material

Please visit https://www.dropbox.com/s/7rwl9m0yutfpo28/BR_GE_supplement.pdf?dl=1