Aggregate Implications of Changing Sectoral Trends

Technical Appendix and Supplementary Material

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Contents

1 Statistical Model of Trend Growth **2**
1.1 Long-Run Correlations and Cross-Correlations of Labor and TFP Growth Within and Across Sectors . . . 2
1.2 Dynamic Factor Model . . . 7
1.3 Estimation . . . 7
1.3.1 The Kalman Smoother Conditional on Known Factor Loadings and Variances . . . 8
1.3.2 Factor Loadings and Variances Conditional on the States . . . 9
1.4 Summary of DFM Posterior . . . 10

2 A Structural Model with Sectoral Linkages in Materials and Investment **21**
2.1 Economic Environment . . . 21
2.2 The Planner's Problem . . . 22
2.3 The Full Set of Equilibrium Conditions . . . 24
2.4 Special Cases with No Growth . . . 25
2.5 Balanced Growth . . . 26

2.5.1 Making the Model Stationary . . . 26
2.5.2 Steady State of the Stationary Environment . . . 30
2.5.3 Steady State Sectoral Value Added Shares in GDP . . . 33
2.5.4 Details of Steady State Calculations . . . 34
2.6 Balanced Growth at the Sectoral and Aggregate Level . . . 38
2.7 Dynamics of the System . . . 39
2.7.1 Linearized Equations: . . . 39
2.7.2 System Reduction: . . . 43
2.8 Solution and Policy Functions in KW (2002) . . . 46
2.9 Time Series Implications for Sectoral and Aggregate Value Added . . . 47
2.10 Solving the Model When Households Have Imperfect Information . . . 48
2.10.1 Computation of the Steady State Kalman Gain . . . 50
2.10.2 Expression for the Driving Processes as an Input to King and Watson (2002) . . . 52

3 Data 52
3.1 Assembling the Sectoral Data . . . 53
3.2 Aggregating KLEMS into Consolidated Sectors . . . 54
3.3 Aggregating the Consolidated Sectors . . . 55

4 Bibliography 57

1 Statistical Model of Trend Growth

1.1 Long-Run Correlations and Cross-Correlations of Labor and TFP Growth Within and Across Sectors

Our interest is on long-run variation and covariation of sectoral growth rates. With this in mind, Tables A2-A4 present estimates and confidence intervals for long-run correlations between the sectoral growth rates of labor and TFP using methods developed in Müller and Watson (2018). These results rely on low-frequency transformations of the data that retain variability for periods longer that 10 years, and discard higher frequency variability. (They use the first $q = 13$ cosine transforms of the data.) The confidence sets are valid for persistence patterns that include $I(0)$, $I(1)$, mixtures of $I(0)$ and $I(1)$ (such as the MUC model), local-to-unity autoregressions, and fractional integration with values of d that satisfy $-0.4 \leq d \leq 1.0$. The point estimates are posterior medians from a bivariate $I(d)$ model where the values of d are uniformly distributed on $[-0.4, 1.0]$. See Müller and Watson (2018) for details.

The sector numbers used in Tables A2-A4 correspond to the following sectors:

Table A1: Sector Names and Numbers

Sector Number	Sector Name
1	Agriculture, Forestry, Fishing, and Hunting
2	Mining
3	Utilities
4	Construction
5	Durable Goods
6	Nondurable Goods
7	Wholesale Trade
8	Retail Trade
9	Transportation and Warehousing
10	Information
11	Finance, Insurance, and Real Estate (x Housing)
12	Professional and Business Services
13	Education, Health Care, and Social Assistance
14	Entertainment and Food Services
15	Other Services (except Government)
16	Housing

Table A2: Long-run Correlations for Sectoral Employment
Posterior Medians and 68% Confidence Sets

(a) Correlations of sectors 1-8 with sectors 1-8

	1	2	3	4	5	6	7	8
1	1.0	0.06 [-0.16,0.31]	-0.30 [-0.65,-0.00]*	0.16 [-0.30,0.45]	-0.50 [-0.80,-0.32]*	-0.38 [-0.75,-0.04]*	-0.03 [-0.65,0.30]	0.20 [-0.30,0.46]
2		1.0	0.09 [-0.15,0.41]	-0.05 [-0.36,0.21]	0.27 [-0.00,0.53]	-0.03 [-0.30,0.21]	0.27 [0.00,0.52]*	-0.16 [-0.45,0.08]
3			1.0	-0.06 [-0.39,0.18]	0.33 [0.03,0.70]*	0.46 [0.21,0.80]*	0.18 [-0.07,0.65]	0.01 [-0.23,0.45]
4				1.0	-0.09 [-0.45,0.35]	-0.43 [-0.65,0.10]	0.34 [0.03,0.65]*	0.42 [0.16,0.64]*
5					1.0	0.52 [0.28,0.75]*	0.35 [0.03,0.65]*	-0.10 [-0.47,0.30]
6						1.0	0.10 [-0.19,0.55]	0.01 [-0.26,0.35]
7							1.0	0.36 [0.04,0.65]*
8								1.0

(b) Correlations of sectors 1-8 with sectors 9-16

	9	10	11	12	13	14	15	16
1	0.30 [0.01,0.65]*	-0.1 [-0.50,0.07]	-0.1 [-0.65,0.13]	0.11 [-0.20,0.41]	0.10 [-0.13,0.35]	0.32 [-0.05,0.53]	0.00 [-0.55,0.32]	-0.02 [-0.21,0.21]
2	-0.0 [-0.30,0.22]	0.0 [-0.25,0.27]	-0.0 [-0.36,0.21]	-0.13 [-0.41,0.08]	0.01 [-0.21,0.30]	-0.10 [-0.41,0.12]	-0.01 [-0.27,0.22]	-0.20 [-0.45,0.02]
3	-0.1 [-0.55,0.10]	0.1 [-0.08,0.44]	0.2 [-0.01,0.70]	-0.00 [-0.28,0.25]	0.02 [-0.21,0.30]	0.21 [-0.04,0.45]	0.21 [-0.07,0.65]	0.06 [-0.13,0.34]
4	-0.2 [-0.51,0.02]	0.2 [-0.01,0.47]	0.65 [0.43,0.79]*	0.40 [0.10,0.64]*	-0.05 [-0.36,0.18]	0.27 [-0.00,0.52]	0.38 [0.07,0.70]*	0.00 [-0.23,0.25]
5	-0.1 [-0.60,0.10]	0.33 [0.02,0.54]*	0.15 [-0.08,0.60]	-0.10 [-0.42,0.12]	-0.21 [-0.46,0.04]	-0.41 [-0.60,-0.03]*	0.03 [-0.25,0.55]	0.10 [-0.08,0.41]
6	0.2 [-0.35,0.50]	0.1 [-0.12,0.41]	-0.16 [-0.49,0.40]	-0.05 [-0.37,0.18]	0.27 [-0.00,0.52]	0.03 [-0.21,0.32]	0.05 [-0.21,0.55]	0.26 [-0.00,0.50]
7	-0.15 [-0.55,0.15]	0.50 [0.27,0.71]*	0.65 [0.38,0.85]*	0.45 [0.20,0.65]*	0.05 [-0.18,0.36]	0.30 [0.01,0.52]*	0.79 [0.60,0.93]*	-0.16 [-0.44,0.12]
8	0.2 [-0.00,0.53]	0.36 [0.05,0.57]*	0.49 [0.20,0.68]*	0.80 [0.68,0.89]*	0.27 [-0.00,0.50]	0.70 [0.49,0.81]*	0.59 [0.38,0.76]*	-0.01 [-0.31,0.21]

(c) Correlations of sectors 9-16 with sectors 9-16

	9	10	11	12	13	14	15	16
9	1.0	0.05 [-0.20,0.36]	-0.32 [-0.55,-0.03]*	0.22 [-0.03,0.50]	0.21 [-0.03,0.45]	0.10 [-0.15,0.40]	-0.00 [-0.35,0.27]	0.16 [-0.05,0.42]
10		1.0	0.41 [0.13,0.58]*	0.42 [0.16,0.64]*	-0.16 [-0.42,0.05]	0.04 [-0.20,0.33]	0.43 [0.18,0.60]*	-0.03 [-0.33,0.18]
11			1.0	0.44 [0.18,0.64]*	-0.16 [-0.46,0.09]	0.32 [0.03,0.53]*	0.64 [0.38,0.85]*	-0.16 [-0.44,0.06]
12				1.0	0.21 [-0.01,0.46]	0.71 [0.53,0.84]*	0.71 [0.55,0.81]*	-0.03 [-0.35,0.20]
13					1.0	0.53 [0.30,0.71]*	0.19 [-0.05,0.46]	0.08 [-0.10,0.38]
14						1.0	0.51 [0.32,0.71]*	0.01 [-0.16,0.33]
15							1.0	-0.21 [-0.45,-0.00]*
16								1.0

Notes: The point estimates are posterior medians from a bivariate $I(d)$ model with $-0.4 \leq d \leq 1$. The bracketed quantities are 68% frequentist confidence intervals. Asterisks highlight confidence intervals that do not include zero. 32% of the pairwise correlations are statistically different from zero at 33% level.The median pairwise point-estimate is 0.10.

Table A3: Long-run Correlations for Sectoral TFP
Posterior Medians and 68% Confidence Sets

(a) Correlations of sectors 1-8 with sectors 1-8

	1	2	3	4	5	6	7	8
1	1.0	-0.10 [-0.39,0.13]	-0.00 [-0.26 ,0.21]	0.23 [0.01,0.45]*	-0.02 [-0.29,0.20]	0.41 [0.13,0.59]*	0.01 [-0.16,0.33]	0.44 [0.21,0.64]*
2		1.0	0.79 [0.62,0.88]*	0.27 [0.00,0.51]*	0.21 [-0.03,0.49]	-0.01 [-0.30,0.26]	0.05 [-0.18,0.35]	0.08 [-0.16,0.36]
3			1.0	0.44 [0.15,0.75]*	0.20 [-0.30,0.49]	0.16 [-0.10,0.55]	-0.00 [-0.28,0.25]	0.15 [-0.08,0.45]
4				1.0	-0.30 [-0.70,0.00]	0.12 [-0.16,0.55]	-0.20 [-0.46,0.05]	0.38 [0.07,0.54]*
5					1.0	0.07 [-0.18,0.39]	0.20 [-0.05,0.44]	0.05 [-0.20,0.36]
6						1.0	0.15 [-0.08,0.42]	0.12 [-0.10,0.41]
7							1.0	0.23 [-0.00,0.50]
8								1.0

(b) Correlations of sectors 1-8 with sectors 9-16

	9	10	11	12	13	14	15	16
1	-0.01 [-0.30,0.16]	-0.16 [-0.43,0.08]	-0.20 [-0.43,0.04]	0.01 [-0.21,0.28]	0.05 [-0.15,0.31]	-0.08 [-0.36,0.08]	0.04 [-0.16,0.31]	0.01 [-0.21,0.28]
2	0.05 [-0.19,0.35]	-0.38 [-0.57,-0.05]*	-0.32 [-0.53,-0.00]*	0.05 [-0.21,0.36]	0.04 [-0.30,0.38]	-0.20 [-0.44,0.04]	0.34 [0.03,0.54]*	0.21 [-0.03,0.49]
3	0.23 [-0.02,0.46]	-0.41 [-0.56,-0.10]*	-0.16 [-0.43,0.05]	0.44 [0.16,0.70]*	0.30 [0.00,0.65]*	-0.13 [-0.41,0.10]	0.23 [-0.03,0.49]	0.27 [-0.03,0.65]
4	0.11 [-0.14,0.36]	-0.23 [-0.45,-0.00]*	-0.18 [-0.43,0.03]	0.41 [0.11,0.70]*	-0.00 [-0.33,0.50]	-0.32 [-0.55,-0.05]*	0.35 [0.03,0.53]*	0.54 [0.34,0.85]*
5	0.08 [-0.15,0.39]	0.03 [-0.20,0.30]	0.06 [-0.15,0.35]	0.03 [-0.45,0.37]	0.34 [-0.30,0.59]	0.44 [0.16,0.60]*	-0.18 [-0.46,0.08]	-0.48 [-0.80,-0.21]*
6	0.34 [0.04,0.55]*	-0.00 [-0.28,0.25]	-0.45 [-0.65,-0.20]*	0.15 [-0.09,0.55]	0.21 [-0.05,0.47]	-0.20 [-0.46,0.05]	0.03 [-0.21,0.33]	-0.07 [-0.39,0.30]
7	-0.00 [-0.29,0.21]	-0.30 [-0.51,-0.00]*	-0.41 [-0.63,-0.08]*	-0.48 [-0.68,-0.21]*	0.16 [-0.13,0.46]	0.01 [-0.21,0.30]	0.36 [0.04,0.56]*	-0.10 [-0.36,0.15]
8	0.00 [-0.23,0.27]	-0.41 [-0.65,-0.14]*	-0.16 [-0.42,0.05]	-0.05 [-0.34,0.17]	0.05 [-0.18,0.36]	-0.01 [-0.30,0.21]	0.30 [0.00,0.52]*	0.21 [-0.04,0.46]

(c) Correlations of sectors 9-16 with sectors 9-16

	9	10	11	12	13	14	15	16
9	1.0	0.05 [-0.15,0.36]	-0.33 [-0.55,-0.01]*	0.42 [0.16,0.63]*	0.06 [-0.16,0.34]	0.23 [-0.01,0.47]	0.01 [-0.23,0.30]	-0.03 [-0.33,0.20]
10		1.0	0.34 [0.02,0.63]*	0.01 [-0.21,0.30]	-0.09 [-0.37,0.13]	0.08 [-0.10,0.38]	-0.36 [-0.60,-0.08]*	-0.33 [-0.53,-0.04]*
11			1.0	0.08 [-0.15,0.42]	-0.03 [-0.30,0.21]	0.41 [0.11,0.64]*	-0.53 [-0.72,-0.31]*	-0.26 [-0.50,0.01]
12				1.0	0.32 [0.01,0.65]*	0.23 [-0.04,0.51]	-0.21 [-0.50,0.05]	0.01 [-0.23,0.50]
13					1.0	0.05 [-0.14,0.33]	-0.26 [-0.50,0.03]	-0.03 [-0.38,0.30]
14						1.0	-0.42 [-0.62,-0.13]*	-0.55 [-0.78,-0.33]*
15							1.0	0.53 [-0.00,0.71]
16								1.0

Notes: See notes to Table A2. 32% of the pairwise correlations are statistically different from zero at 33% level.The median pairwise point-estimate is 0.03.

Table A4: Long-run Correlations for Sectoral Employment (rows) and TFP (columns)
Posterior Medians and 68% Confidence Sets

(a) Correlations of Employment sectors 1-8 with TFP sectors 1-8

	1	2	3	4	5	6	7	8
1	-0.21 [-0.43,0.03]	-0.27 [-0.49,0.01]	-0.53 [-0.75,-0.30]*	-0.48 [-0.80,-0.21]*	-0.00 [-0.30,0.45]	-0.42 [-0.75,-0.16]*	0.27 [-0.20,0.46]	-0.13 [-0.42,0.10]
2	-0.56 [-0.72,-0.34]*	-0.29 [-0.50,-0.00]*	-0.29 [-0.50,0.00]	-0.41 [-0.54,-0.09]*	-0.00 [-0.29,0.25]	-0.34 [-0.56,-0.02]*	-0.44 [-0.65,-0.16]*	-0.60 [-0.74,-0.37]*
3	0.06 [-0.14,0.33]	-0.18 [-0.50,0.08]	-0.01 [-0.32,0.27]	0.10 [-0.19,0.55]	-0.60 [-0.80,-0.38]*	0.08 [-0.18,0.55]	-0.16 [-0.44,0.06]	-0.34 [-0.54,-0.03]*
4	0.14 [-0.08,0.42]	-0.44 [-0.64,-0.16]*	-0.34 [-0.57,0.10]	-0.34 [-0.56,-0.03]*	0.16 [-0.10,0.45]	0.50 [0.21,0.70]*	0.53 [0.29,0.71]*	0.08 [-0.13,0.38]
5	-0.01 [-0.30,0.21]	-0.03 [-0.34,0.21]	0.19 [-0.06,0.65]	0.32 [0.01,0.70]*	-0.25 [-0.60,0.00]	-0.00 [-0.27,0.50]	-0.33 [-0.55,-0.01]*	0.15 [-0.08,0.42]
6	0.01 [-0.21,0.23]	-0.00 [-0.29,0.28]	0.18 [-0.08,0.55]	0.44 [0.15,0.75]*	-0.50 [-0.80,-0.27]*	-0.00 [-0.30,0.45]	-0.18 [-0.46,0.08]	0.01 [-0.23,0.29]
7	-0.20 [-0.49,0.01]	-0.42 [-0.60,-0.12]*	-0.23 [-0.49,0.35]	-0.11 [-0.45,0.40]	-0.27 [-0.52,0.25]	0.15 [-0.12,0.60]	-0.00 [-0.27,0.25]	-0.05 [-0.37,0.17]
8	-0.00 [-0.29,0.21]	-0.18 [-0.55,0.08]	-0.27 [-0.52,0.01]	-0.11 [-0.55,0.16]	-0.16 [-0.45,0.15]	0.06 [-0.19,0.35]	0.54 [0.33,0.72]*	0.05 [-0.19,0.35]

(b) Correlations of Employment sectors 1-8 with TFP sectors 9-16

	9	10	11	12	13	14	15	16
1	-0.42 [-0.65,-0.14]*	0.25 [0.01,0.46]*	0.16 [-0.03,0.60]	-0.49 [-0.80,-0.27]*	-0.10 [-0.55,0.17]	0.05 [-0.16,0.26]	0.00 [-0.50,0.27]	-0.12 [-0.55,0.17]
2	0.08 [-0.13,0.37]	0.43 [0.20,0.66]*	0.50 [0.23,0.68]*	0.09 [-0.13,0.41]	-0.16 [-0.46,0.20]	0.38 [0.08,0.63]*	-0.54 [-0.72,-0.31]*	-0.20 [-0.45,0.05]
3	0.13 [-0.10,0.41]	-0.10 [-0.39,0.12]	-0.21 [-0.46,0.01]	0.15 [-0.10,0.55]	-0.08 [-0.45,0.30]	-0.20 [-0.44,0.05]	0.03 [-0.21,0.33]	0.30 [0.01,0.55]*
4	0.04 [-0.18,0.33]	0.00 [-0.21,0.28]	-0.06 [-0.37,0.14]	-0.20 [-0.50,0.08]	0.25 [-0.02,0.50]	0.18 [-0.05,0.42]	-0.13 [-0.42,0.15]	-0.34 [-0.54,-0.03]*
5	0.53 [0.33,0.72]*	-0.13 [-0.41,0.08]	-0.01 [-0.45,0.21]	0.48 [0.21,0.75]*	0.00 [-0.27,0.50]	0.19 [-0.05,0.48]	-0.00 [-0.27,0.28]	0.30 [0.00,0.65]*
6	0.40 [0.10,0.54]*	0.00 [-0.23,0.23]	-0.19 [-0.43,0.05]	0.27 [-0.01,0.65]	-0.21 [-0.50,0.30]	-0.13 [-0.41,0.10]	0.35 [0.03,0.54]*	0.34 [0.03,0.65]*
7	-0.00 [-0.23,0.40]	-0.01 [-0.23,0.20]	0.05 [-0.20,0.35]	0.03 [-0.23,0.55]	-0.03 [-0.34,0.50]	0.01 [-0.21,0.26]	-0.05 [-0.38,0.20]	-0.03 [-0.50,0.21]
8	-0.00 [-0.32,0.22]	-0.14 [-0.41,0.07]	-0.42 [-0.60,-0.12]*	-0.51 [-0.71,-0.21]*	-0.02 [-0.35,0.25]	-0.03 [-0.34,0.21]	0.27 [-0.01,0.52]	-0.13 [-0.55,0.16]

(c) Correlations of Employment sectors 9-16 with TFP sectors 1-8

	1	2	3	4	5	6	7	8
9	-0.34 [-0.54,-0.04]*	0.18 [-0.08,0.46]	-0.03 [-0.50,0.23]	-0.21 [-0.65,0.03]	0.03 [-0.21,0.50]	-0.44 [-0.64,-0.18]*	0.30 [0.00,0.52]*	0.04 [-0.21,0.34]
10	0.01 [-0.16,0.30]	-0.26 [-0.50,0.01]	-0.03 [-0.33,0.21]	-0.03 [-0.32,0.22]	-0.27 [-0.52,0.01]	-0.00 [-0.28,0.28]	0.13 [-0.08,0.41]	0.27 [-0.00,0.50]
11	0.13 [-0.10,0.42]	-0.30 [-0.60,-0.01]*	-0.27 [-0.50,0.30]	-0.10 [-0.43,0.45]	-0.10 [-0.50,0.18]	0.49 [0.21,0.75]*	0.23 [-0.03,0.50]	0.05 [-0.17,0.36]
12	0.00 [-0.25,0.21]	-0.18 [-0.42,0.05]	-0.28 [-0.51,0.00]	-0.10 [-0.36,0.13]	-0.07 [-0.40,0.16]	0.06 [-0.15,0.35]	0.44 [0.20,0.65]*	0.13 [-0.08,0.41]
13	-0.02 [-0.31,0.20]	-0.07 [-0.37,0.15]	-0.25 [-0.49,0.01]	-0.03 [-0.45,0.20]	-0.16 [-0.46,0.10]	-0.00 [-0.27,0.26]	-0.01 [-0.33,0.21]	-0.23 [-0.48,0.01]
14	0.00 [-0.21,0.30]	-0.28 [-0.50,0.00]	-0.42 [-0.60,-0.12]*	-0.05 [-0.37,0.17]	-0.38 [-0.57,-0.05]*	0.07 [-0.16,0.38]	0.32 [0.00,0.53]*	-0.10 [-0.41,0.08]
15	0.00 [-0.21,0.20]	-0.41 [-0.70,-0.10]*	-0.30 [-0.50,0.20]	-0.01 [-0.45,0.27]	-0.27 [-0.49,0.15]	0.21 [-0.03,0.55]	0.10 [-0.13,0.37]	0.01 [-0.35,0.27]
16	0.08 [-0.08,0.38]	0.01 [-0.21,0.28]	0.03 [-0.20,0.30]	0.13 [-0.12,0.33]	-0.05 [-0.33,0.13]	-0.00 [-0.25,0.22]	0.30 [-0.00,0.51]	0.21 [-0.02,0.45]

(d) Correlations of Employment sectors 9-16 with TFP sectors 9-16

	9	10	11	12	13	14	15	16
9	-0.05 [-0.34,0.16]	-0.03 [-0.31,0.19]	-0.05 [-0.37,0.18]	-0.50 [-0.75,-0.21]*	-0.27 [-0.55,0.01]	0.01 [-0.21,0.30]	0.49 [-0.00,0.70]	0.03 [-0.45,0.38]
10	-0.03 [-0.35,0.20]	-0.39 [-0.61,-0.10]*	0.00 [-0.23,0.27]	-0.20 [-0.49,0.25]	0.08 [-0.16,0.36]	0.02 [-0.16,0.32]	-0.00 [-0.28,0.28]	0.05 [-0.21,0.35]
11	0.04 [-0.20,0.45]	-0.14 [-0.43,0.07]	-0.33 [-0.53,-0.03]*	-0.08 [-0.42,0.40]	0.27 [0.00,0.55]*	-0.05 [-0.34,0.20]	-0.08 [-0.44,0.18]	-0.03 [-0.37,0.25]
12	-0.30 [-0.53,0.00]	-0.29 [-0.51,0.00]	-0.30 [-0.52,-0.00]*	-0.53 [-0.71,-0.30]*	-0.13 [-0.41,0.10]	-0.10 [-0.45,0.10]	0.32 [0.01,0.53]*	-0.02 [-0.45,0.21]
13	0.01 [-0.21,0.30]	0.37 [0.03,0.58]*	-0.01 [-0.32,0.19]	-0.14 [-0.45,0.08]	-0.57 [-0.76,-0.41]*	-0.08 [-0.38,0.10]	0.21 [-0.04,0.45]	-0.23 [-0.50,0.03]
14	-0.27 [-0.51,0.01]	0.00 [-0.21,0.27]	-0.33 [-0.54,-0.01]*	-0.52 [-0.71,-0.05]*	-0.27 [-0.50,0.01]	-0.42 [-0.60,-0.05]*	0.36 [0.03,0.56]*	0.00 [-0.45,0.27]
15	-0.16 [-0.40,0.25]	-0.05 [-0.30,0.13]	-0.12 [-0.38,0.11]	-0.13 [-0.42,0.40]	-0.01 [-0.33,0.45]	-0.10 [-0.40,0.30]	0.03 [-0.20,0.50]	-0.04 [-0.55,0.19]
16	0.16 [-0.04,0.42]	-0.08 [-0.38,0.08]	-0.13 [-0.41,0.07]	-0.04 [-0.31,0.16]	-0.23 [-0.46,-0.00]*	0.01 [-0.20,0.30]	0.58 [0.33,0.77]*	0.20 [-0.04,0.45]

Notes: See notes to Table A2. 25% of the pairwise correlations are statistically different from zero at 33% level. The median pairwise point-estimate is -0.03. The median diagonal correlation is -0.16 and 7/16 (44%) of the diagonal entries are statistically different from zero at the 33% level.

1.2 Dynamic Factor Model

Let x denote labor, ℓ or TFP, z. The empirical model is given by

$$\Delta \ln(x_{j,t}) = \lambda_{j,\tau}\tau_{c,t} + \lambda_{j,\varepsilon}\varepsilon_{c,t} + \tau_{j,t} + \varepsilon_{j,t},$$

$$\tau_{c,t} = \tau_{c,t-1} + \sigma_{\Delta\tau,c} \times \eta_{\tau,c,t},$$

$$\varepsilon_{c,t} = \sigma_{\varepsilon,c} \times \eta_{\varepsilon,c,t},$$

$$\tau_{j,t} = \tau_{j,t-1} + \sigma_{\Delta\tau,j} \times \eta_{\tau,j,t},$$

$$\varepsilon_{j,t} = \sigma_{\varepsilon,j} \times \eta_{\varepsilon,j,t},$$

where $(\eta_{\tau,c,t}, \eta_{\varepsilon,c,t}, \{\eta_{\tau,j,t}, \eta_{\varepsilon,j,t}\}_{j=1}^{n}) \sim^{iid} N(0, I_{2(n+1)})$.

1.3 Estimation

The full system of equations may be recast in state-space form as

$$\underbrace{\Delta \ln(x_t)}_{\substack{y_t \\ (n\times 1)}} = \underbrace{\begin{bmatrix} \lambda_\varepsilon & \lambda_\tau & I \end{bmatrix}}_{\substack{H' \\ n\times(n+2)}} \underbrace{\begin{bmatrix} \varepsilon_{c,t} \\ \tau_{c,t} \\ \tau_t \end{bmatrix}}_{\substack{s_t \\ (n+2)\times 1}} + \underbrace{\sigma_\varepsilon \odot \eta_{\varepsilon,t}}_{\substack{w_t \\ n\times 1}}, \tag{1}$$

$$\underbrace{\begin{bmatrix} \varepsilon_{c,t} \\ \tau_{c,t} \\ \tau_t \end{bmatrix}}_{\substack{s_t \\ (n+2)\times 1}} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_n \end{bmatrix}}_{\substack{F \\ (n+2)\times(n+2)}} \underbrace{\begin{bmatrix} \varepsilon_{c,t-1} \\ \tau_{c,t-1} \\ \tau_{t-1} \end{bmatrix}}_{\substack{s_{t-1} \\ (n+2)\times 1}} + \underbrace{\begin{bmatrix} \sigma_{\varepsilon,c} \times \eta_{\varepsilon,c,t} \\ \sigma_{\Delta\tau,c} \times \eta_{\Delta\tau,c,t} \\ \sigma_{\Delta\tau} \odot \eta_{\Delta\tau,t} \end{bmatrix}}_{\substack{v_t \\ (n+2)\times 1}}. \tag{2}$$

$$w_t \sim N\left(0, \underbrace{diag(\sigma_\varepsilon^2)}_{R}\right)$$

$$v_t \sim N\left(0, \underbrace{diag(\sigma_{\varepsilon,c}^2, \sigma_{\Delta\tau,c}^2, \sigma_{\Delta\tau}^2)}_{Q}\right)$$

Bayes' rule implies that

$$\Pr(\Theta|y_{1:T}) = \Pr(y_{1:T}|\Theta)\Pr(\Theta)\,/\Pr(y_{1:T}),$$

where $\Pr(\Theta)$ is the prior distribution, $\Pr(y_{1:T}|\Theta)$ is the likelihood, $\Pr(y_{1:T})$ is the marginal likelihood, and $\Pr(\Theta|y_{1:T})$ is the posterior distribution we are interested in estimating. The data are denoted by $y_{1:T} = (y_1, ..., y_T)$ and $\Theta = [\zeta_t, \theta]$ with

$$\zeta_t = \left\{\Delta\tau_{c,t}^x, \varepsilon_{c,t}^x, \left\{\Delta\tau_{j,t}^x, \varepsilon_{j,t}^x\right\}_{j=1}^n\right\},\ \zeta_t \sim^{iid} N(0, \Sigma_\zeta),$$

and

$$\theta = \left\{\lambda_{j,\tau}^x, \lambda_{j,\varepsilon}^x, \left(\sigma_{j,\epsilon}^x\right)^2, \left(\sigma_{j,\Delta\tau}^x\right)^2\right\}_{j=1}^n.$$

We estimate the posterior distribution

$$\Pr(\zeta_t, \theta|y_{1:T}),$$

by way of Gibbs sampling in two steps. The first Gibbs step draws $\zeta_t|\theta, y_{1:T}$. The second Gibbs step draws $\theta|\zeta_t, y_{1:T}$.

1.3.1 The Kalman Smoother Conditional on Known Factor Loadings and Variances

Step 1: Draw $\zeta_t|\theta, y_{1:T}$.

With known factor loadings and variances, equations (1) and (2) are interpreted as the observation equation and state equation respectively in a Kalman filtering context,

$$y_t = H' s_t + w_t, \qquad w_t \sim N\left(0, R\right),$$

$$s_t = F s_{t-1} + v_t, \qquad v_t \sim N\left(0, Q\right).$$

The linear-Gaussian structure implies that $s_{1:T}|y_{1:T}$ is normally distributed, and the goal is to obtain draws from this distribution. As shown in Carter and Kohn (1994), this can be achieved as follows:

- Use the Kalman filter to obtain $s_{T|T} = E(s_T|y_{1:T})$ and $P_{T|T} = var(s_T|y_{1:T})$.
- Draw s_T from $N(s_{T|T}, P_{T|T})$.
- Note that the distribution of $s_t|\left(y_{1:T}, s_{t+1}, ..., s_T\right)$ depends only on $\left(y_{1:t}, s_{t+1}\right)$ and $s_t|(y_{1:t}, s_{t+1}) \sim N(\mu_t, \Sigma_t)$ where $\mu_t = s_{t|t} - P_{t|t}F'P^{-1}_{t+1|t}(s_{t+1} - s_{t+1|t})$ and $\Sigma_t = P_{t|t} - P_{t|t}F'P^{-1}_{t+1|t}FP_{t|t}$. Recursively draw $s_{T-1}, s_{T-2}, ..., s_1$.

Priors for s_0 are set by initializing the mean and covariance matrices of the Kalman filter using the normalization $\tau_{c,0} = \varepsilon_{c,0} = 0$ and a diffuse prior for τ_0:

$$\begin{bmatrix} \varepsilon_{c,0} \\ \tau_{c,0} \\ \tau_0 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \kappa I_n \end{bmatrix} \right).$$

where $\kappa = 10^{12}$ approximates the diffuse prior.

1.3.2 Factor Loadings and Variances Conditional on the States

Step 2: Draw $\theta|\zeta_t, y_{1:T}$.

This step is divided into 2 sub-steps.

Step 2a: Draw $\lambda^x_{j,\tau}, \lambda^x_{j,\varepsilon}|\left(\zeta_t, \left(\sigma^x_{j,\varepsilon}\right)^2, \left(\sigma^x_{j,\Delta\tau}\right)^2, y_{1:T}\right)$.
Write

$$(y_t - \tau_t) = \begin{bmatrix} \varepsilon_{c,t} & \tau_{c,t} \end{bmatrix} \begin{bmatrix} \lambda_{\varepsilon,t} \\ \lambda_{\tau,t} \end{bmatrix} + \tilde{\varepsilon}_t$$

$$\begin{bmatrix} \lambda_{\varepsilon,t} \\ \lambda_{\tau,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_{\varepsilon,t-1} \\ \lambda_{\tau,t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which is a linear-Gaussian state-space model with state $s_t = \begin{bmatrix} \lambda_{\varepsilon,t} \\ \lambda_{\tau,t} \end{bmatrix}$. Because s_t is time invariant, we can draw $s_T | \left(\zeta_t, \left(\sigma^x_{j,\varepsilon}\right)^2, \left(\sigma^x_{j,\Delta\tau}\right)^2, y_{1:T}\right) \sim N(s_{T|T}, P_{T|T})$, where $s_{T|T}$ and $P_{T|T}$ are computed from the Kalman filter. We impose the normalization $\sum_j \lambda_{j,\tau} \geq 0$ and $\sum_j \lambda_{j,\varepsilon} \geq 0$, by drawing from the truncated normal (implemented by redrawing from the normal until the constraint is satisfied).

Priors are set by initializing the mean and covariance matrices of the Kalman filter such that

$$\lambda_\tau \sim N\left(0, I_n\right) \text{ and } \lambda_\varepsilon \sim N\left(0, 4^2\right).$$

Step 2b: Draw $\left(\sigma^x_{j,\varepsilon}\right)^2, \left(\sigma^x_{j,\Delta\tau}\right)^2 | \left(\zeta_t, \lambda^x_{j,\tau}, \lambda^x_{j,\varepsilon}, y_{1:T}\right)$. These are inverse Gamma draws with prior

$$\sigma^x_{j,\cdot} \sim IG\left(T_{prior}, \frac{T * \omega^2_{prior}}{2}\right)$$

which implies posterior draws from

$$\sigma^x_{j,\varepsilon} \sim IG\left(T_{prior} + \frac{T}{2}, \frac{1}{2}\sum_{t=1}^{T}\left(y_{j,t} - \tau_{j,t} - \lambda_{j,\tau}\tau_{c,t} - \lambda_{j,\varepsilon}\varepsilon_{c,t}\right)^2 + \frac{T * \omega^2_{prior}}{2}\right),$$

and

$$\sigma^x_{j,\Delta\tau} \sim IG\left(T_{prior} + \frac{T}{2}, \frac{1}{2}\sum_{t=1}^{T}\left(\tau_{j,t} - \tau_{j,t-1}\right)^2 + \frac{T * \omega^2_{prior}}{2}\right).$$

1.4 Summary of DFM Posterior

The model was estimated using draws from two independent Gibbs runs, where each included 505k draws with the first 5k discarded. Table A5 summarizes the posteriors for $(\lambda_{j,\tau}, \lambda_{j,\epsilon}, \sigma_{j,\Delta\tau}, \sigma_{j,\epsilon})$. Figure A1 shows the estimated trends.

Table A5: Posterior summaries for model parameters
Median (68% credible interval) [90% credible interval]

A. Labor

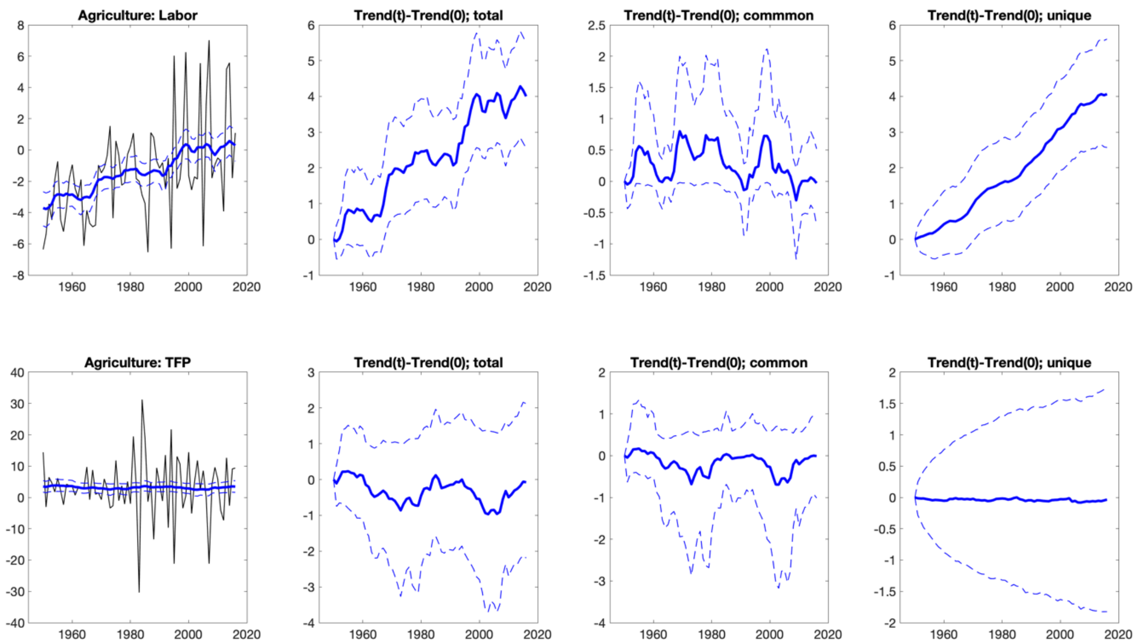
Sector	λ_τ	λ_ϵ	$\sigma_{j,\Delta\tau}$	$\sigma_{j,\epsilon}$
Agriculture, forestry, fishing, and hunting	0.33 (-0.03 0.69) [-0.29 0.94]	0.74 (0.19 1.27) [-0.21 1.67]	0.32 (0.23 0.45) [0.18 0.58]	2.70 (2.48 2.97) [2.34 3.19]
Mining	0.25 (-0.38 0.90) [-0.84 1.38]	-0.16 (-1.24 1.00) [-2.01 1.80]	0.41 (0.21 1.91) [0.15 3.55]	5.94 (5.01 6.61) [3.93 7.09]
Utilities	0.05 (-0.20 0.30) [-0.38 0.48]	0.63 (0.27 0.99) [0.03 1.25]	0.25 (0.18 0.36) [0.14 0.47]	1.82 (1.65 2.00) [1.55 2.14]
Construction	0.79 (0.45 1.07) [-0.60 1.27]	0.37 (-0.04 0.78) [-0.32 1.08]	0.23 (0.16 0.36) [0.13 0.50]	1.99 (1.81 2.19) [1.70 2.35]
Durable goods	-0.55 (-0.89 -0.14) [-1.12 0.71]	0.40 (0.00 0.80) [-0.29 1.06]	0.51 (0.33 0.78) [0.24 0.98]	1.70 (1.48 1.94) [1.33 2.12]
Nondurable goods	-0.36 (-0.57 -0.09) [-0.72 0.53]	0.60 (0.36 0.84) [0.18 1.03]	0.27 (0.20 0.38) [0.16 0.47]	1.11 (0.99 1.24) [0.92 1.35]
Wholesale trade	0.20 (-0.01 0.42) [-0.17 0.57]	0.09 (-0.23 0.40) [-0.46 0.63]	0.19 (0.15 0.26) [0.12 0.34]	1.70 (1.55 1.86) [1.47 1.98]
Retail trade	0.27 (0.06 0.45) [-0.12 0.59]	0.71 (0.47 0.94) [0.30 1.14]	0.18 (0.14 0.25) [0.12 0.31]	1.15 (1.05 1.28) [0.98 1.37]
Transportation and warehousing	-0.21 (-0.43 0.06) [-0.59 0.34]	0.24 (-0.05 0.52) [-0.25 0.72]	0.28 (0.20 0.41) [0.15 0.53]	1.47 (1.34 1.62) [1.25 1.74]
Information	0.68 (0.21 1.10) [-0.27 1.38]	0.47 (0.06 0.88) [-0.22 1.19]	1.12 (0.81 1.42) [0.56 1.62]	1.54 (1.32 1.81) [1.18 2.06]
FIRE (ex Housing)	0.69 (0.43 0.92) [-0.48 1.09]	0.45 (0.14 0.77) [-0.09 0.99]	0.24 (0.17 0.33) [0.14 0.43]	1.43 (1.29 1.58) [1.20 1.70]
Professional and business services	0.67 (0.36 0.96) [-0.32 1.17]	0.84 (0.46 1.21) [0.19 1.47]	0.22 (0.16 0.31) [0.13 0.41]	1.55 (1.39 1.72) [1.29 1.85]
Education, health care, and social assistance	-0.09 (-0.37 0.22) [-0.58 0.45]	0.83 (0.46 1.20) [0.17 1.48]	0.24 (0.17 0.36) [0.14 0.48]	1.85 (1.67 2.04) [1.55 2.20]
Arts and entertainment	0.49 (0.18 0.79) [-0.14 1.00]	1.36 (1.03 1.68) [0.80 1.94]	0.21 (0.15 0.30) [0.13 0.40]	1.37 (1.22 1.55) [1.10 1.68]
Other services (excl. gov.)	0.47 (0.18 0.72) [-0.15 0.90]	0.88 (0.58 1.19) [0.36 1.40]	0.25 (0.18 0.33) [0.15 0.41]	1.36 (1.23 1.51) [1.14 1.63]
Housing	-0.68 (-1.22 -0.02) [-1.60 0.83]	-0.37 (-1.25 0.51) [-1.84 1.15]	0.31 (0.19 0.61) [0.14 1.02]	4.58 (4.17 5.02) [3.88 5.38]

B. TFP

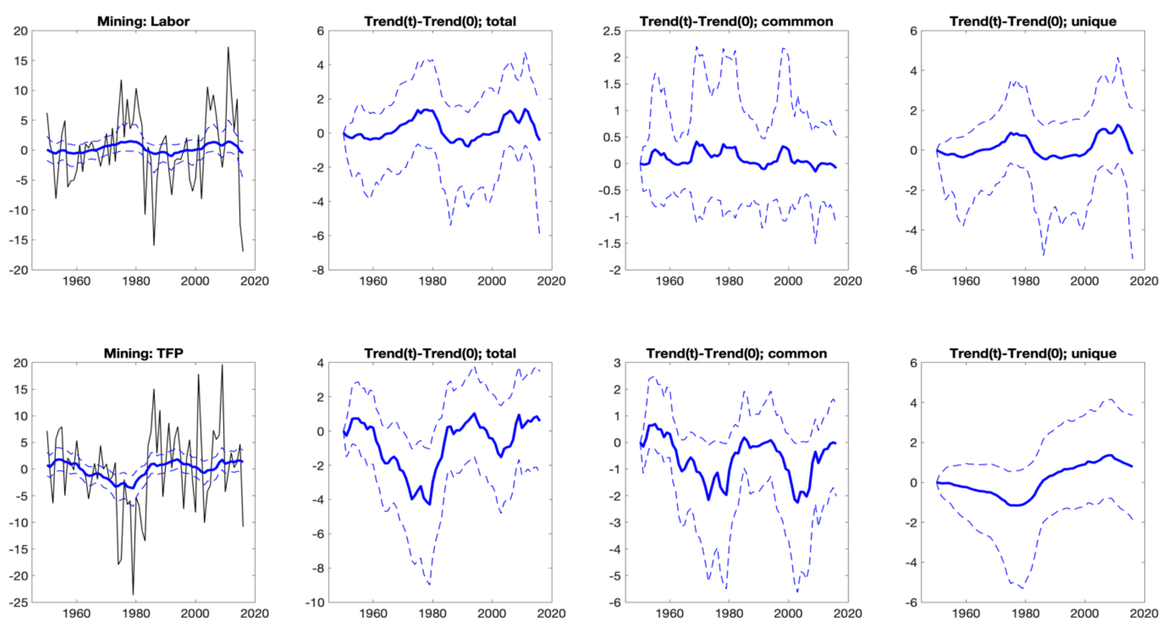
Sector	λ_τ	λ_ϵ	$\sigma_{j,\Delta\tau}$	$\sigma_{j,\epsilon}$
Agriculture, forestry, fishing, and hunting	0.27 (-0.43 0.96) [-0.92 1.49]	1.11 (-0.55 2.77) [-1.75 3.99]	0.26 (0.17 0.41) [0.14 0.59]	9.50 (8.72 10.35) [8.24 11.07]
Mining	0.69 (-0.61 1.61) [-1.24 2.17]	-0.06 (-1.53 1.45) [-2.59 2.46]	0.43 (0.22 1.04) [0.16 1.79]	7.12 (6.40 7.84) [5.92 8.42]
Utilities	0.20 (-0.24 0.64) [-0.57 0.97]	0.84 (-0.29 1.95) [-1.17 2.78]	0.25 (0.17 0.38) [0.14 0.53]	4.52 (4.12 4.95) [3.79 5.27]
Construction	0.57 (-0.65 1.07) [-0.99 1.33]	0.60 (0.13 1.04) [-0.20 1.39]	0.38 (0.24 0.60) [0.17 0.79]	2.08 (1.87 2.32) [1.72 2.51]
Durable goods	-0.11 (-0.70 0.91) [-1.02 1.30]	1.24 (0.75 1.74) [0.34 2.09]	0.47 (0.27 0.76) [0.19 0.98]	2.12 (1.84 2.40) [1.67 2.61]
Nondurable goods	0.11 (-0.27 0.48) [-0.54 0.76]	0.62 (-0.11 1.38) [-0.72 1.89]	0.26 (0.18 0.41) [0.14 0.58]	3.28 (3.00 3.61) [2.81 3.86]
Wholesale trade	0.11 (-0.27 0.53) [-0.57 0.86]	1.44 (0.75 2.12) [0.17 2.62]	0.24 (0.17 0.37) [0.13 0.52]	3.11 (2.80 3.45) [2.56 3.70]
Retail trade	0.20 (-0.21 0.56) [-0.50 0.82]	1.01 (0.43 1.57) [-0.03 1.97]	0.23 (0.16 0.35) [0.13 0.47]	2.39 (2.15 2.65) [1.97 2.84]
Transportation and warehousing	0.01 (-0.47 0.58) [-0.80 0.98]	1.29 (0.68 1.89) [0.26 2.33]	0.29 (0.19 0.49) [0.15 0.73]	2.67 (2.38 2.98) [2.16 3.22]
Information	-0.22 (-0.75 0.57) [-1.04 0.98]	0.32 (-0.31 0.98) [-0.76 1.46]	0.25 (0.17 0.40) [0.13 0.57]	3.29 (3.00 3.62) [2.80 3.86]
FIRE (ex Housing)	-0.23 (-0.70 0.44) [-1.02 0.83]	-1.00 (-1.62 -0.39) [-2.05 0.08]	0.23 (0.16 0.35) [0.13 0.48]	3.03 (2.75 3.34) [2.56 3.57]
Professional and business services	0.04 (-0.27 0.39) [-0.50 0.62]	0.61 (0.22 0.99) [-0.06 1.27]	0.26 (0.19 0.39) [0.15 0.51]	1.97 (1.78 2.17) [1.66 2.33]
Education, health care, and social assistance	0.01 (-0.31 0.36) [-0.54 0.60]	0.37 (-0.02 0.77) [-0.29 1.05]	0.37 (0.24 0.54) [0.19 0.69]	2.00 (1.81 2.21) [1.69 2.37]
Arts and entertainment	-0.24 (-0.69 0.65) [-0.97 1.01]	0.57 (-0.03 1.14) [-0.45 1.56]	0.21 (0.16 0.32) [0.12 0.43]	2.80 (2.55 3.08) [2.40 3.29]
Other services (excl. gov.)	0.42 (-0.48 0.82) [-0.77 1.05]	1.12 (0.65 1.57) [0.30 1.87]	0.22 (0.16 0.33) [0.13 0.44]	1.93 (1.71 2.15) [1.56 2.33]
Housing	0.21 (-0.34 0.42) [-0.49 0.54]	0.02 (-0.17 0.21) [-0.31 0.35]	0.19 (0.14 0.25) [0.12 0.32]	1.03 (0.94 1.13) [0.89 1.22]

Figure A1: Posterior estimates and 68% (pointwise) credible sets for trends

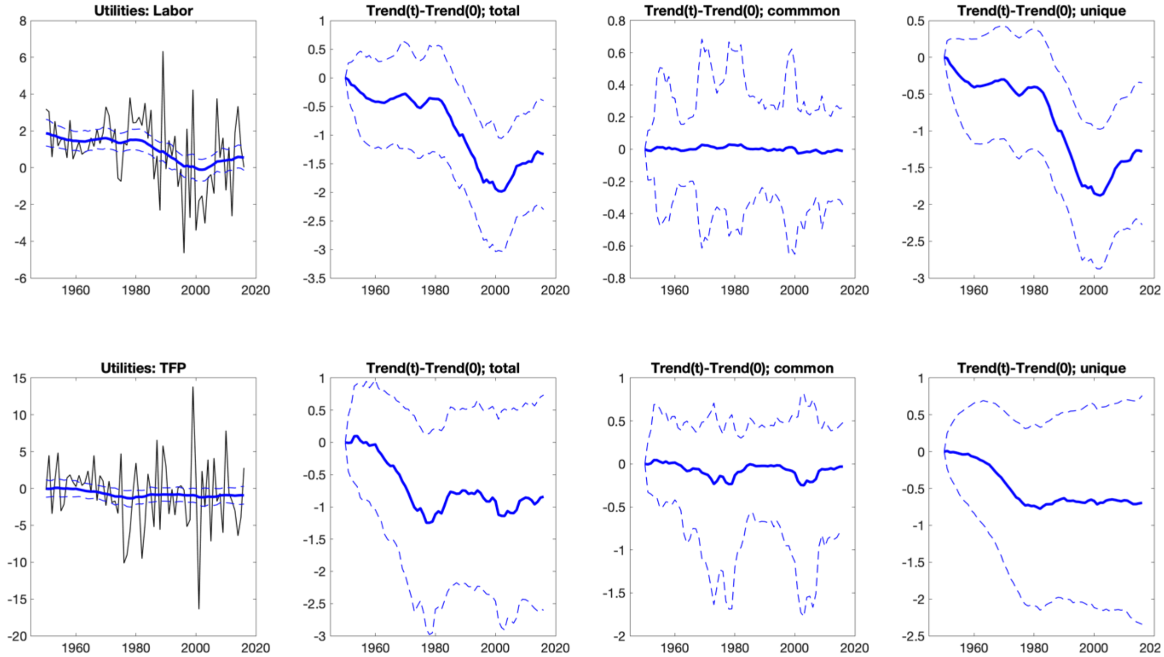
A. Agriculture, forestry, fishing, and hunting



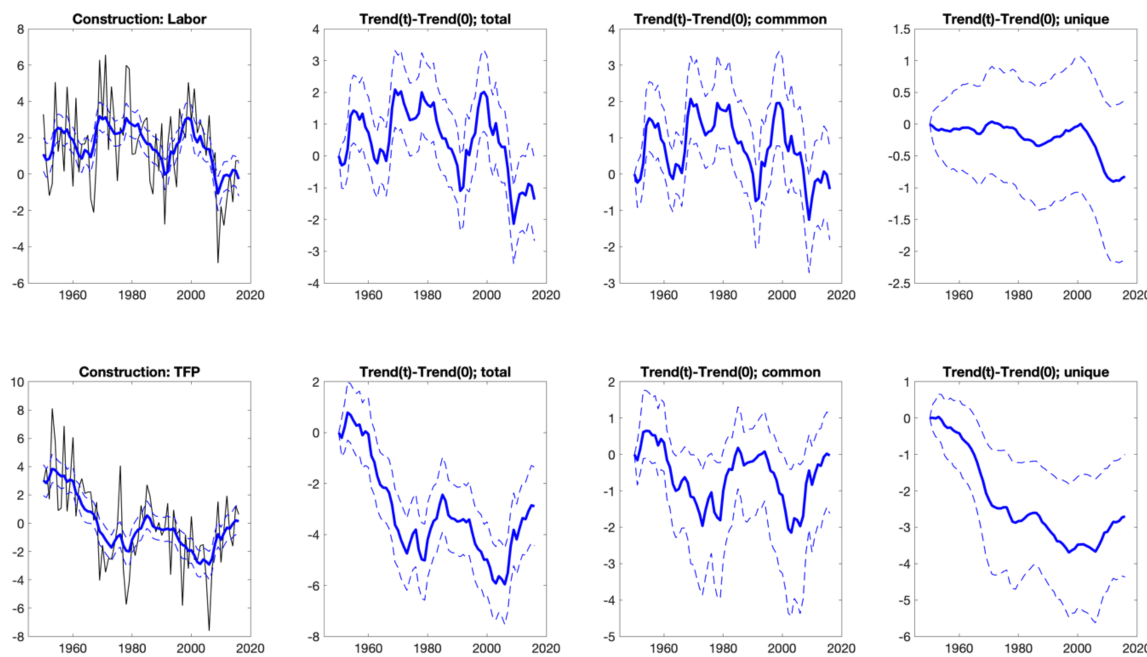
B. Mining



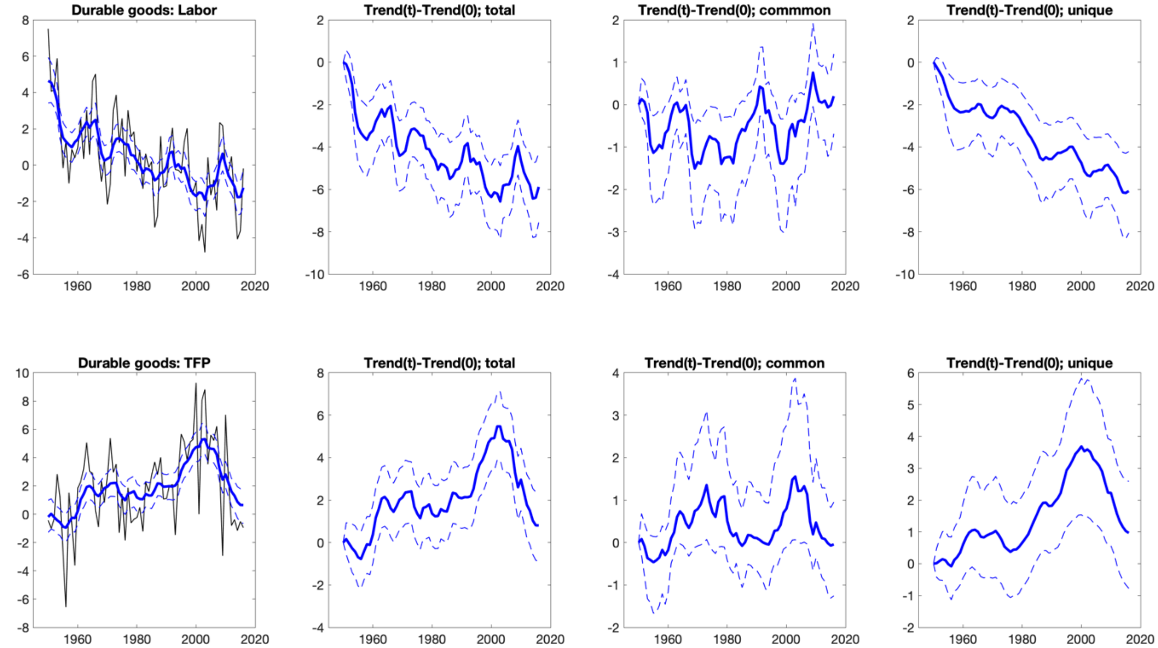
C. Utilities



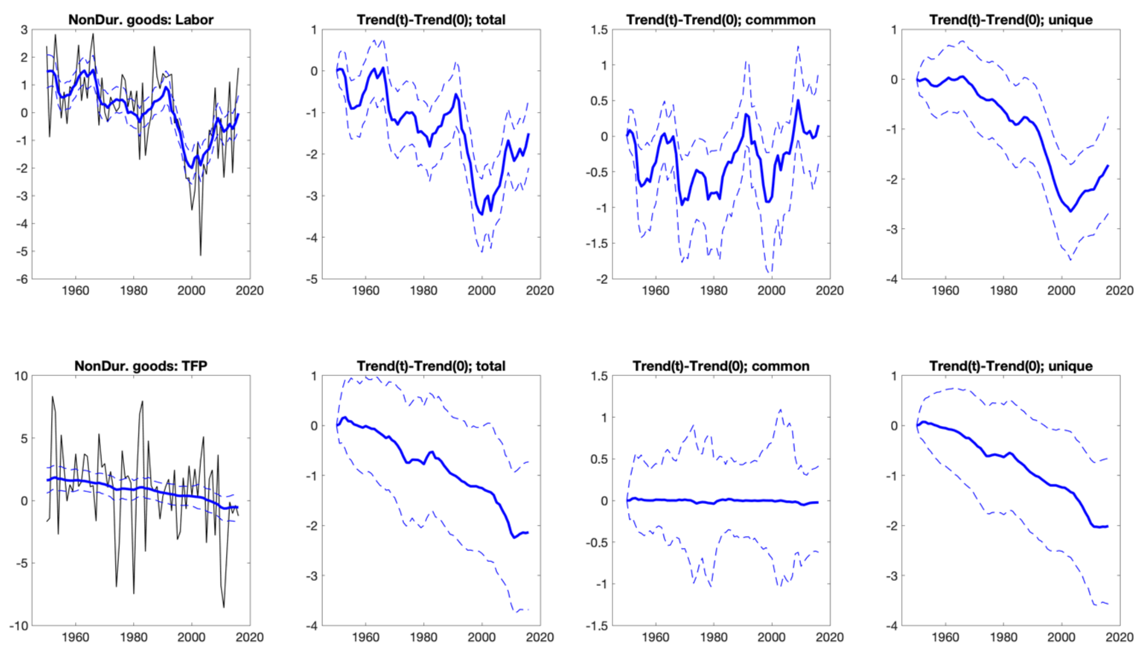
D. Construction



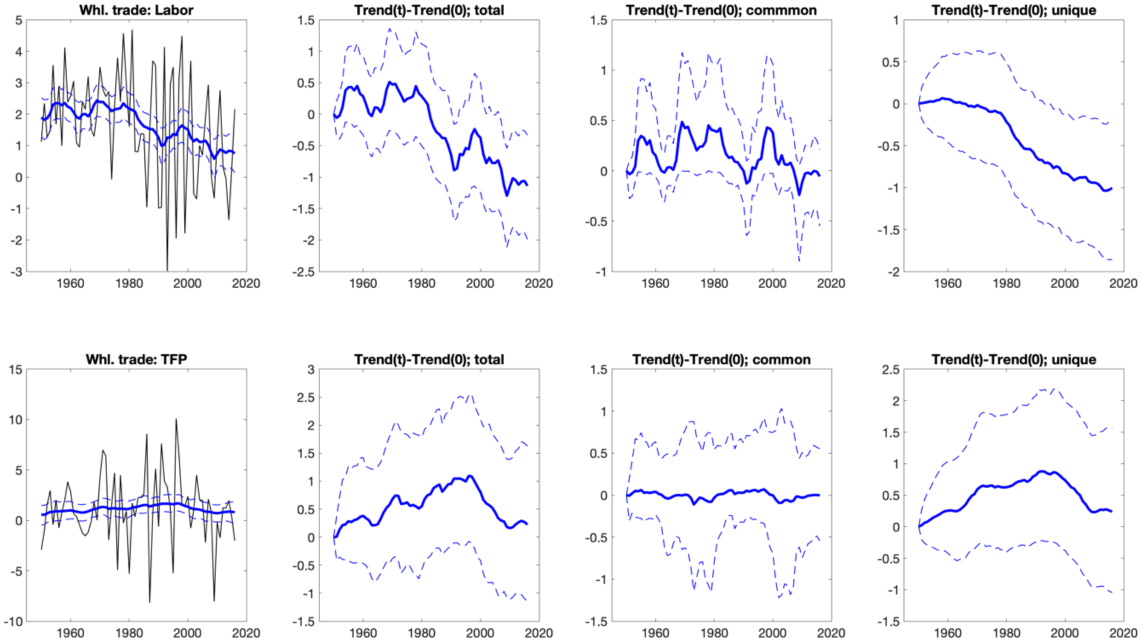
E. Durable goods



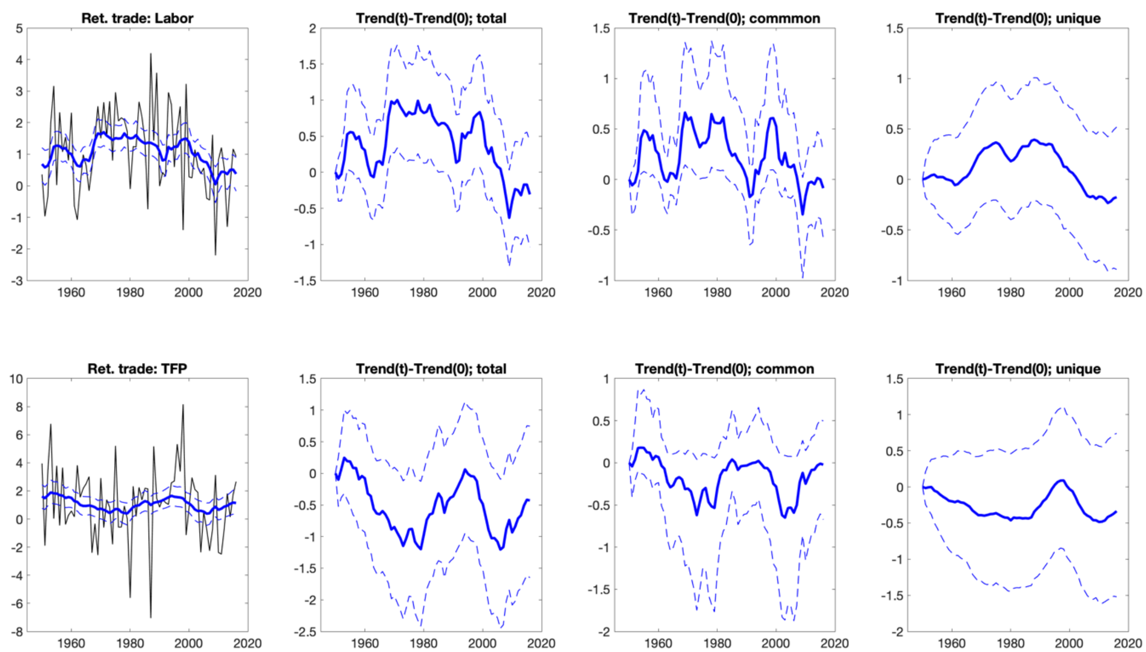
F. Nondurable goods



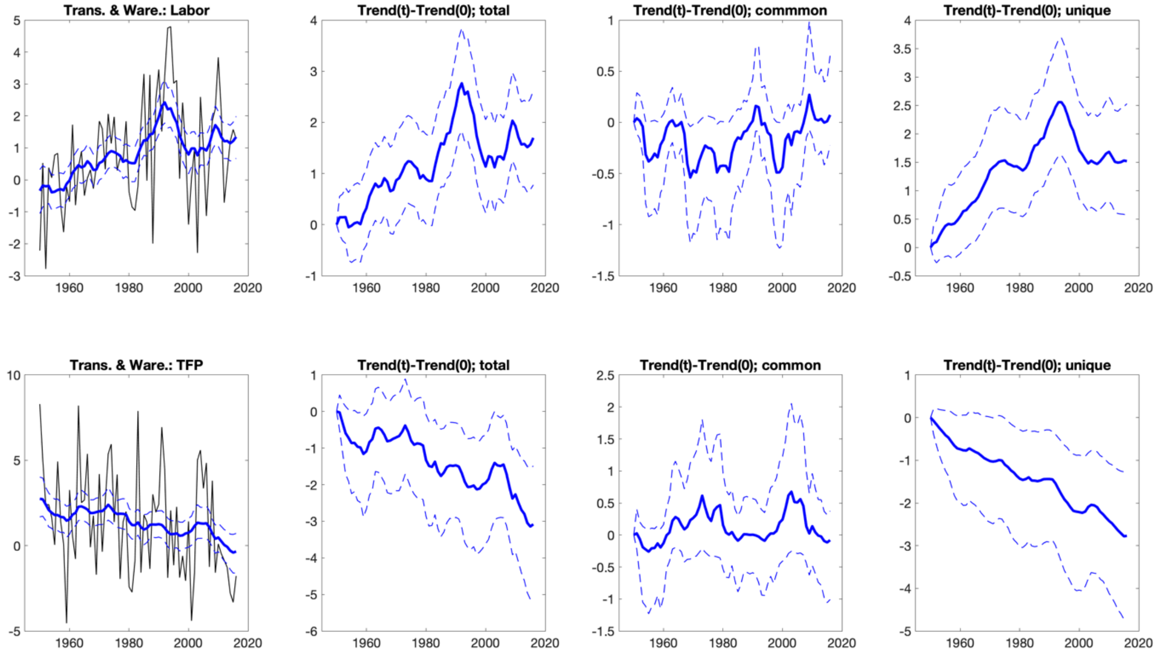
G. Wholesale trade



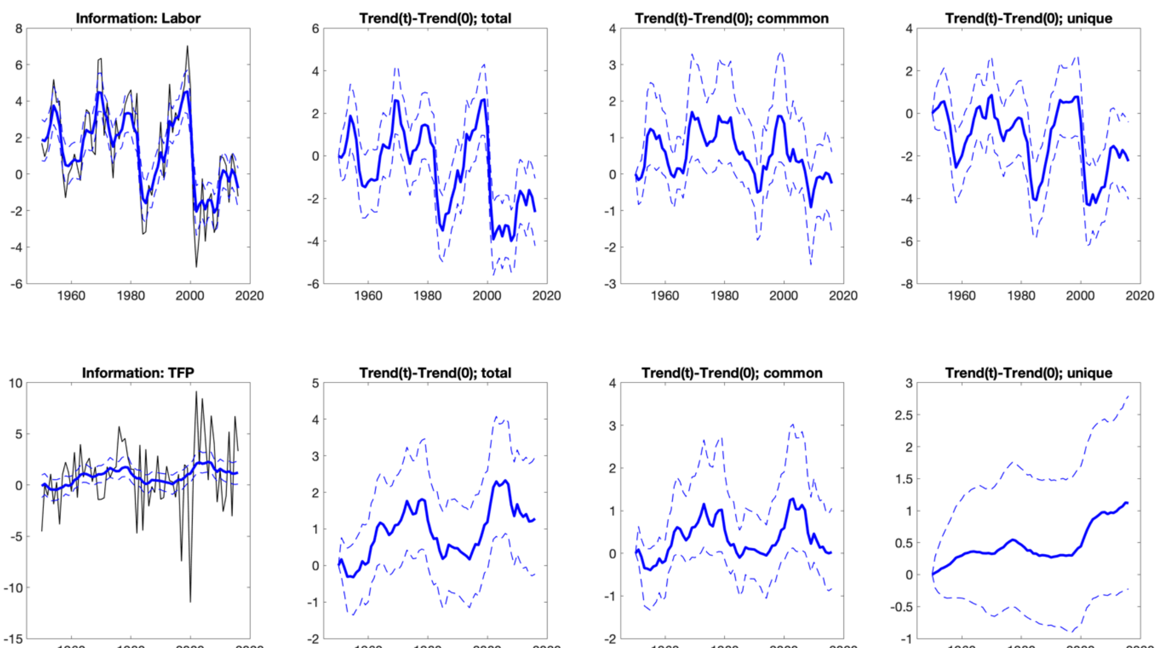
H. Retail trade



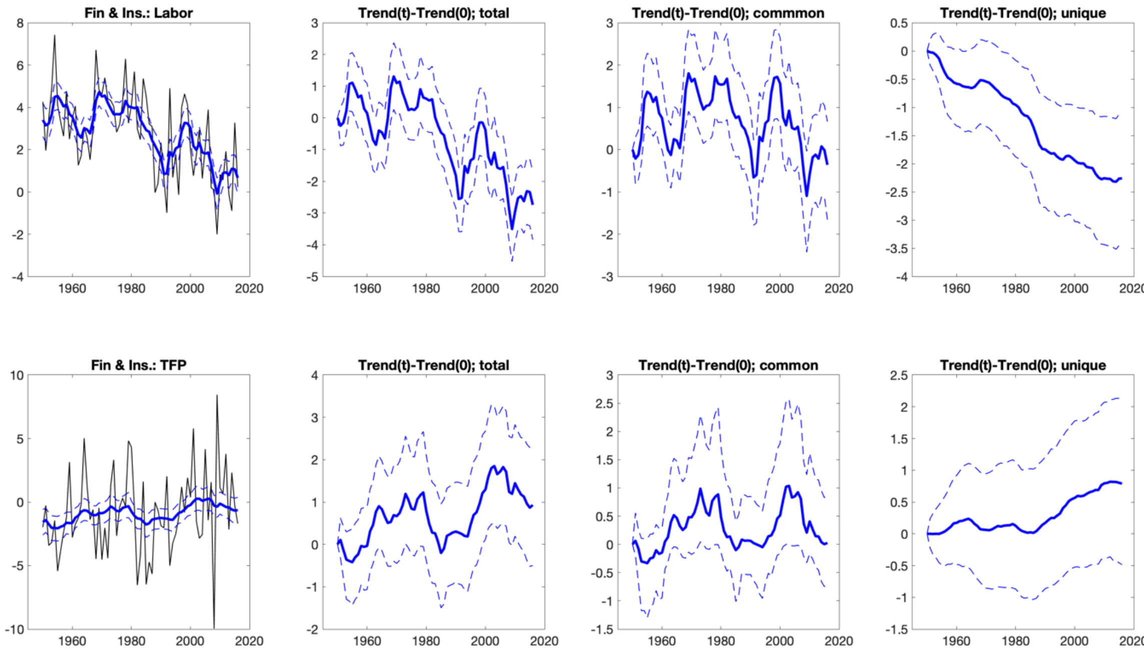
I. Transportation and warehousing



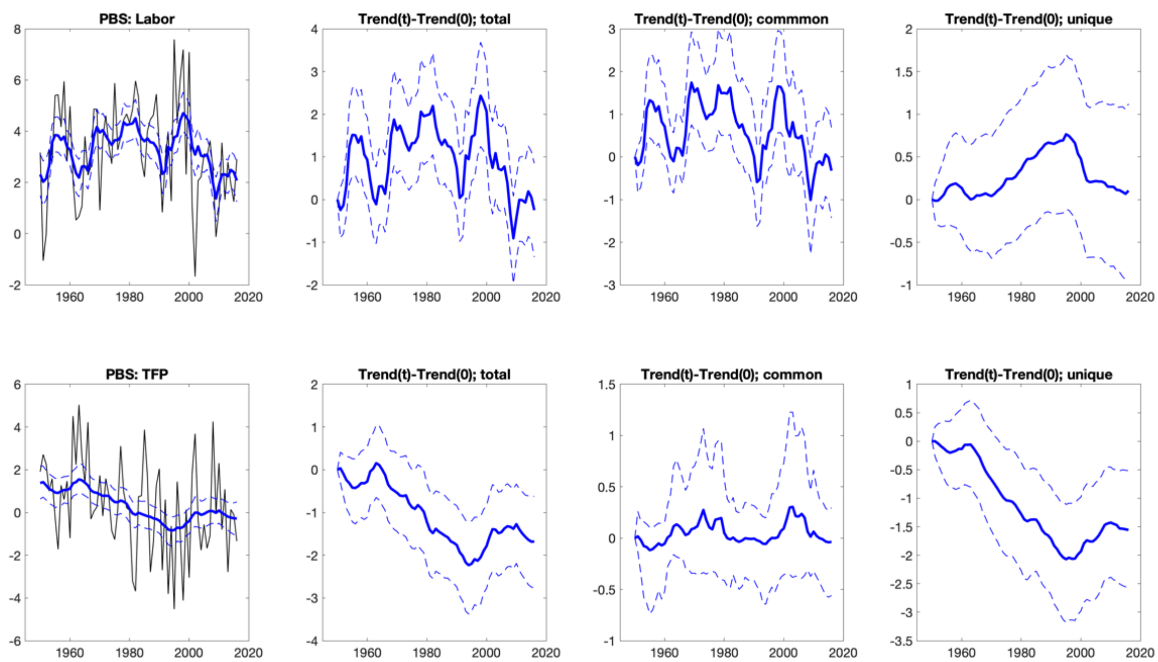
J. Information



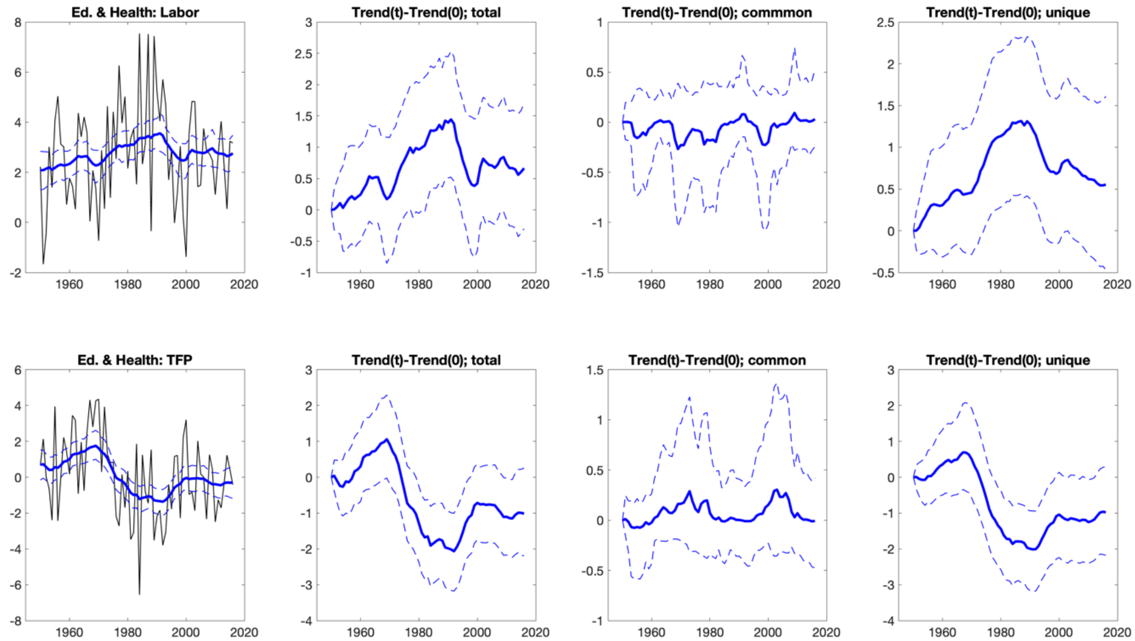
K. FIRE (ex Housing)



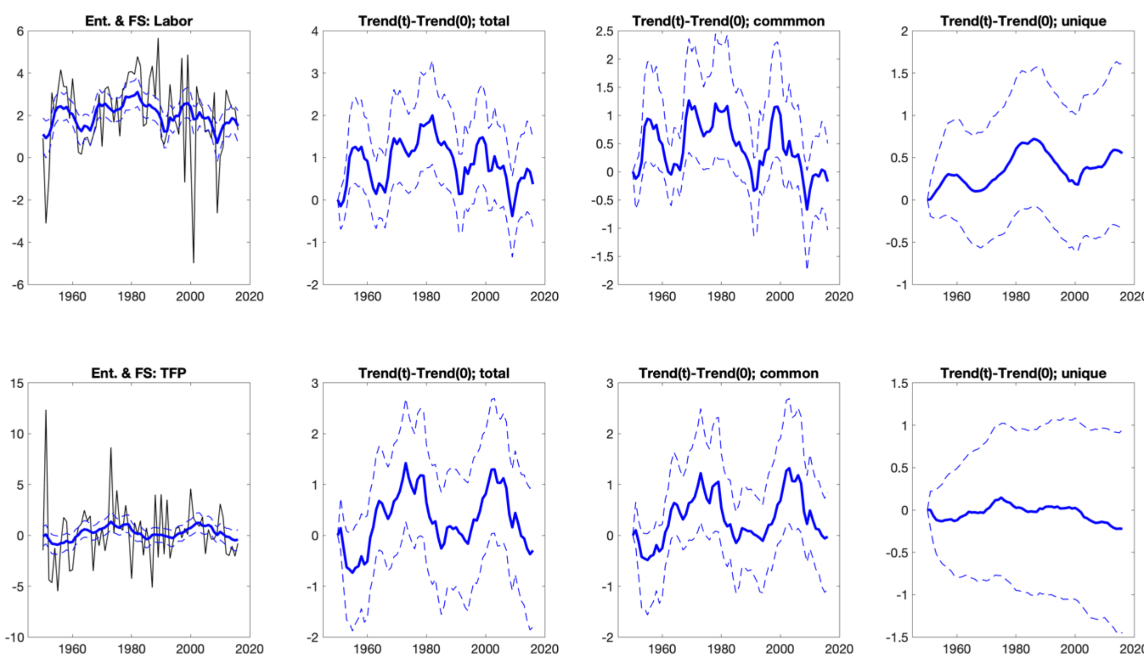
L. Professional and business services



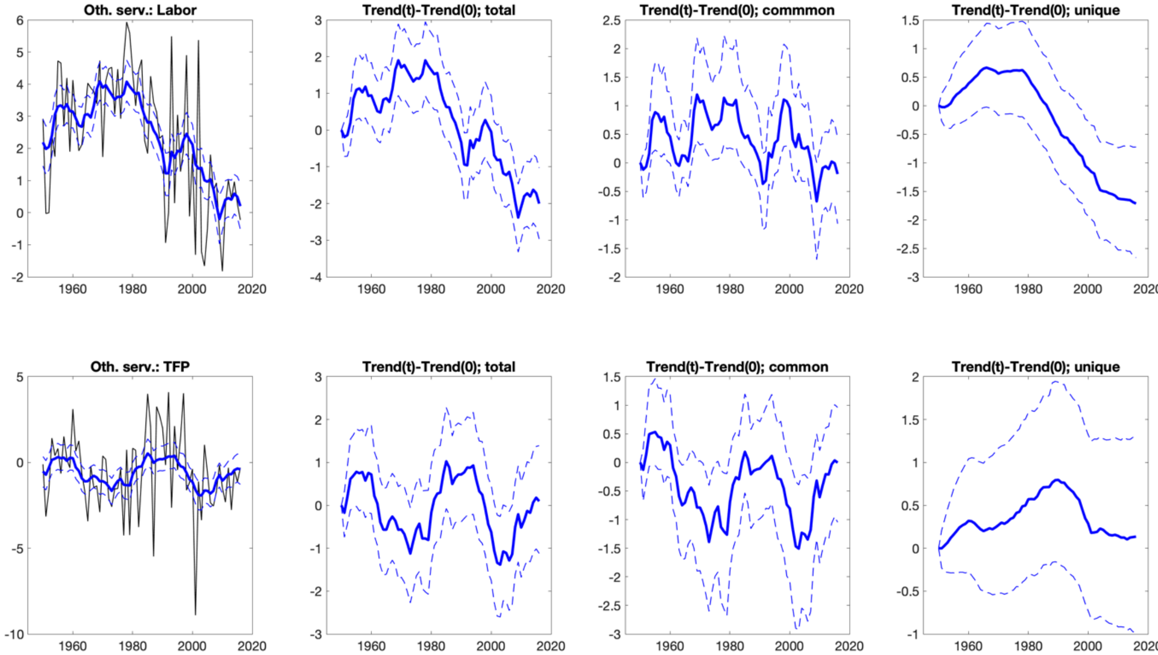
M. Education, health care, and social assistance



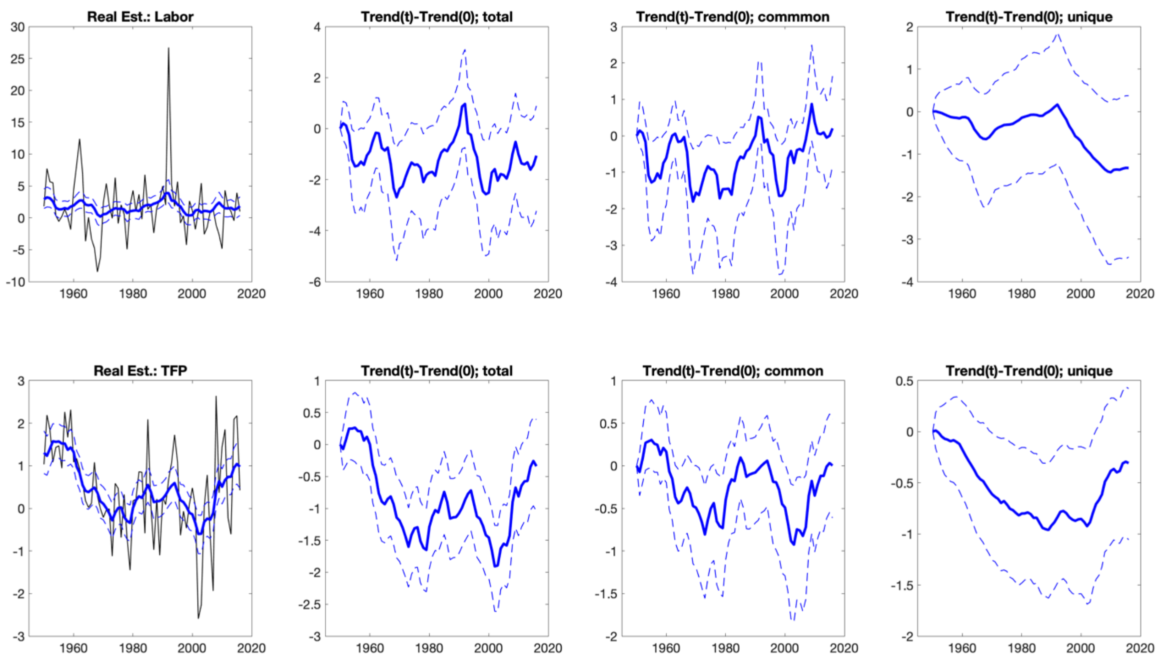
N. Arts and entertainment



O. Other services



P. Housing



2 A Structural Model with Sectoral Linkages in Materials and Investment

2.1 Economic Environment

Preferences are given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t C_t,$$

$$C_t = \prod_{j=1}^{n} \left(\frac{c_{j,t}}{\theta_j} \right)^{\theta_j}, \ \sum_{j=1}^{n} \theta_j = 1, \ \theta_j \geq 0$$

where C_t represents an aggregate consumption bundle taken to be the numeraire good.

The production side of the economy is described by

$$y_{j,t} = \left(\frac{v_{j,t}}{\gamma_j} \right)^{\gamma_j} \left(\frac{m_{j,t}}{1-\gamma_j} \right)^{(1-\gamma_j)}, \ \gamma_j \in [0,1],$$

$$m_{j,t} = \prod_{i=1}^{n} \left(\frac{m_{ij,t}}{\phi_{ij}} \right)^{\phi_{ij}}, \ \sum_{i=1}^{n} \phi_{ij} = 1, \ \phi_{ij} \geq 0,$$

$$v_{j,t} = z_{j,t} \left(\frac{k_{j,t}}{\alpha_j} \right)^{\alpha_j} \left(\frac{\ell_{j,t}}{1-\alpha_j} \right)^{1-\alpha_j}, \ \alpha_j \in [0,1].$$

We assume that labor supply follows an exogenous process.

Capital accumulation in each sector follows

$$k_{j,t+1} = x_{j,t} + (1-\delta_j) k_{j,t}$$

$$x_{j,t} = \prod_{i=1}^{n} \left(\frac{x_{ij,t}}{\omega_{ij}} \right)^{\omega_{ij}}, \ \sum_{i=1}^{n} \omega_{ij} = 1, \ \omega_{ij} \geq 0.$$

Goods market clearing requires that

$$c_{j,t} + \sum_{i=1}^{n} m_{ji,t} + \sum_{i=1}^{n} x_{ji,t} = y_{j,t}.$$

Finally, since labor input is taken to be exogenous, we define

$$A_{j,t} = z_{j,t} \left(\frac{\ell_{j,t}}{1-\alpha_j} \right)^{1-\alpha_j},$$

and express value added in sector j as

$$v_{j,t} = A_{j,t} \left(\frac{k_{j,t}}{\alpha_j} \right)^{\alpha_j}.$$

We then express the driving process for $A_{j,t}$ as

$$\Delta \ln A_{j,t} = \Delta \ln z_{j,t} + (1 - \alpha_j) \Delta \ln \ell_{j,t},$$

where

$$\Delta \ln z_{j,t} = \lambda^z_{j,\tau} \tau^z_{c,t} + \tau^z_{j,t} + \lambda^z_{j,\varepsilon} \varepsilon^z_{c,t} + \varepsilon^z_{j,t},$$

$$\tau^z_{c,t} = (1 - \rho) g^z_c + \rho \tau^z_{c,t-1} + \eta^z_{c,t},$$

$$\tau^z_{j,t} = (1 - \rho) g^z_j + \rho \tau^z_{j,t-1} + \eta^z_{j,t},$$

$$\Delta \ln \ell_{j,t} = \lambda^\ell_{j,\tau} \tau^\ell_{c,t} + \tau^\ell_{j,t} + \lambda^\ell_{j,\varepsilon} \varepsilon^\ell_{c,t} + \varepsilon^\ell_{j,t},$$

$$\tau^\ell_{c,t} = (1 - \rho) g^\ell_c + \rho \tau^\ell_{c,t-1} + \eta^\ell_{c,t},$$

$$\tau^\ell_{j,t} = (1 - \rho) g^\ell_j + \rho \tau^\ell_{j,t-1} + \eta^\ell_{j,t},$$

where the disturbances have the properties described in section 1.

We assume that ρ is arbitrarily close to 1 in which case the processes for $\Delta \ln z_{j,t}$ and $\Delta \ln \ell_{j,t}$ approach those described in section 1.

Notation: Let $\Theta = (\theta_1, ..., \theta_n)$, $\Gamma_d = diag\{\gamma_j\}$, $\Phi = \{\phi_{ij}\}$, $\Omega = \{\omega_{ij}\}$, $\alpha_d = diag\{\alpha_j\}$, $\delta_d = diag\{\delta_j\}$.

2.2 The Planner's Problem

$$\begin{aligned}
\max \ \mathcal{L} = & \sum_{t=0}^{\infty} \beta^t \prod_{j=1}^{n} \left(\frac{c_{j,t}}{\theta_j} \right)^{\theta_j} \\
& + \sum_{t=0}^{\infty} \beta^t \sum_{j=1}^{n} p^y_{j,t} \left[\left(\frac{v_{j,t}}{\gamma_j} \right)^{\gamma_j} \left(\frac{m_{j,t}}{1 - \gamma_j} \right)^{(1-\gamma_j)} - c_{j,t} - \sum_{i=1}^{n} m_{ji,t} - \sum_{i=1}^{n} x_{ji,t} \right] \\
& + \sum_{t=0}^{\infty} \beta^t \sum_{j=1}^{n} p^m_{j,t} \left[\prod_{i=1}^{n} \left(\frac{m_{ij,t}}{\phi_{ij}} \right)^{\phi_{ij}} - m_{j,t} \right] \\
& + \sum_{t=0}^{\infty} \beta^t \sum_{j=1}^{n} p^v_{j,t} \left[A_{j,t} \left(\frac{k_{j,t}}{\alpha_j} \right)^{\alpha_j} - v_{j,t} \right]
\end{aligned}$$

$$+\sum_{t=0}^{\infty}\beta^t\sum_{j=1}^{n}p^x_{j,t}\left[\prod_{i=1}^{n}\left(\frac{x_{ij,t}}{\omega_{ij}}\right)^{\omega_{ij}}+(1-\delta_j)k_{j,t}-k_{j,t+1}\right]$$

This yields:

$$\frac{\theta_j C_t}{c_{j,t}}=p^y_{j,t}, \tag{3}$$

which also defines the ideal price index,

$$1=\prod_{j=1}^{n}\left(p^y_{j,t}\right)^{\theta_j}.$$

Moreover, we have that

$$\gamma_j\frac{p^y_{j,t}y_{j,t}}{v_{j,t}}=p^v_{j,t},$$

and

$$(1-\gamma_j)\frac{p^y_{j,t}y_{j,t}}{m_{j,t}}=p^m_{j,t},$$

which define a price index for gross output,

$$p^y_{j,t}=\left(p^v_{j,t}\right)^{\gamma_j}\left(p^m_{j,t}\right)^{1-\gamma_j}.$$

In addition,

$$\phi_{ij}\frac{p^m_{j,t}m_{j,t}}{m_{ij,t}}=p^y_{i,t},$$

which gives material prices in terms of gross output prices,

$$p^m_{j,t}=\prod_{i=1}^{n}\left(p^y_{i,t}\right)^{\phi_{ij}},$$

and

$$\omega_{ij}\frac{p^x_{j,t}x_{j,t}}{x_{ij,t}}=p^y_{i,t},$$

which gives prices for capital in each sector in terms of gross output prices,

$$p^x_{j,t}=\prod_{i=1}^{n}\left(p^y_{i,t}\right)^{\omega_{ij}}.$$

Finally, we have an Euler equation

$$p^x_{j,t}=\beta\mathbb{E}_t\left[\alpha_j\frac{p^v_{j,t+1}v_{j,t+1}}{k_{j,t+1}}+p^x_{j,t+1}(1-\delta_j)\right].$$

Value added in sector j in this economy is $p^y_{j,t}y_{j,t} - \sum_i p^y_{i,t}m_{ij,t} = p^y_{j,t}y_{j,t} - \sum_i(1-\gamma_j)\phi_{ij}p^y_{j,t}y_{j,t} = \gamma_j p^y_{j,t}y_{j,t} = p^v_{j,t}v_{j,t}$. GDP is then given by $\sum_j p^v_{j,t}v_{j,t}$. It is also the case that $p^y_{j,t}y_{j,t} - \sum_i p^y_{j,t}m_{ji,t} = p^y_{j,t}c_{j,t} + \sum_i p^y_{j,t}x_{ji,t}$.

2.3 The Full Set of Equilibrium Conditions

For clarity, we collect in this subsection the full set of equilibrium conditions. The economic environment is described by,

$$c_{j,t} + \sum_{i=1}^{n} m_{ji,t} + \sum_{i=1}^{n} x_{ji,t} = y_{j,t},\ \forall j,$$

$$x_{j,t} = \prod_{i=1}^{n}\left(\frac{x_{ij,t}}{\omega_{ij}}\right)^{\omega_{ij}},\ \forall j,$$

$$k_{j,t+1} = x_{j,t} + (1-\delta)k_{j,t},\ \forall j,\ \text{and } k_{j,0} \text{ given},$$

$$v_{j,t} = A_{j,t}\left(\frac{k_{j,t}}{\alpha_j}\right)^{\alpha_j},\ \forall j,$$

$$m_{j,t} = \prod_{i=1}^{n}\left(\frac{m_{ij,t}}{\phi_{ij}}\right)^{\phi_{ij}},\ \forall j,$$

$$y_{j,t} = \left(\frac{v_{j,t}}{\gamma_j}\right)^{\gamma_j}\left(\frac{m_{j,t}}{1-\gamma_j}\right)^{1-\gamma_j},\ \forall j.$$

The first-order conditions from the planner's problem are,

$$\frac{\theta_j C_t}{c_{j,t}} = p^y_{j,t},\ \forall j,$$

$$C_t = \prod_{j=1}^{n}\left(\frac{c_{j,t}}{\theta_j}\right)^{\theta_j},$$

$$\gamma_j\frac{p^y_{j,t}y_{j,t}}{v_{j,t}} = p^v_{j,t},\ \forall j,$$

$$(1-\gamma_j)\frac{p^y_{j,t}y_{j,t}}{m_{j,t}} = p^m_{j,t},\ \forall j,$$

$$\phi_{ij}\frac{p^m_{j,t}m_{j,t}}{m_{ij,t}} = p^y_{i,t},\ \forall i,j,$$

$$\omega_{ij}\frac{p^x_{j,t}x_{j,t}}{x_{ij,t}} = p^y_{i,t},\ \forall i,j,$$

$$p_{j,t}^x = \beta \mathbb{E}_t \left[\alpha_j \frac{p_{j,t+1}^v v_{j,t+1}}{k_{j,t+1}} + p_{j,t+1}^x (1-\delta_j) \right] \ \forall j$$

The exogenous sectoral processes driving productivity are

$$\Delta \ln A_{j,t} = \Delta \ln z_{j,t} + (1-\alpha_j) \Delta \ln \ell_{j,t},$$

$$\Delta \ln z_{j,t} = \lambda_{j,\tau}^z \tau_{c,t}^z + \tau_{j,t}^z + \lambda_{j,\varepsilon}^z \varepsilon_{c,t}^z + \varepsilon_{j,t}^z,$$
$$\tau_{c,t}^z = (1-\rho) g_c^z + \rho \tau_{c,t-1}^z + \eta_{c,t}^z,$$
$$\tau_{j,t}^z = (1-\rho) g_j^z + \rho \tau_{j,t-1}^z + \eta_{j,t}^z,$$
$$\Delta \ln \ell_{j,t} = \lambda_{j,\tau}^\ell \tau_{c,t}^\ell + \tau_{j,t}^\ell + \lambda_{j,\varepsilon}^\ell \varepsilon_{c,t}^\ell + \varepsilon_{j,t}^\ell,$$
$$\tau_{c,t}^\ell = (1-\rho) g_c^\ell + \rho \tau_{c,t-1}^\ell + \eta_{c,t}^\ell,$$
$$\tau_{j,t}^\ell = (1-\rho) g_j^\ell + \rho \tau_{j,t-1}^\ell + \eta_{j,t}^\ell.$$

There are $2n^2+15n+3$ equations, with unknowns given by $\{y_{j,t}, c_{j,t}, m_{j,t}, x_{j,t}, v_{j,t}, k_{j,t+1}, A_{j,t}, \tau_{j,t}, p_{j,t}^y, p_{j,t}^m, p_{j,t}^x\}_{j=1}^n$, $\{m_{ij,t}, x_{ij,t}\}_{i,j=1}^n$, and C_t.

2.4 Special Cases with No Growth

Consider the case where $\alpha_j = 0 \ \forall j$ and structural changes embodied in $\{z_{j,t}, \ell_{j,t}\}$ are modeled as stationary processes in levels. Then $A_{j,t}$ is also stationary in levels rather than growth rates. Aggregate value added or GDP, V_t, is then given by the aggregate consumption bundle, C_t, and

$$\frac{\partial \ln V_t}{\partial \ln A_{j,t}} = s_j^v, \tag{4}$$

where s_j^v is sector j's value added share in GDP, and where these shares may be summarized in a vector, $s^v = (s_1^v, ..., s_n^v)$, given by $s^v = \Theta(I - (I - \Gamma_d)\Phi')^{-1}\Gamma_d$.[1]

When $\alpha_j > 0$ for some j, the economy becomes dynamic and, absent shocks, converges to a steady state in levels in the long-run. Getting rid of the t subscripts to denote variables in that steady state, and letting $A = (A_1, ..., A_n)$ stand for the long-run vector of composite exogenous sectoral states, we have that in the limit as $\beta \to 1$,

$$\frac{\partial \ln V}{\partial \ln A_j} = \eta s_j^v, \tag{5}$$

where η is an adjustment factor approximately equal to the inverse of the mean employment

[1] The object $\Theta(I - (I - \Gamma_d)\Phi')^{-1}\Gamma_d$ is the *influence vector* highlighted by Acemoglu et al. (2012).

share across sectors. In particular, when sectors use capital with the same intensity, $\alpha_j = \alpha$ $\forall j$, $\eta = \frac{1}{1-\alpha}$. The influence vector in this case, s^v, is given by $\Theta[\Gamma_d^{-1}(I - (I - \Gamma_d)\Phi') - \alpha_d\Omega']^{-1}/\Theta[\Gamma_d^{-1}(I - (I - \Gamma_d)\Phi') - \alpha_d\Omega']^{-1}\mathbf{1}$, where $\mathbf{1}$ is a unit vector of size n. When $\beta < 1$, the influence vector also depends on sectoral depreciation rates, δ_j, and equation (5) holds as an approximation that depends on $\frac{\beta}{1-\beta(1-\delta_j)} \times \delta_j \approx 1$ for standard calibrations of β. See section 2.5.3 for a general formulation of steady state value added shares where equation (5) arises as a special case.

2.5 Balanced Growth

Consider the case where all variables are growing at a constant rate along a non-stochastic steady state path, $\varepsilon^i_{c,t} = \varepsilon^i_{j,t} = 0$ for $i = z$, ℓ, $\eta^i_{c,t} = \eta^i_{j,t} = 0$ for $i = z$, ℓ, and $\tau^i_{j,t} = \tau^i_{j,t-1}$, $\tau^i_{c,t} = \tau^i_{c,t-1}$ for $i = z$, ℓ, and all j and t. Then,

$$\Delta \ln A_{j,t} \equiv \overline{g}_j = \lambda^z_{j,\tau} g^z_c + g^z_j + (1 - \alpha_j)\left(\lambda^\ell_{j,\tau} g^\ell_c + g^\ell_j\right) \tag{6}$$

so that

$$\widetilde{A}_{j,t} = \frac{A_{j,t}}{A_{j,t-1}} = e^{\overline{g}_j} \approx 1 + \overline{g}_j.$$

for reasonable growth rates.

2.5.1 Making the Model Stationary

We normalize the model's variables with respect to sector-specific factors, $\mu_{j,t}$, determined below, to yield a system of equations that is stationary in the normalized variables. If all growth rates are constant, the resource constraint in any individual sector implies that all the variables in that constraint must grow at the same rate. Thus, define $\widetilde{y}_{j,t} = y_{j,t}/\mu_{j,t}$, $\widetilde{c}_{j,t} = c_{j,t}/\mu_{j,t}$, $\widetilde{m}_{ji,t} = m_{ji,t}/\mu_{j,t}$, and $\widetilde{x}_{ji,t} = x_{ji,t}/\mu_{j,t}$. Then, the economy's resource constraint becomes

$$\widetilde{c}_{j,t} + \sum_{i=1}^{n} \widetilde{m}_{ji,t} + \sum_{i=1}^{n} \widetilde{x}_{ji,t} = \widetilde{y}_{j,t}.$$

Given the above definitions, the production of investment goods may be re-written as

$$\widetilde{x}_{j,t} = \prod_{i=1}^{n} \left(\frac{\widetilde{x}_{ij,t}}{\omega_{ij}}\right)^{\omega_{ij}},$$

where $\widetilde{x}_{j,t} = x_{j,t}/\prod_{i=1}^{n}\mu_{i,t}^{\omega_{ij}}$. Under this normalization, the capital accumulation equation is

$$k_{j,t+1} = \widetilde{x}_{j,t}\prod_{i=1}^{n}\mu_{i,t}^{\omega_{ij}} + (1-\delta_j)k_{j,t},$$

and so becomes

$$\widetilde{k}_{j,t+1} = \widetilde{x}_{j,t} + (1-\delta_j)\widetilde{k}_{j,t}\prod_{i=1}^{n}\left(\frac{\mu_{i,t-1}}{\mu_{i,t}}\right)^{\omega_{ij}},$$

where $\widetilde{k}_{j,t+1} = k_{j,t+1}/\prod_{i=1}^{n}\mu_{i,t}^{\omega_{ij}}$.

The expression for value added may be written as

$$v_{j,t} = A_{j,t}\left(\frac{\widetilde{k}_{j,t}\prod_{i=1}^{n}\mu_{i,t-1}^{\omega_{ij}}}{\alpha_j}\right)^{\alpha_j},$$

so that, defining $\widetilde{v}_{j,t} = v_{j,t}/A_{j,t}\left(\prod_{i=1}^{n}\mu_{i,t-1}^{\omega_{ij}}\right)^{\alpha_j}$, it becomes

$$\widetilde{v}_{j,t} = \left(\frac{\widetilde{k}_{j,t}}{\alpha_j}\right)^{\alpha_j}.$$

The composite bundle of materials used in sector j may be expressed as

$$\widetilde{m}_{j,t} = \prod_{i=1}^{n}\left(\frac{\widetilde{m}_{ij,t}}{\phi_{ij}}\right)^{\phi_{ij}},$$

with $\widetilde{m}_{j,t} = m_{j,t}/\prod_{i=1}^{n}\mu_{i,t}^{\phi_{ij}}$.

Under our normalization, gross output may be written as

$$\widetilde{y}_{j,t}\mu_{j,t} = \left(\frac{\widetilde{v}_{j,t}A_{j,t}\prod_{i=1}^{n}\mu_{i,t-1}^{\alpha_j\omega_{ij}}}{\gamma_j}\right)^{\gamma_j}\left(\frac{\widetilde{m}_{j,t}\prod_{i=1}^{n}\mu_{i,t}^{\phi_{ij}}}{1-\gamma_j}\right)^{1-\gamma_j},$$

which, collecting terms, gives

$$\widetilde{y}_{j,t} = \left(\frac{\widetilde{v}_{j,t}}{\gamma_j}\right)^{\gamma_j} \left(\frac{\widetilde{m}_{j,t}}{1-\gamma_j}\right)^{1-\gamma_j} \left[\frac{A_{j,t}^{\gamma_j}}{\mu_{j,t}} \prod_{i=1}^{n} \mu_{i,t-1}^{\gamma_j \alpha_j \omega_{ij}} \mu_{i,t}^{(1-\gamma_j)\phi_{ij}}\right].$$

We can now use the expression in square brackets to solve for the normalizing factors, $\mu_{j,t}$, as a function of the model's underlying parameters.

First, re-write the term in square brackets as

$$\frac{A_{j,t}^{\gamma_j}}{\mu_{j,t}} \left(\prod_{i=1}^{n} \frac{\mu_{i,t-1}^{\gamma_j \alpha_j \omega_{ij}}}{\mu_{i,t}^{\gamma_j \alpha_j \omega_{ij}}}\right) \left(\prod_{i=1}^{n} \mu_{i,t}^{(1-\gamma_j)\phi_{ij}} \mu_{i,t}^{\gamma_j \alpha_j \omega_{ij}}\right),$$

where this last expression involves the growth rate of $\mu_{i,t}$. Thus, without loss of generality with respect to growth rates, we choose $\mu_{j,t}$ such that

$$\frac{A_{j,t}^{\gamma_j}}{\mu_{j,t}} \prod_{i=1}^{n} \mu_{i,t}^{\gamma_j \alpha_j \omega_{ij} + (1-\gamma_j)\phi_{ij}} = 1.$$

Taking logs of both sides of the above expression, we have

$$\gamma_j \ln A_{j,t} - \ln \mu_{j,t} + \sum_{i=1}^{n} \left(\gamma_j \alpha_j \omega_{ij} + (1-\gamma_j)\phi_{ij}\right) \ln \mu_{i,t} = 0,$$

or in vector form,

$$\Gamma_d \ln A_t - \ln \mu_t + \Gamma_d \alpha_d \Omega' \ln \mu_t + (I - \Gamma_d)\Phi' \ln \mu_t = 0,$$

which gives us

$$\ln \mu_t = \Xi' \ln A_t, \tag{7}$$

where

$$\Xi' = \left(I - \Gamma_d \alpha_d \Omega' - (I - \Gamma_d)\Phi'\right)^{-1} \Gamma_d,$$

with $\Xi = \{\xi_{ij}\}$.

Going back to equation (6), and writing the vector of productivity growth rates as $\Delta \ln A_t = \overline{g}_a$, it follows that

$$\Delta \ln \mu_t = \Xi' \overline{g}_a = \Xi' \left(\underbrace{\lambda_\tau^z g_c^z + g^z}_{\overline{g}_z} + (I - \alpha_d) \underbrace{\left(\lambda_\tau g_c^\ell + g^\ell\right)}_{\overline{g}_\ell} \right).$$

Note: $\Gamma_d^{-1}\left(I - \Gamma_d \alpha_d \Omega' - (I - \Gamma_d)\Phi'\right)\mathbf{1} = (I - \alpha_d)\mathbf{1}$.

Let $\mu^v_{j,t}$ denote the normalizing factor for value added in sector j, $A_{j,t}\left(\prod_{i=1}^{n}\mu_{i,t-1}^{\omega_{ij}}\right)^{\alpha_j}$, defined above,

$$\mu^v_{j,t} = A_{j,t}\left(\prod_{j=1}^{n}\mu_{i,t-1}^{\omega_{ij}}\right)^{\alpha_j}.$$

Then, using equation (7), we have that

$$\ln \mu^v_t = \ln A_t + \alpha_d \Omega' \Xi' \ln A_{t-1},$$

or

$$\Delta \ln \mu^v_t = \left[I + \alpha_d \Omega' \underbrace{\left(I - \alpha_d \Gamma_d \Omega' - (I - \Gamma_d)\Phi'\right)^{-1}\Gamma_d}_{\Xi'}\right]\overline{g}_a. \tag{8}$$

Equation (7) gives us the normalizing factor in the resource constraint for sector i, $\ln \mu_{i,t} = \sum_{k=1}^{n}\xi_{ki}\ln A_{k,t}$, so that

$$\mu_{i,t} = \prod_{k=1}^{n} A_{k,t}^{\xi_{ki}}.$$

The normalized capital accumulation equation then becomes

$$\widetilde{k}_{j,t+1} = \widetilde{x}_{j,t} + (1 - \delta_j)\widetilde{k}_{j,t}\prod_{i=1}^{n}\left(\prod_{k=1}^{n}\widetilde{A}_{k,t}^{-\omega_{ij}\xi_{ki}}\right)$$

where $\widetilde{A}_{k,t} = A_{k,t}/A_{k,t-1}$. Similarly, normalized gross output in sector j may be written as

$$\widetilde{y}_{j,t} = \left(\frac{\widetilde{v}_{j,t}}{\gamma_j}\right)^{\gamma_j}\left(\frac{\widetilde{m}_{j,t}}{1 - \gamma_j}\right)^{1-\gamma_j}\left(\prod_{i=1}^{n}\prod_{k=1}^{n}\widetilde{A}_{k,t}^{-\xi_{ki}\gamma_j\alpha_j\omega_{ij}}\right).$$

Normalized aggregate consumption, $\widetilde{C}_t = C_t / \prod_{j=1}^{n}\mu_{j,t}^{\theta_j}$, solves

$$\widetilde{C}_t = \prod_{j=1}^{n}\left(\frac{\widetilde{c}_{j,t}}{\theta_j}\right)^{\theta_j}.$$

The planner's first-order conditions may now also be easily expressed in normalized terms,

$$\frac{\theta_j \widetilde{C}_t}{\widetilde{c}_{j,t}} = \widetilde{p}^y_{j,t} \equiv \frac{p^y_{j,t}\mu_{j,t}}{\prod_{k=1}^{n}\mu_{k,t}^{\theta_k}}, \ \forall j,$$

$$\gamma_j \frac{\widetilde{p}^y_{j,t}\widetilde{y}_{j,t}}{\widetilde{v}_{j,t}} = \widetilde{p}^v_{j,t} \equiv \frac{p^v_{j,t} A_{j,t} \prod_{k=1}^{n}\mu_{k,t-1}^{\omega_{kj}\alpha_j}}{\prod_{k=1}^{n}\mu_{k,t}^{\theta_k}} \ \forall j,$$

$$(1-\gamma_j)\frac{\widetilde{p}^y_{j,t}\widetilde{y}_{j,t}}{\widetilde{m}_{j,t}} = \widetilde{p}^m_{j,t} \equiv p^m_{j,t}\prod_{k=1}^{n}\mu_{k,t}^{\phi_{kj}-\theta_k}, \ \forall j,$$

$$\phi_{ij}\frac{\widetilde{p}^m_{j,t}\widetilde{m}_{j,t}}{\widetilde{m}_{ij,t}} = \widetilde{p}^y_{i,t}, \ \forall i, j,$$

$$\frac{p^x_{j,t}}{\prod_{k=1}^{n}\mu_{k,t}^{\theta_k-\omega_{kj}}} \equiv \widetilde{p}^x_{j,t} = \frac{\widetilde{p}^y_{i,t}\widetilde{x}_{ij,t}}{\omega_{ij}\widetilde{x}_{j,t}}, \ \forall i, j,$$

$$\widetilde{p}^x_{j,t} = \beta \mathbb{E}_t \prod_{i=1}^{n}\prod_{k=1}^{n}\widetilde{A}_{k,t+1}^{\xi_{ki}(\theta_i-\omega_{ij})}\left[\alpha_j \frac{\widetilde{p}^v_{j,t+1}\widetilde{v}_{j,t+1}}{\widetilde{k}_{j,t+1}}\left(\prod_{i=1}^{n}\prod_{k=1}^{n}\widetilde{A}_{k,t+1}^{\xi_{ki}\omega_{ij}}\right) + \widetilde{p}^x_{j,t+1}(1-\delta)\right].$$

Finally, the normalized driving processes may be expressed as

$$\ln \widetilde{A}_{j,t} = \Delta \ln z_{j,t} + (1-\alpha_j)\Delta \ln \ell_{j,t}$$

$$\begin{aligned}
\Delta \ln z_{j,t} &= \lambda^z_{j,\tau}\tau^z_{c,t} + \tau^z_{j,t} + \lambda^z_{j,\varepsilon}\varepsilon^z_{c,t} + \varepsilon^z_{j,t}, \\
\tau^z_{c,t} &= (1-\rho)g^z_c + \rho\tau^z_{c,t-1} + \eta^z_{c,t}, \\
\tau^z_{j,t} &= (1-\rho)g^z_j + \rho\tau^z_{j,t-1} + \eta^z_{j,t}, \\
\Delta \ln \ell_{j,t} &= \lambda^\ell_{j,\tau}\tau^\ell_{c,t} + \tau^\ell_{j,t} + \lambda^\ell_{j,\varepsilon}\varepsilon^\ell_{c,t} + \varepsilon^\ell_{j,t}, \\
\tau^\ell_{c,t} &= (1-\rho)g^\ell_c + \rho\tau^\ell_{c,t-1} + \eta^\ell_{c,t}, \\
\tau^\ell_{j,t} &= (1-\rho)g^\ell_j + \rho\tau^\ell_{j,t-1} + \eta^\ell_{j,t}.
\end{aligned}$$

2.5.2 Steady State of the Stationary Environment

The relationships between price indices hold in normalized form,

$$\sum_j \theta_j \ln \widetilde{p}^y_j = 0,$$

$$\ln \widetilde{p}^y_j = \gamma_j \ln \widetilde{p}^v_j + (1-\gamma_j) \ln \widetilde{p}^m_j + \gamma_j \alpha_j \sum_{i=1}^{n} \sum_{k=1}^{n} \omega_{ij} \xi_{ki} \ln \widetilde{A}_k,$$

$$\ln \widetilde{p}^m_j = \sum_i \phi_{ij} \ln \widetilde{p}^y_i,$$

$$\ln \widetilde{p}^x_j = \sum_i \omega_{ij} \ln \widetilde{p}^y_i.$$

In matrix form, these are

$$\Theta \ln \widetilde{p}^y = 0,$$

$$\ln \widetilde{p}^y = \Gamma_d \ln \widetilde{p}^v + (I - \Gamma_d) \ln \widetilde{p}^m + \alpha_d \Gamma_d \Omega' \Xi' \ln \widetilde{A},$$

$$\ln \widetilde{p}^m = \Phi' \ln \widetilde{p}^y,$$

$$\ln \widetilde{p}^x = \Omega' \ln \widetilde{p}^y.$$

The definition of value added gives

$$\ln \widetilde{v}_{j,t} = \alpha_j \ln \left(\frac{\widetilde{k}_j}{\alpha_j} \right),$$

while the normalized Euler equation implies

$$\widetilde{p}^x_j \left[1 - (1-\delta_j)\beta \prod_{i=1}^{n} \prod_{k=1}^{n} (1+\overline{g}_k)^{\xi_{ki}(\theta_i - \omega_{ij})} \right] = \beta \alpha_j \frac{\widetilde{p}^v_j \widetilde{v}_j}{\widetilde{k}_j} \left(\prod_{i=1}^{n} \prod_{k=1}^{n} (1+\overline{g}_k)^{\xi_{ki}\theta_i} \right)$$

so that

$$\frac{\widetilde{k}_j}{\alpha_j} = \left(\frac{\widetilde{p}^v_j \widetilde{v}_j}{\widetilde{p}^x_j} \right) \left[\frac{\beta \prod_{i=1}^{n} \prod_{k=1}^{n} (1+\overline{g}_k)^{\xi_{ki}\theta_i}}{1 - \beta \left[\prod_{i=1}^{n} \prod_{k=1}^{n} (1+\overline{g}_k)^{\xi_{ki}(\theta_i - \omega_{ij})} \right] (1-\delta_j)} \right].$$

Combining these expressions yields

$$(1-\alpha_j) \ln \left(\widetilde{p}^v_j \widetilde{v}_j \right) = \ln \widetilde{p}^v_j - \alpha_j \ln \widetilde{p}^x_j + \alpha_j \ln \Delta_j,$$

where $\Delta_j = \dfrac{\beta \prod_{i=1}^{n}\prod_{k=1}^{n}(1+\bar{g}_k)^{\xi_{ki}\theta_i}}{1-\beta\left[\prod_{i=1}^{n}\prod_{k=1}^{n}(1+\bar{g}_k)^{\xi_{ki}(\theta_i-\omega_{ij})}\right](1-\delta_j)}$. In matrix form, we then have

$$(I-\alpha_d)\ln(\widetilde{p}^v.\times\widetilde{v}) = \ln\widetilde{p}^v - \alpha_d\ln\widetilde{p}^x + \alpha_d\ln\Delta,$$

where $(\widetilde{p}^v.\times\widetilde{v}) = \{\widetilde{p}^v_j\widetilde{v}_j\}$ and $\Delta = \{\Delta_j\}$.

Using the normalized price indices

$$\begin{aligned}(I-\alpha_d)\ln(\widetilde{p}^v.\times\widetilde{v}) &= \Gamma_d^{-1}(I-(I-\Gamma_d)\Phi')\ln\widetilde{p}^y - \alpha_d\Omega'\ln\widetilde{p}^y + \alpha_d\ln\Delta - \alpha_d\Omega'\Xi'\ln\widetilde{A}\\ &= \Pi\ln\widetilde{p}^y - \alpha_d\Omega'\ln\widetilde{p}^y + \alpha_d\ln\Delta - \alpha_d\Omega'\Xi'\ln\widetilde{A}\end{aligned}$$

where $\Pi = \Gamma_d^{-1}(I-(I-\Gamma_d)\Phi')$. It follows that

$$\ln\widetilde{p}^y = (\Pi-\alpha_d\Omega')^{-1}\left[(I-\alpha_d)\ln(\widetilde{p}^v.\times\widetilde{v}) - \alpha_d\ln\Delta + \alpha_d\Omega'\Xi'\ln\widetilde{A}\right].$$

From the normalized resource constraint, we have that

$$\widetilde{p}^y_j\widetilde{y}_j = \widetilde{p}^y_j\widetilde{c}_j + \sum_{i=1}^{n}\widetilde{p}^y_j\widetilde{m}_{ji} + \sum_{i=1}^{n}\widetilde{p}^y_j\widetilde{x}_{ji},$$

or alternatively

$$\frac{\widetilde{p}^v_j\widetilde{v}_j}{\gamma_j} = \theta_j\widetilde{C} + \sum_{i=1}^{n}\phi_{ji}(1-\gamma_i)\frac{\widetilde{p}^v_i\widetilde{v}_i}{\gamma_i} + \sum_{i=1}^{n}\omega_{ji}\left[1-(1-\delta_i)\prod_{k=1}^{n}\prod_{\ell=1}^{n}(1+\bar{g}_\ell)^{-\omega_{ki}\xi_{\ell k}}\right]\Delta_i\alpha_i\widetilde{p}^v_i\widetilde{v}_i.$$

In matrix form, this last expression becomes

$$\Gamma_d^{-1}(\widetilde{p}^v.\times\widetilde{v}) = \Theta'\widetilde{C} + \Phi(I-\Gamma_d)\Gamma_d^{-1}(\widetilde{p}^v.\times\widetilde{v}) + \Omega G_d\alpha_d(\widetilde{p}^v.\times\widetilde{v}),$$

where $G_d = diag\left\{\left[1-(1-\delta_i)\prod_{k=1}^{n}\prod_{\ell=1}^{n}(1+\bar{g}_\ell)^{-\omega_{ki}\xi_{\ell k}}\right]\Delta_i\right\}$. Therefore,

$$\begin{aligned}\frac{(\widetilde{p}^v.\times\widetilde{v})}{\widetilde{C}} &= [(I-\Phi(I-\Gamma_d))\Gamma_d^{-1} - \Omega G_d\alpha_d]^{-1}\Theta'\\ &= (\Pi' - \Omega G_d\alpha_d)^{-1}\Theta'\\ &\equiv \psi.\end{aligned}$$

Taking logs,

$$\ln\left(\widetilde{p}^{v}. \times \widetilde{v}\right) = \ln \psi + \mathbf{1} \ln \widetilde{C}, \tag{9}$$

and, substituting this expression into the equation for gross output prices, we obtain

$$\ln \widetilde{p}^{y} = (\Pi - \alpha_d \Omega')^{-1} \left[(I - \alpha_d)(\ln \psi + \mathbf{1} \ln \widetilde{C}) - \alpha_d \ln \Delta + \alpha_d \Omega' \Xi' \ln \widetilde{A} \right].$$

Recall that the normalized ideal consumption price index implies $\Theta \ln \widetilde{p}^{y} = 0$. It follows that

$$0 = \Theta(\Pi - \alpha_d \Omega')^{-1} \left[(I - \alpha_d)(\ln \psi + \mathbf{1} \ln \widetilde{C}) - \alpha_d \ln \Delta + \alpha_d \Omega' \Xi' \ln \widetilde{A} \right],$$

which gives aggregate consumption as a function of model parameters only,

$$\ln \widetilde{C} = \frac{\Theta(\Pi - \alpha_d \Omega')^{-1} \left[\alpha_d \ln \Delta - \alpha_d \Omega' \Xi' \ln \widetilde{A} - (I - \alpha_d) \ln \psi \right]}{\Theta(\Pi - \alpha_d \Omega')^{-1} (I - \alpha_d) \mathbf{1}}. \tag{10}$$

Note: $\Theta(\Pi - \alpha_d \Omega')^{-1}(I - \alpha_d)\mathbf{1} = 1$. Therefore,

$$\ln \widetilde{C} = \Theta(\Pi - \alpha_d \Omega')^{-1} \left[\alpha_d \ln \Delta - \alpha_d \Omega' \Xi' \ln \widetilde{A} - (I - \alpha_d) \ln \psi \right].$$

Normalized GDP is then given by

$$\widetilde{V} = \mathbf{1}' \left(\widetilde{p}^{v}. \times \widetilde{v}\right) = \mathbf{1}' \psi \widetilde{C}.$$

All other normalized prices and allocations, as well as shares, are functions of the model's parameters and can now be computed recursively.

2.5.3 Steady State Sectoral Value Added Shares in GDP

The vector of sectoral value added shares or influence vector is given by

$$s^{v} = \frac{[(I - \Phi(I - \Gamma_d))\Gamma_d^{-1} - \Omega G_d \alpha_d]^{-1} \Theta'}{\mathbf{1}'[(I - \Phi(I - \Gamma_d))\Gamma_d^{-1} - \Omega G_d \alpha_d]^{-1} \Theta'},$$

where

$$G_d = diag \left\{ \left[1 - (1 - \delta_i) \prod_{k=1}^{n} \prod_{\ell=1}^{n} (1 + \overline{g}_{\ell})^{-\omega_{ki} \xi_{\ell k}} \right] \Delta_i \right\},$$

with $\Delta_i = \frac{\beta \prod_{j=1}^{n}\prod_{k=1}^{n}(1+\overline{g}_k)^{\theta_j \xi_{kj}}}{1-\beta(1-\delta_i)\left[\prod_{j=1}^{n}\prod_{k=1}^{n}(1+\overline{g}_k)^{\xi_{kj}(\theta_j-\omega_{ji})}\right]}$.

Consider the case where no sector is growing in the long-run, $\overline{g}_j = 0 \ \forall j$, and where elements of A_t are instead stationary in levels with steady state values given by the vector A. Then, detrending is immaterial and the model's steady state may be solved using the set of equilibrium conditions in section 2.3 directly. In that steady state,

$$s^v = \frac{[(I - \Phi(I - \Gamma_d))\Gamma_d^{-1} - \Omega\Delta_d\delta_d\alpha_d]^{-1}\Theta'}{\mathbf{1}'[(I - \Phi(I - \Gamma_d))\Gamma_d^{-1} - \Omega\Delta_d\delta_d\alpha_d]^{-1}\Theta'},$$

where $\Delta_d = diag\left(\frac{\beta}{1-\beta(1-\delta_j)}\right)$, and

$$\frac{d\ln V}{d\ln A} = \frac{d\ln C}{d\ln A} = \frac{([I - \Phi(I - \Gamma_d)]\Gamma_d^{-1} - \Omega\alpha_d)^{-1}\Theta'}{\mathbf{1}'(I - \alpha_d)([I - \Phi'(I - \Gamma_d)]\Gamma_d^{-1} - \Omega\alpha_d)^{-1}\Theta'}.$$

Therefore, in the limit where $\beta \to 1$, so that $\left(\frac{\beta}{1-\beta(1-\delta_j)}\right)\delta_j \to 1$,

$$\frac{d\ln V}{d\ln A_j} = \eta s_j^v,$$

where η is an adjustment factor related to the inverse mean of employment shares in each sector. When sectors use capital with the same intensity, $\alpha_j = \alpha \ \forall j$, $\eta = \frac{1}{1-\alpha}$.

2.5.4 Details of Steady State Calculations

step 1. Compute $G_d = diag\left\{\left[1 - (1-\delta_i)\prod_{k=1}^{n}\prod_{\ell=1}^{n}(1+\overline{g}_\ell)^{-\omega_{ki}\xi_{\ell k}}\right]\Delta_i\right\}$. Let

$$T_i = \prod_{k=1}^{n}\prod_{\ell=1}^{n}(1+\overline{g}_\ell)^{-\omega_{ki}\xi_{\ell k}}$$

so that

$$\ln T_i = -\sum_{k=1}^{n}\sum_{\ell=1}^{n}\omega_{ki}\xi_{\ell k}\ln(1+\overline{g}_\ell),$$

or, in matrix notation,

$$\ln T = -\Omega'\Xi'\ln(1+\overline{g}).$$

In matrix form,

$$\{1-(1-\delta_i)\prod_{k=1}^{n}\prod_{\ell=1}^{n}(1+\overline{g}_\ell)^{-\omega_{ki}\xi_{\ell k}}\} = \mathbf{1}-(I-\delta_d)e^{\ln(T)}.$$

Next, consider the term,

$$\Delta_i = \frac{\beta\prod_{j=1}^{n}\prod_{k=1}^{n}(1+\overline{g}_k)^{\theta_j\xi_{kj}}}{1-\beta(1-\delta_i)\left[\prod_{j=1}^{n}\prod_{k=1}^{n}(1+\overline{g}_k)^{\xi_{kj}(\theta_j-\omega_{ji})}\right]},$$

and let T_{Δ_1} denote the double product in the numerator,

$$T_{\Delta_1,i} = \prod_{j=1}^{n}\prod_{k=1}^{n}(1+\overline{g}_k)^{\xi_{kj}\theta_j},$$

so that

$$\ln T_{\Delta_1,i} = \sum_{j=1}^{n}\sum_{k=1}^{n}\theta_j\xi_{kj}\ln(1+\overline{g}_k).$$

We write this term in matrix notation as

$$\ln T_{\Delta_1} = \Theta^r\Xi'\ln(1+\overline{g}),$$

where

$$\Theta^r = \begin{bmatrix} \theta_1 & ... & \theta_n \\ ... & ... & ... \\ \theta_1 & ... & \theta_n \end{bmatrix}.$$

Observe that $\ln T_{\Delta_1}$ is an $N\times 1$ vector where each element is the same for all i. Let T_{Δ_2} denote the double product in the denominator

$$T_{\Delta_2,i} = \prod_{j=1}^{n}\prod_{k=1}^{n}(1+\overline{g}_k)^{\xi_{kj}(\theta_j-\omega_{ji})},$$

so that

$$\ln T_{\Delta_2,i} = \sum_{j=1}^{n}\sum_{k=1}^{n}\xi_{kj}\left(\theta_j-\omega_{ji}\right)\ln(1+\overline{g}_k)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \xi_{kj}\theta_j \ln(1+\overline{g}_k) - \sum_{j=1}^{n} \sum_{k=1}^{n} \omega_{ji}\xi_{kj} \ln(1+\overline{g}_k).$$

Then, in matrix notation, we have that

$$\ln T_{\Delta_2} = [\Theta^r \Xi' - \Omega' \Xi'] \ln(1+\overline{g}).$$

It follows that

$$\{\Delta_i\} = \beta e^{\ln\left(T_{\Delta_1}\right)} ./ \left(\mathbf{1} - \beta(I-\delta_d) e^{\ln\left(T_{\Delta_2}\right)}\right).$$

Moreover

$$G_d = diag\left\{\left[\mathbf{1} - (1-\delta_d)e^{\ln(T_2)}\right] . \times \{\Delta_i\}\right\}.$$

step 2. Compute

$$\psi = \left(\Pi' - \Omega G_d \alpha_d\right)^{-1} \Theta'.$$

step 3. Compute aggregate consumption, equation (10).

$$\ln \widetilde{C} = \frac{\Theta(\Pi - \alpha_d \Omega')^{-1}\left[\alpha_d \ln \Delta - \alpha_d \Omega' \Xi' \ln \widetilde{A} - (I-\alpha_d)\ln \psi\right]}{\Theta(\Pi - \alpha_d \Omega')^{-1}(I-\alpha_d)\mathbf{1}}.$$

step 4. Compute aggregate GDP, $G\widetilde{D}P = \mathbf{1}'\left(\widetilde{p}^v . \times \widetilde{v}\right) = \mathbf{1}'\psi\widetilde{C}$.

step 5. Compute nominal sectoral value added, $(\widetilde{p}^v . \times \widetilde{v}) = \psi\widetilde{C}$.

step 6. Compute nominal sectoral gross output, $\{\widetilde{p}_j^y \widetilde{y}_j\} = \Gamma_d^{-1}\left(\widetilde{p}^v . \times \widetilde{v}\right)$.

step 7. Compute nominal sectoral consumption. $\{\widetilde{p}_j^y \widetilde{c}_j\} = \Theta' \widetilde{C}$.

step 8. Compute nominal material inputs,

$$\begin{bmatrix} \widetilde{p}^y_1\widetilde{m}_{11} & ... & \widetilde{p}^y_1\widetilde{m}_{1N} \\ ... & & ... \\ \widetilde{p}^y_N\widetilde{m}_{N1} & ... & \widetilde{p}^y_N\widetilde{m}_{NN} \end{bmatrix} = \begin{bmatrix} \phi_{11} & ... & \phi_{1N} \\ ... & & ... \\ \phi_{N1} & ... & \phi_{NN} \end{bmatrix} \begin{bmatrix} (1-\gamma_1) & ... & 0 \\ ... & & ... \\ 0 & ... & (1-\gamma_N) \end{bmatrix}$$
$$\times \begin{bmatrix} \gamma_1 & ... & 0 \\ ... & & ... \\ 0 & ... & \gamma_N \end{bmatrix}^{-1} \begin{bmatrix} (\widetilde{p}^v_1. \times \widetilde{v}_1) & ... & 0 \\ ... & & ... \\ 0 & ... & (\widetilde{p}^v_n. \times \widetilde{v}_n) \end{bmatrix}.$$

step 9. Compute nominal sectoral investment,

$$\begin{bmatrix} \widetilde{p}^y_1\widetilde{x}_{11} & ... & \widetilde{p}^y_1\widetilde{x}_{1N} \\ ... & & ... \\ \widetilde{p}^y_N\widetilde{x}_{N1} & ... & \widetilde{p}^y_N\widetilde{x}_{NN} \end{bmatrix} = \begin{bmatrix} \omega_{11} & ... & \omega_{1N} \\ ... & & ... \\ \omega_{N1} & ... & \omega_{NN} \end{bmatrix} \begin{bmatrix} \alpha_1 & ... & 0 \\ ... & & ... \\ 0 & ... & \alpha_N \end{bmatrix}$$
$$\times \begin{bmatrix} G_{d,1} & ... & 0 \\ ... & & ... \\ 0 & ... & G_{d,N} \end{bmatrix} \begin{bmatrix} (\widetilde{p}^v_1. \times \widetilde{v}_1) & ... & 0 \\ ... & & ... \\ 0 & ... & (\widetilde{p}^v_n. \times \widetilde{v}_n) \end{bmatrix}.$$

step 10. Compute gross output shares using nominal values above,

$$s_{c_j} = \frac{\widetilde{c}_j}{\widetilde{y}_j} = \frac{\widetilde{p}^y_j\widetilde{c}_j}{\widetilde{p}^y_j\widetilde{y}_j},\ s_{m_{ji}} = \frac{\widetilde{m}_{ji}}{\widetilde{y}_j} = \frac{\widetilde{p}^y_j\widetilde{m}_{ji}}{\widetilde{p}^y_j\widetilde{y}_j},\ \text{and}\ s_{x_{ji}} = \frac{\widetilde{x}_{ji}}{\widetilde{y}_j} = \frac{\widetilde{p}^y_j\widetilde{x}_{ji}}{\widetilde{p}^y_j\widetilde{y}_j}.$$

step 11. Compute gross output prices and value added prices,

$$\ln \widetilde{p}^y = (\Pi - \alpha_d\Omega')^{-1}\left[(I-\alpha_d)\ln(\widetilde{p}^v. \times \widetilde{v}) - \alpha_d \ln \Delta + \alpha_d\Omega'\Xi' \ln \widetilde{A}\right],$$
$$\ln \widetilde{p}^v = \Pi \ln \widetilde{p}^y - \alpha_d\Omega'\Xi' \ln \widetilde{A}.$$

step 12. Compute steady state (normalized) consumption, capital, and investment,

$$\widetilde{c}_j = \frac{\widetilde{p}^y_j\widetilde{c}_j}{\widetilde{p}^y_j},$$

and

$$\ln \widetilde{p}_j^v \widetilde{v}_j - \ln \widetilde{p}_j^v = \alpha_j \ln\left(\frac{\widetilde{k}_j}{\alpha_j}\right)$$
$$\Rightarrow \widetilde{k}_j = \alpha_j e^{\frac{1}{\alpha_j}\left(\ln \widetilde{p}_j^v \widetilde{v}_j - \ln \widetilde{p}_j^v\right)},$$

while investment is given by

$$\widetilde{x}_j = \left(1 - (1-\delta_j)\prod_{i=1}^{n}\prod_{k=1}^{n}(1+\overline{g}_k)^{-\omega_{ij}\xi_{ki}}\right)\widetilde{k}_j,$$

or in terms of the notation introduced earlier in step 1,

$$\widetilde{x} = \left(\mathbf{1} - (I-\delta_d)e^{\ln(T)}\right) . \times \widetilde{k}.$$

2.6 Balanced Growth at the Sectoral and Aggregate Level

As derived above, sectoral value added growth is given by

$$\Delta \ln \mu_t^v = [I + \alpha_d \Omega' \Xi']\, \overline{g}_a,$$

where

$$\Xi' = (I - \Gamma_d \alpha_d \Omega' - (I - \Gamma_d)\Phi')^{-1}\, \Gamma_d$$

with j^{th} element $\overline{g}_j + \sum_{i=1}^{n} \alpha_j \omega_{ij} \sum_{k=1}^{n} \xi_{ki} \overline{g}_k$, and where $\overline{g}_a$ denotes the vector of productivity growth rates, $\Delta \ln A_t$.

The Divisia index describing aggregate GDP growth is

$$\Delta \ln V_t = \sum_{j=1}^{n} s_{j,t}^v \Delta \ln v_{j,t},$$

where $\Delta \ln v_{j,t}$ denotes the growth rate of real value added in sector j, and $s_{j,t}$, is the share of sector j in nominal value added,

$$s_{j,t}^v = \frac{p_{j,t}^v v_{j,t}}{\sum_{j=1}^{n} p_{j,t}^v v_{j,t}}.$$

Hence, the balanced growth rate of real aggregate GDP is

$$\Delta \ln V_t = \sum_{j=1}^{n} s_j^v \left[\overline{g}_j + \sum_{i=1}^{n} \alpha_j \omega_{ij} \sum_{k=1}^{n} \xi_{ki} \overline{g}_k \right]. \tag{11}$$

Using the expressions for the normalized variables, we have

$$\begin{aligned} p_{j,t}^v v_{j,t} &= \left(\frac{\widetilde{p}_{j,t}^v \prod_{k=1}^{n} \mu_{k,t}^{\theta_k}}{A_{j,t} \prod_{k=1}^{n} \mu_{k,t-1}^{\omega_{kj}\alpha_j}} \right) \left(\widetilde{v}_{j,t} A_{j,t} \prod_{i=1}^{n} \mu_{i,t-1}^{\omega_{ij}\alpha_j} \right) \\ &= \widetilde{p}_{j,t}^v \widetilde{v}_{j,t} \prod_{k=1}^{n} \mu_{k,t}^{\theta_k}, \end{aligned}$$

so that

$$s_{j,t}^v = \frac{\widetilde{p}_{j,t}^v \widetilde{v}_{j,t} \prod_{k=1}^{n} \mu_{k,t}^{\theta_k}}{\sum_{j=1}^{n} \widetilde{p}_{j,t}^v \widetilde{v}_{j,t} \prod_{k=1}^{n} \mu_{k,t}^{\theta_k}} = \frac{\widetilde{p}_{j,t}^v \widetilde{v}_{j,t}}{\sum_{j=1}^{n} \widetilde{p}_{j,t}^v \widetilde{v}_{j,t}}.$$

Then, along a balanced growth path where $\widetilde{p}_{j,t}^v \widetilde{v}_{j,t}$ is constant in every sector, nominal shares in value added are also constant and given by

$$s_j^v = \frac{\widetilde{p}_j^v \widetilde{v}_j}{\mathbf{1}' \left(\widetilde{p}^v . \times \widetilde{v} \right)}, \tag{12}$$

where $(\widetilde{p}^v . \times \widetilde{v})$ is obtained through equations (9) and (10).

2.7 Dynamics of the System

2.7.1 Linearized Equations:

The "hat" notation stands for percent deviation from steady state for the normalized variables (i.e. for some variable, x_t, and with some abuse of notation, $\widehat{x}_t = \widehat{\widetilde{x}}_t = \ln(\widetilde{x}_t / x^*)$).

The linearized version of the resource constraint is

$$s_{c_j} \widehat{c}_{j,t} + \sum_{i=1}^{n} s_{m_{ji}} \widehat{m}_{ji,t} + \sum_{i=1}^{n} s_{x_{ji}} \widehat{x}_{ji,t} = \widehat{y}_{j,t},$$

where $s_{c_j} = \widetilde{c}_j / \widetilde{y}_j$, $s_{m_{ji}} = \widetilde{m}_{ji} / \widetilde{y}_j$, and $s_{x_{ji}} = \widetilde{x}_{ji} / \widetilde{y}_j$. The production of investment goods in linearized form becomes

$$\widehat{x}_{j,t} = \sum_{i=1}^{n} \omega_{ij} \widehat{x}_{ij,t}.$$

The capital accumulation equation in linearized form becomes

$$\widehat{k}_{j,t+1} = \kappa_{x_j} x_{j,t} + (1-\delta_j)\kappa_{k_j} \widehat{k}_{j,t} - (1-\delta_j)\kappa_{k_j} \sum_{i=1}^{n}\sum_{k=1}^{n} \omega_{ij} \xi_{ki} \widehat{A}_{k,t},$$

where $\kappa_{x_j} = \widetilde{x}_j / \widetilde{k}_j$, $\kappa_{k_j} = \prod_{i=1}^{n}\prod_{k=1}^{n} (1+\overline{g}_k)^{-\omega_{ij}\xi_{ki}}$. Real value added in linearized form becomes

$$\widehat{v}_{j,t} = \alpha_j \widehat{k}_{j,t}.$$

The composite material bundle used in sector j takes the form

$$\widehat{m}_{j,t} = \sum_{i=1}^{n} \phi_{ij} \widehat{m}_{ij,t}.$$

Gross output production in linearized form is

$$\widehat{y}_{j,t} = \gamma_j \widehat{v}_{j,t} + (1-\gamma_j)\widehat{m}_{j,t} - \gamma_j \alpha_j \sum_{i=1}^{n}\sum_{k=1}^{n} \omega_{ij} \xi_{ki} \widehat{A}_{k,t}.$$

Aggregate consumption in linearized form is

$$\widehat{C}_t = \sum_{j=1}^{n} \theta_j \widehat{c}_{j,t}.$$

In linearized form, the first-order conditions are:

$$\widehat{C}_t - \widehat{c}_{j,t} = \widehat{p}^{y}_{j,t},$$

$$\widehat{p}^{y}_{j,t} + \widehat{y}_{j,t} - \widehat{v}_{j,t} = \widehat{p}^{v}_{j,t},$$

$$\widehat{p}^{y}_{j,t} + \widehat{y}_{j,t} - \widehat{m}_{j,t} = \widehat{p}^{m}_{j,t},$$

$$\widehat{p}^{m}_{j,t} + \widehat{m}_{j,t} - \widehat{m}_{ij,t} = \widehat{p}^{y}_{i,t},$$

$$\widehat{p}^{y}_{i,t} + \widehat{x}_{ij,t} - \widehat{x}_{j,t} = \widehat{p}^{x}_{j,t},$$

and

$$\begin{aligned}\widehat{p}^x_{j,t} &= \sum_{i=1}^n\sum_{k=1}^n \xi_{ki}(\theta_i-\omega_{ij})\mathbb{E}_t\left(\widehat{A}_{k,t+1}\right)\\ &+\pi_{1,j}\mathbb{E}_t\left[\widehat{p}^v_{j,t+1}+\widehat{v}_{j,t+1}-\widehat{k}_{j,t+1}+\sum_{i=1}^n\sum_{k=1}^n\xi_{ki}\omega_{ij}\widehat{A}_{k,t+1}\right]\\ &+\pi_{2,j}\mathbb{E}_t\left(\widehat{p}^x_{j,t+1}\right),\end{aligned}$$

where $\pi_{1,j}=1-\pi_{2,j}$ and $\pi_{2,j}=\beta(1-\delta_j)\prod_{i=1}^n\prod_{k=1}^n(1+\overline{g}_k)^{\xi_{ki}(\theta_i-\omega_{ij})}$.

Finally, the linearized driving processes are given by:

$$\begin{aligned}\widehat{A}_{j,t} &= \ln\left(\frac{\widetilde{A}_{j,t}}{1+\overline{g}_j}\right)=\ln\left(\widetilde{A}_{j,t}\right)-\overline{g}_j\\ &=\Delta\ln z_{j,t}-\lambda^z_{j,\tau}g^z_c-g^z_j+(1-\alpha_j)\left(\Delta\ln\ell_{j,t}-\lambda^\ell_{j,\tau}g^\ell_c-g^\ell_j\right)\\ &=\lambda^z_{j,\tau}\underbrace{\left(\tau^z_{c,t}-g^z_c\right)}_{\widehat{\tau}^z_{c,t}}+\underbrace{\tau^z_{j,t}-g^z_j}_{\widehat{\tau}^z_{j,t}}+\lambda^z_{j,\varepsilon}\varepsilon_{c,t}+\varepsilon^z_{j,t}\\ &+(1-\alpha_j)\left(\lambda^\ell_{j,\tau}\underbrace{\left(\tau^\ell_{c,t}-g^\ell_c\right)}_{\widehat{\tau}^\ell_{c,t}}+\underbrace{\tau^\ell_{j,t}-g^\ell_j}_{\widehat{\tau}^\ell_{j,t}}+\lambda^\ell_{j,\varepsilon}\varepsilon_{c,t}+\varepsilon^\ell_{j,t}\right)\end{aligned}$$

$$\widehat{\tau}^z_{c,t}=\rho\widehat{\tau}^z_{c,t-1}+\eta^z_{c,t},$$

$$\widehat{\tau}^z_{j,t}=\rho\widehat{\tau}^z_{j,t-1}+\eta^z_{j,t},$$

$$\widehat{\tau}^\ell_{c,t}=\rho\widehat{\tau}^\ell_{c,t-1}+\eta^\ell_{c,t},$$

$$\widehat{\tau}^\ell_{j,t}=\rho\widehat{\tau}^\ell_{j,t-1}+\eta^\ell_{j,t},$$

Define $\widehat{c}_t=(\widehat{c}_{1,t}...,\widehat{c}_{n,t})'$, $\widehat{y}_t=(\widehat{y}_{1,t}...,\widehat{y}_{n,t})'$, $\widehat{x}_t=(\widehat{x}_{1,t}...,\widehat{x}_{n,t})'$, $\widehat{m}_t=(\widehat{m}_{1,t}...,\widehat{m}_{n,t})'$, and analogous vectors for prices. In addition, define $\widehat{M}_t=(\widehat{m}_{11,t},\widehat{m}_{12,t},...,\widehat{m}_{nn,t})'$ and $\widehat{X}_t=(\widehat{x}_{11,t},\widehat{x}_{12,t},...,\widehat{x}_{nn,t})'$. Then, in matrix form, the above system of equations reads as

$$S_c\widehat{c}_t+S_m\widehat{M}_t+S_x\widehat{X}_t=\widehat{y}_t,$$

where

$$S_c=\begin{bmatrix}s_{c_1} & ... & 0\\ \vdots & \ddots & \vdots\\ 0 & ... & s_{c_n}\end{bmatrix},\ S_m=\begin{bmatrix}s_{m_{11}} & s_{m_{12}} & ... & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & ... & s_{m_{n(n-1)}} & s_{m_{nn}}\end{bmatrix},$$

and

$$S_x = \begin{bmatrix} s_{x_{11}} & s_{x_{12}} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s_{x_{n(n-1)}} & s_{x_{nn}} \end{bmatrix},$$

$$\widehat{x}_t = \widetilde{\Omega}\widehat{X}_t,$$

where

$$\widetilde{\Omega}_{n\times n^2} = \begin{bmatrix} \omega_{11} & 0 & \dots & 0 & \omega_{21} & 0 & \dots & 0 & \dots & \omega_{n1} & 0 & \dots & 0 \\ 0 & \omega_{12} & \dots & 0 & 0 & \omega_{22} & \dots & 0 & \dots & 0 & \omega_{n2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \dots & \dots & \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & \omega_{1n} & 0 & 0 & \dots & \omega_{2n} & \dots & 0 & 0 & \dots & \omega_{nn} \end{bmatrix},$$

$$\widehat{k}_{t+1} = \kappa_x\widehat{x}_t + (I-\delta_d)\kappa_k\widehat{k}_t - (I-\delta_d)\kappa_k\Omega^{'}\Xi'\widehat{A}_t$$

$$\widehat{v}_t = \alpha_d\widehat{k}_t,$$

$$\widehat{m}_t = \widetilde{\Phi}\widehat{M_t},$$

where

$$\widetilde{\Phi}_{n\times n^2} = \begin{bmatrix} \phi_{11} & 0 & \dots & 0 & \phi_{21} & 0 & \dots & 0 & \dots & \phi_{n1} & 0 & \dots & 0 \\ 0 & \phi_{12} & \dots & 0 & 0 & \phi_{22} & \dots & 0 & \dots & 0 & \phi_{n2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \dots & \dots & \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & \phi_{1n} & 0 & 0 & \dots & \phi_{2n} & \dots & 0 & 0 & \dots & \phi_{nn} \end{bmatrix},$$

$$\widehat{y}_t = \Gamma_d\widehat{v}_t + (I-\Gamma_d)\widehat{m}_t - \Gamma_d\alpha_d\Omega'\Xi'\widehat{A}_t,$$

$$\widehat{C}_t = \Theta\widehat{c}_t,$$

where $\Theta = (\theta_1, ..., \theta_n)$,

$$\mathbf{1}_n\widehat{C}_t - \widehat{c}_t = \widehat{p}_t^y,$$

$$\widehat{p}_t^y + \widehat{y}_t - \widehat{v}_t = \widehat{p}_t^v,$$

$$\widehat{p}_t^y + \widehat{y}_t - \widehat{m}_t = \widehat{p}_t^m,$$

$$\widehat{M_t} = \mathbf{M}_1\widehat{p}_t^m - \mathbf{M}_2\widehat{p}_t^y + \mathbf{M}_1\widehat{m}_t,$$

where

$$\mathbf{M}_1 = 1_{n\times 1}\otimes I \text{ and } \mathbf{M}_2 = I\otimes 1_{n\times 1},$$

$$\widehat{X}_t = \mathbf{M}_1 \widehat{p}_t^x - \mathbf{M}_2 \widehat{p}_t^y + \mathbf{M}_1 \widehat{x}_t,$$

and finally the Euler equations,

$$\begin{aligned}
\widehat{p}_t^x &= (\Theta^r - \Omega')\Xi' \mathbb{E}_t \widehat{A}_{t+1} \\
&\quad + (I - \pi_2)\, \mathbb{E}_t \left[\widehat{p}_{t+1}^v + \widehat{v}_{t+1} - \widehat{k}_{t+1} + \Omega' \Xi' \widehat{A}_{t+1} \right] \\
&\quad + \pi_2 \mathbb{E}_t \widehat{p}_{t+1}^x,
\end{aligned}$$

where

$$\pi_2 = \begin{bmatrix} \pi_{2,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \pi_{2,n} \end{bmatrix}.$$

2.7.2 System Reduction:

We now substitute out the flow variables in order to arrive at a dynamic system of equations expressed in terms of the basic state and co-state variables only.

Starting with the equation for investment, we have

$$\widehat{x}_t = \widetilde{\Omega} \left(\mathbf{M}_1 \widehat{p}_t^x - \mathbf{M}_2 \widehat{p}_t^y + \mathbf{M}_1 \widehat{x}_t \right),$$

where $\widetilde{\Omega}\mathbf{M}_1 = I$ and $\widetilde{\Omega}\mathbf{M}_2 = \Omega'$. Then,

$$\widehat{p}_t^x = \Omega' \widehat{p}_t^y.$$

Similarly, from the equations governing the choice of material inputs,

$$\widehat{m}_t = \widetilde{\Phi} \left(\mathbf{M}_1 \widehat{p}_t^m - \mathbf{M}_2 \widehat{p}_t^y + \mathbf{M}_1 \widehat{m}_t \right),$$

where $\widetilde{\Phi}\mathbf{M}_1 = I$ and $\widetilde{\Phi}\mathbf{M}_2 = \Phi'$, we have

$$\widehat{p}_t^m = \Phi' \widehat{p}_t^y.$$

The Euler equation then implies

$$\begin{aligned}
\Omega' \widehat{p}_t^y &= (\Theta^r - \Omega' + (I - \pi_2)\, \Omega')\Xi' \mathbb{E}_t \widehat{A}_{t+1} \\
&\quad + (I - \pi_2)\, \mathbb{E}_t \left[\widehat{p}_{t+1}^y + \widehat{y}_{t+1} - \widehat{k}_{t+1} \right] \\
&\quad + \pi_2 \mathbb{E}_t \left[\Omega' \widehat{p}_{t+1}^y \right].
\end{aligned}$$

Observe that $\mathbf{1}_n\Theta = \Theta^r$, and since $\widehat{C}_t = \Theta\widehat{c}_t$, $\mathbf{1}_n\widehat{C}_t - \widehat{c}_t = \widehat{p}_t^y$ gives

$$(\Theta^r - I)\,\widehat{c}_t = \widehat{p}_t^y,$$

so that the Euler equation becomes

$$\begin{aligned}\Omega'\left(\Theta^r - I\right)\widehat{c}_t &= (\Theta^r - \Omega' + \left(I - \pi_2\right)\Omega')\Xi'\mathbb{E}_t\widehat{A}_{t+1}\\ &+ \left(I - \pi_2\right)\mathbb{E}_t\left[\widehat{y}_{t+1} - \widehat{k}_{t+1}\right]\\ &+ \left[\left(I - \pi_2\right) + \pi_2\Omega'\right]\left(\Theta^r - I\right)\mathbb{E}_t\widehat{c}_{t+1}.\end{aligned} \tag{13}$$

From the equation describing production, we have

$$\widehat{y}_t = \Gamma_d\alpha_d\widehat{k}_t + \left(I - \Gamma_d\right)\left(\widehat{p}_t^y + \widehat{y}_t - \widehat{p}_t^m\right) - \Gamma_d\alpha_d\Omega'\Xi'\widehat{A}_t,$$

and using the derivation for $\widehat{p}_t^m$ above,

$$\widehat{y}_t = \alpha_d\widehat{k}_t + \Gamma_d^{-1}(I - \Gamma_d)(I - \Phi')(\Theta^r - I)\widehat{c}_t - \alpha_d\Omega'\Xi'\widehat{A}_t,$$

which we express as

$$\widehat{y}_t = \alpha_d\widehat{k}_t + Q_c\widehat{c}_t - Q_a\widehat{A}_t,$$

with $Q_c = \Gamma_d^{-1}(I - \Gamma_d)(I - \Phi')(\Theta^r - I)$ and $Q_a = \alpha_d\Omega'\Xi'$. Substituting for $\widehat{y}_{t+1}$ in the Euler equation (13) gives

$$\begin{aligned}\Omega'\left(\Theta^r - I\right)\widehat{c}_t &= \left[(\Theta^r - \pi_2\Omega')\Xi' - \left(I - \pi_2\right)Q_a\right]\mathbb{E}_t\widehat{A}_{t+1}\\ &+ \left(I - \pi_2\right)\left[\alpha_d - I\right]\mathbb{E}_t\widehat{k}_{t+1}\\ &+ \left(\left[\left(I - \pi_2\right) + \pi_2\Omega'\right]\left(\Theta^r - I\right) + \left(I - \pi_2\right)Q_c\right)\mathbb{E}_t\widehat{c}_{t+1},\end{aligned} \tag{14}$$

an equation in terms of states, $\widehat{k}$, $\widehat{A}$, and co-states, $\widehat{p}^y = \left(\Theta^r - I\right)\widehat{c}$, only.

Turning to the resource constraint, we have

$$\begin{aligned}\widehat{y}_t &= S_c\widehat{c}_t\\ &+ S_m\left(\mathbf{M}_1\widehat{p}_t^m - \mathbf{M}_2\widehat{p}_t^y + \mathbf{M}_1\widehat{m}_t\right)\\ &+ S_x\left(\mathbf{M}_1\widehat{p}_t^x - \mathbf{M}_2\widehat{p}_t^y + \mathbf{M}_1\widehat{x}_t\right),\end{aligned} \tag{15}$$

where $\widehat{x}_t$ in the capital accumulation satisfies

$$\widehat{x}_t = \kappa_x^{-1}\widehat{k}_{t+1} - (I-\delta_d)\kappa_x^{-1}\kappa_k\widehat{k}_t + (I-\delta_d)\kappa_x^{-1}\kappa_k\Omega'\Xi'\widehat{A}_t.$$

Substituting for $\widehat{x}_t$ in equation (15), and using the fact that $\widehat{p}_t^y + \widehat{y}_t = \widehat{p}_t^m + \widehat{m}_t$ while $\widehat{p}_t^x = \Omega'\widehat{p}_t^y$ and $\widehat{p}_t^m = \Phi'\widehat{p}_t^y$, we obtain

$$\begin{aligned}(I - S_m\mathbf{M}_1)\,\widehat{y}_t &= S_c\widehat{c}_t + S_m\left(\mathbf{M}_1\widehat{p}_t^y - \mathbf{M}_2\widehat{p}_t^y\right) \\ &\quad + S_x\left(\mathbf{M}_1\Omega'\widehat{p}_t^y - \mathbf{M}_2\widehat{p}_t^y + \mathbf{M}_1\left[\kappa_x^{-1}\widehat{k}_{t+1} - (I-\delta_d)\kappa_x^{-1}\kappa_k\widehat{k}_t + (I-\delta_d)\kappa_x^{-1}\kappa_k\Omega'\Xi'\widehat{A}\right]\right),\end{aligned}$$

or

$$\begin{aligned}(I - S_m\mathbf{M}_1)\,\widehat{y}_t &= S_c\widehat{c}_t + \left[S_m\left(\mathbf{M}_1 - \mathbf{M}_2\right) + S_x\left(\mathbf{M}_1\Omega' - \mathbf{M}_2\right)\right]\widehat{p}_t^y \\ &\quad + S_x\mathbf{M}_1\kappa_x^{-1}\widehat{k}_{t+1} - (I-\delta_d)S_x\mathbf{M}_1\kappa_x^{-1}\kappa_k\widehat{k}_t + (I-\delta_d)S_x\mathbf{M}_1\kappa_x^{-1}\kappa_k\Omega'\Xi'\widehat{A_t}.\end{aligned}$$

Substituting for $\widehat{y}_t$ and $\widehat{p}_t^y$ in this last expression, the economy resource constraint becomes

$$\begin{aligned}&(I - S_m\mathbf{M}_1)\left(\alpha_d\widehat{k}_t + Q_c\widehat{c}_t - Q_a\widehat{A}_t\right) \\ &= S_c\widehat{c}_t + \left[S_m\left(\mathbf{M}_1 - \mathbf{M}_2\right) + S_x\left(\mathbf{M}_1\Omega' - \mathbf{M}_2\right)\right]\left(\Theta^r - I\right)\widehat{c}_t \\ &+ S_x\mathbf{M}_1\kappa_x^{-1}\widehat{k}_{t+1} - (I-\delta_d)S_x\mathbf{M}_1\kappa_x^{-1}\kappa_k\widehat{k}_t + (I-\delta_d)S_x\mathbf{M}_1\kappa_x^{-1}\kappa_k\Omega'\Xi'\widehat{A_t},\end{aligned} \tag{16}$$

an equation in terms of $\widehat{k}$, $\widehat{A}$, and $\widehat{c}$ only.

We summarize equations (14) and (16) as

$$\begin{aligned}&\begin{bmatrix} \left[(I-\pi_2) + \pi_2\Omega'\right]\left(\Theta^r - I\right) + \left(I-\pi_2\right)Q_c & \left(I-\pi_2\right)\left[\alpha_d - I\right] \\ 0 & S_x\mathbf{M}_1\kappa_x^{-1}\end{bmatrix}\mathbb{E}_t\begin{bmatrix}\widehat{c}_{t+1} \\ \widehat{k}_{t+1}\end{bmatrix} \\ &= \begin{bmatrix}\Omega'\left(\Theta^r - I\right) & 0 \\ \mathbf{B}_{21} & \mathbf{B}_{22}\end{bmatrix}\begin{bmatrix}\widehat{c}_t \\ \widehat{k}_t\end{bmatrix} \\ &+ \begin{bmatrix} -\left[(\Theta^r - \pi_2\Omega')\Xi' - \left(I-\pi_2\right)Q_a\right] \\ 0\end{bmatrix}\mathbb{E}_t\widehat{A}_{t+1} \\ &+ \begin{bmatrix} 0 \\ -\left(I - S_m\mathbf{M}_1\right)Q_a - (I-\delta_d)S_x\mathbf{M}_1\kappa_x^{-1}\kappa_k\Omega'\Xi'\end{bmatrix}\widehat{A}_t,\end{aligned}$$

where

$$\mathbf{B}_{21} = \left(I - S_m\mathbf{M}_1\right)Q_c - S_c - \left[S_m\left(\mathbf{M}_1 - \mathbf{M}_2\right) + S_x\left(\mathbf{M}_1\Omega' - \mathbf{M}_2\right)\right]\left(\Theta^r - I\right),$$

and

$$\mathbf{B}_{22} = (I - S_m \mathbf{M}_1)\, \alpha_d + (I - \delta_d) S_x \mathbf{M}_1 \kappa_x^{-1} \kappa_k.$$

The dynamics of the above system, along with the driving process,

$$\widehat{A}_t = \widehat{z}_t + (I - \alpha_d)\, \widehat{\ell}_t,$$

$$\widehat{z}_t = \widehat{\tau}_t^z + \varepsilon_t^z + \lambda_\tau^z \widehat{\tau}_{c,t}^z + \lambda_\varepsilon^z \varepsilon_{c,t}^z,$$

$$\widehat{\ell}_t = \widehat{\tau}_t^\ell + \varepsilon_t^\ell + \lambda_\tau^\ell \widehat{\tau}_{c,t}^\ell + \lambda_\varepsilon^\ell \varepsilon_{c,t}^\ell,$$

$$\widehat{\tau}_{c,t}^z = \rho \widehat{\tau}_{c,t-1}^z + \eta_{c,t}^z,$$

$$\widehat{\tau}_t^z = \rho \widehat{\tau}_{t-1}^z + \eta_t^z,$$

$$\widehat{\tau}_{c,t}^\ell = \rho \widehat{\tau}_{c,t-1}^\ell + \eta_{c,t}^\ell,$$

$$\widehat{\tau}_t^\ell = \rho \widehat{\tau}_{t-1}^\ell + \eta_t^\ell,$$

may be solved using standard linear rational expectations solution toolkits, including in this case King and Watson - KW - (2002).

2.8 Solution and Policy Functions in KW (2002)

In the notation of KW (2002), the Markov decision and policy rules take the form,

$$\begin{bmatrix} \widehat{c}_t \\ \widehat{k}_t \\ \widehat{A}_t \\ \widehat{z}_t \\ \widehat{\ell}_t \end{bmatrix} = \begin{bmatrix} \pi_{ck} & \pi_{c\tau^z} & \pi_{c\varepsilon^z} & \pi_{c\tau_c^z} & \tau_{c\varepsilon_c^z} & \pi_{c\tau^\ell} & \pi_{c\varepsilon^\ell} & \pi_{c\tau_c^\ell} & \pi_{c\varepsilon_c^\ell} \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & I & \lambda_\tau^z & \lambda_\varepsilon^z & I - \alpha_d & I - \alpha_d & (I - \alpha_d)\, \lambda_\tau^\ell & (I - \alpha_d)\, \lambda_\varepsilon^\ell \\ 0 & I & I & \lambda_\tau^z & \lambda_\varepsilon^z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & I & \lambda_\tau^\ell & \lambda_\varepsilon^\ell \end{bmatrix} \begin{bmatrix} \widehat{k}_t \\ \delta_{1,t} \\ \delta_{2,t} \\ \delta_{3,t} \\ \delta_{4,t} \\ \delta_{5,t} \\ \delta_{6,t} \\ \delta_{7,t} \\ \delta_{8,t} \end{bmatrix},$$

$$\begin{bmatrix} \widehat{k}_t \\ \delta_{1,t} \\ \delta_{2,t} \\ \delta_{3,t} \\ \delta_{4,t} \\ \delta_{5,t} \\ \delta_{6,t} \\ \delta_{7,t} \\ \delta_{8,t} \end{bmatrix} = \begin{bmatrix} m_k & m_{\tau^z} & m_{\varepsilon^z} & m_{\tau_c^z} & m_{\varepsilon_c^z} & m_{\tau^\ell} & m_{\varepsilon^\ell} & m_{\tau_c^\ell} & m_{\varepsilon_c^\ell} \\ 0 & \rho_{\delta_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{\delta_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_{\delta_5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{\delta_7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{k}_{t-1} \\ \delta_{1,t-1} \\ \delta_{2,t-1} \\ \delta_{3,t-1} \\ \delta_{4,t-1} \\ \delta_{5,t-1} \\ \delta_{6,t-1} \\ \delta_{7,t-1} \\ \delta_{8,t-1} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_t^z \\ \varepsilon_t^z \\ \eta_{c,t}^z \\ \varepsilon_{c,t}^z \\ \eta_t^\ell \\ \varepsilon_t^\ell \\ \eta_{c,t}^\ell \\ \varepsilon_{c,t}^\ell \end{bmatrix},$$

where $\delta_{1,t} = \widehat{\tau}_t^z$, $\delta_{2,t} = \varepsilon_t^z$, $\delta_{3,t} = \widehat{\tau}_{c,t}^z$, $\delta_{4,t} = \varepsilon_{c,t}^z$, $\delta_{5,t} = \widehat{\tau}_t^\ell$, $\delta_{6,t} = \varepsilon_t^\ell$, $\delta_{7,t} = \widehat{\tau}_{c,t}^\ell$, $\delta_{8,t} = \varepsilon_{c,t}^\ell$, and $\rho_{\delta_1} = \rho_{\delta_3} = \rho_{\delta_5} = \rho_{\delta_7} \approx I$.

2.9 Time Series Implications for Sectoral and Aggregate Value Added

We wish to recover the implications of the model for sectoral value added growth, $\Delta \ln v_{j,t}$, and aggregate GDP growth, $\Delta \ln V_t$. Observe that $\Delta \ln \widetilde{k}_{j,t} = \Delta \widehat{k}_{j,t}$, where $\Delta \widehat{k}_{j,t}$ follows from the Markov decision rules, and

$$\Delta \ln \widetilde{v}_{j,t} = \alpha_j \Delta \ln \widetilde{k}_{j,t}.$$

By definition of $\widetilde{v}_{j,t}$,

$$\begin{aligned} \Delta \ln v_{j,t} &= \Delta \ln \widetilde{v}_{j,t} + \Delta \ln A_{j,t} + \sum_{i=1}^{n} \alpha_j \omega_{ij} \Delta \ln \mu_{i,t-1} \\ &= \alpha_j \Delta \widehat{k}_{j,t} + \widehat{A}_{j,t} + \overline{g}_j + \sum_{i=1}^{n} \alpha_j \omega_{ij} \Delta \ln \mu_{i,t-1} \end{aligned}$$

or in vector form,

$$\Delta \ln v_t = \alpha_d \Delta \widehat{k}_t + \widehat{A}_t + \overline{g}_a + \alpha_d \Omega' \Delta \ln \mu_{t-1}.$$

The sector-specific detrending factors, $\mu_{j,t}$, solve

$$\Delta \ln \mu_t = \Xi' \Delta \ln A_t,$$

where

$$\Xi' = \left(I - \Gamma_d \alpha_d \Omega' - (I - \Gamma_d)\Phi'\right)^{-1} \Gamma_d.$$

Then,

$$\Delta \ln v_t = \alpha_d \Delta \widehat{k}_t + \widehat{A}_t + \overline{g}_a + \alpha_d \Omega' \Xi' \left(\widehat{A}_{t-1} + \overline{g}_a\right)$$

or

$$\Delta \ln v_t = \left(I + \alpha_d \Omega' \Xi'\right) \overline{g}_a + \alpha_d \Delta \widehat{k}_t + \widehat{A}_t + \alpha_d \Omega' \Xi' \widehat{A}_{t-1}.$$

When all shocks are set to zero, so that $\Delta \widehat{k}_t = \widehat{A}_t = \widehat{A}_{t-1} = 0$, we recover the sectoral balanced growth paths for value added,

$$\Delta \ln v_t = \left(I + \alpha_d \Omega' \Xi'\right) \overline{g}_a.$$

2.10 Solving the Model When Households Have Imperfect Information

Thus far, the representative household is assumed to have full information with respect to transitory and permanent shocks, as well as between idiosyncratic and common shocks, that affect the economy. In contrast, we now consider an imperfect information case in which the representative household cannot distinguish between permanent and transitory components of exogenous changes to the environment. In this alternative scenario, the household faces an additional filtering problem in which it must infer estimates of these components in deciding how much to consume and save.

With imperfect information, the objective function of the representative household is given by

$$\widetilde{\mathbb{E}}_0 \sum_{t=0}^{\infty} \beta^t C_t,$$

where the 'tilde' over the expectations operator accounts for the fact that the information set does not allow households to distinguish between persistent and transitory shocks. It can be shown that the filtering problem that households then face does not affect the calculation of eigenvalues that determine whether the dynamic equilibrium is saddle-path stable or, alternatively, unique. This filtering problem, however, does lead to different Markov decision rules insofar as expectations of future shocks now involve estimates of the unobserved exogenous states.

In vector form, the composite driving processes for $\hat{z}_t$ and $\hat{\ell}_t$ are

$$\hat{z}_t = \lambda_\tau^z \hat{\tau}_{c,t}^z + \hat{\tau}_t^z + \lambda_\varepsilon^z \varepsilon_{c,t}^z + \varepsilon_t^z,$$

$$\hat{\tau}_{c,t}^z = \rho I_n \hat{\tau}_{c,t-1}^z + \eta_{c,t}^z,$$

$$\hat{\tau}_t^z = \rho I_n \hat{\tau}_{t-1}^z + \eta_t^z,$$

and

$$\hat{\ell}_t = \lambda_\tau^\ell \hat{\tau}_{c,t}^\ell + \hat{\tau}_t^\ell + \lambda_\varepsilon^\ell \varepsilon_{c,t}^\ell + \varepsilon_t^\ell,$$

$$\hat{\tau}_{c,t}^\ell = \rho I_n \hat{\tau}_{c,t-1}^\ell + \eta_{c,t}^\ell,$$

$$\hat{\tau}_t^\ell = \rho I_n \hat{\tau}_{t-1}^\ell + \eta_t^\ell.$$

Given the Gaussian nature of the shocks and the above linear processes, estimates of the unobserved exogenous states, $\{\tau_{c,t}^x, \tau_t^x\}$, $x = z,\ \ell$, are formed using the Kalman filter with observables $\hat{z}_t$ and $\hat{\ell}_t$.

For a given variable $x = z,\ \ell$, the observation error is

$$u_t^x = \lambda_\varepsilon^x \varepsilon_{c,t}^x + \varepsilon_t^x,$$

with the observation equation written as

$$\hat{x}_t = \underbrace{\begin{bmatrix} \lambda_\tau^x & I_n \end{bmatrix}}_{H} \begin{bmatrix} \hat{\tau}_{c,t}^x \\ \hat{\tau}_t^x \end{bmatrix} + u_t^x,\ \ u_t^x \sim N\left(0, \underbrace{\Sigma_\varepsilon^x + \sigma_{\varepsilon_c}^x \lambda_\varepsilon^x \lambda_\varepsilon^{x\prime}}_{R^x}\right).$$

The corresponding state equation is

$$\begin{bmatrix} \hat{\tau}_{c,t}^x \\ \hat{\tau}_t^x \end{bmatrix} = \underbrace{\begin{bmatrix} \rho & 0 \\ 0 & \rho I_n \end{bmatrix}}_{F} \begin{bmatrix} \hat{\tau}_{c,t-1}^x \\ \hat{\tau}_{t-1}^x \end{bmatrix} + \begin{bmatrix} \eta_{c,t}^x \\ \eta_t^x \end{bmatrix}, \qquad \begin{bmatrix} \eta_{c,t}^x \\ \eta_t^x \end{bmatrix} \sim N\left(0, Q^x\right).$$

Given a steady-state Kalman gain matrix, K^x, the Kalman updating follows

$$\begin{bmatrix} \hat{\tau}_{c,t|t-1}^x \\ \hat{\tau}_{t|t-1}^x \end{bmatrix} = \begin{bmatrix} \rho & 0 \\ 0 & \rho I_n \end{bmatrix} \begin{bmatrix} \hat{\tau}_{c,t-1|t-1}^x \\ \hat{\tau}_{t-1|t-1}^x \end{bmatrix},$$

$$\begin{bmatrix} \hat{\tau}^x_{c,t|t} \\ \hat{\tau}^x_{t|t} \end{bmatrix} = \begin{bmatrix} \hat{\tau}^x_{c,t|t-1} \\ \hat{\tau}^x_{t|t-1} \end{bmatrix} + K^x a^x_t,$$

where a^x_t is the forecast error,

$$a^x_t = \hat{x}_t - \hat{x}_{t|t-1},$$

and

$$\hat{x}_{t|t-1} = \begin{bmatrix} \lambda^x_\tau & I_n \end{bmatrix} \begin{bmatrix} \hat{\tau}^x_{c,t|t-1} \\ \hat{\tau}^x_{t|t-1} \end{bmatrix} = \begin{bmatrix} \lambda^x_\tau & I_n \end{bmatrix} \begin{bmatrix} \rho & 0 \\ 0 & \rho I_n \end{bmatrix} \begin{bmatrix} \hat{\tau}^x_{c,t-1|t-1} \\ \hat{\tau}^x_{t-1|t-1} \end{bmatrix}.$$

It follows that a household with imperfect information who is unable to distinguish between persistent and transitory shocks faces the following effective driving process,

$$\begin{bmatrix} \hat{\tau}^x_{c,t|t} \\ \hat{\tau}^x_{t|t} \end{bmatrix} = \begin{bmatrix} \rho I_n & 0 \\ 0 & \rho I_n \end{bmatrix} \begin{bmatrix} \hat{\tau}^x_{c,t-1|t-1} \\ \hat{\tau}^x_{t-1|t-1} \end{bmatrix} + K^x a^x_t,$$

$$a^x_t = \hat{x}_t - \begin{bmatrix} \lambda^x_\tau & I_n \end{bmatrix} \begin{bmatrix} \rho I_n & 0 \\ 0 & \rho I_n \end{bmatrix} \begin{bmatrix} \hat{\tau}^x_{c,t-1|t-1} \\ \hat{\tau}^x_{t-1|t-1} \end{bmatrix},$$

where $x = z,\ \ell$, and $\hat{x}_t$ is treated as the shock. Put alternatively, the imperfect information problem introduces estimates of the unobserved exogenous states as state variables (rather than the states themselves) and modifies the driving processes the representative household faces. However, given these modified processes, the model may be solved using standard linear rational expectations solution toolkits as before.

2.10.1 Computation of the Steady State Kalman Gain

The observation and state equations for $x = z,\ \ell$, are respectively,

$$x_t = H\tau^x_t + u^x_t,\ u^x_t \sim N\left(0, R^x\right),$$

and

$$\tau^x_t = F\tau^x_{t-1} + \eta^x_t,\ \eta^x_t \sim N\left(0, Q^x\right),$$

where $\tau^x_t = (\tau^x_{c,t}, \tau^x_t)'$ and H and F are defined above (for simplicity, we leave out the "ˆ" notation over the variables).

The prediction equations are:

- $\tau^x_{t|t-1} = F\tau^x_{t-1|t-1}$,
- $P_{t|t-1} = E[(\tau^x_t - \tau^x_{t|t-1})(\tau^x_t - \tau^x_{t|t-1})'|x^{t-1}] = FP_{t-1|t-1}F' + Q^x$,

- $x_{t|t-1} = H\tau^x_{t|t-1}$,
- $S_{t|t-1} = E[(x_t - x_{t|t-1})(x_t - x_{t|t-1})'|x^{t-1}] = HP_{t|t-1}H' + R^x$.

The updating equations are:

- $\tau^x_{t|t} = \tau^x_{t|t-1} + \underbrace{P_{t|t-1}H'S^{-1}_{t|t-1}}_{K^x_t}(x_t - x_{t|t-1})$,
- $P_{t|t} = P_{t|t-1} - \underbrace{P_{t|t-1}H'S^{-1}_{t|t-1}}_{K^x_t}HP_{t|t-1}$.

The steady state Kalman gain is then given by $K^x = PH'S^{-1}$ where P and S respectively solve

$$P = FPF' + Q^x,$$

$$S = HPH' + R^x.$$

We obtain Q^x and R^x by computing covariances of the shocks from the MUC model,

$$Q^x = Var\left(\left[\begin{array}{c} \eta^x_{c,t} \\ \eta^x_t \end{array}\right]\right),$$

$$R^x = Var\left(\lambda^x_{\varepsilon^x}\varepsilon^x_{c,t} + \varepsilon^x_t\right).$$

We then compute the steady state Kalman gain by iterating on the following equations where, given iteration $j-1$,

$$P_1 = FP_0^{(j-1)}F' + Q^x,$$

$$S = HP_1^{(j-1)}H' + R^x,$$

$$K^{x(j)} = P_0^{(j-1)}H'S^{-1},$$

and

$$P_0^{(j)} = P_1 - K^{x(j)}HP_1.$$

We stop the iterations when $\left\|P_0^{(j)} - P_0^{(j-1)}\right\| < 10^{-8}$. The steady state Kalman gain is then $K^{x(j)}$.

2.10.2 Expression for the Driving Processes as an Input to King and Watson (2002)

The driving process in this case is

$$\begin{bmatrix} \hat{z}_t \\ \hat{\ell}_t \end{bmatrix} = \begin{bmatrix} 0 & \rho I_n & 0 & \lambda_\tau^z \rho & I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho I_n & 0 & \lambda_\tau^\ell \rho & I_n \end{bmatrix} \begin{bmatrix} \hat{\tau}^z_{t|t} \\ \hat{\tau}^z_{t-1|t-1} \\ \hat{\tau}^z_{c,t|t} \\ \hat{\tau}^z_{c,t-1|t-1} \\ a^z_t \\ \hat{\tau}^\ell_{t|t} \\ \hat{\tau}^\ell_{t-1|t-1} \\ \hat{\tau}^\ell_{c,t|t} \\ \hat{\tau}^\ell_{c,t-1|t-1} \\ a^\ell_t \end{bmatrix}$$

$$\begin{bmatrix} \hat{\tau}^z_{t|t} \\ \hat{\tau}^z_{t-1|t-1} \\ \hat{\tau}^z_{c,t|t} \\ \hat{\tau}^z_{c,t-1|t-1} \\ a^z_t \\ \hat{\tau}^\ell_{t|t} \\ \hat{\tau}^\ell_{t-1|t-1} \\ \hat{\tau}^\ell_{c,t|t} \\ \hat{\tau}^\ell_{c,t-1|t-1} \\ a^\ell_t \end{bmatrix} = \begin{bmatrix} \rho I_n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\tau}^z_{t-1|t-1} \\ \hat{\tau}^z_{t-2|t-2} \\ \hat{\tau}^z_{c,t-1|t-1} \\ \hat{\tau}^z_{c,t-2|t-2} \\ a^z_{t-1} \\ \hat{\tau}^\ell_{t-1|t-1} \\ \hat{\tau}^\ell_{t-2|t-2} \\ \hat{\tau}^\ell_{c,t-1|t-1} \\ \hat{\tau}^\ell_{c,t-2|t-2} \\ a^\ell_{t-1} \end{bmatrix} + \begin{bmatrix} K_{2z} & 0 \\ 0 & 0 \\ K_{1z} & 0 \\ 0 & 0 \\ I & 0 \\ 0 & K_{2\ell} \\ 0 & 0 \\ 0 & K_{1\ell} \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} a^z_t \\ a^\ell_t \end{bmatrix}$$

3 Data

The raw data from KLEMS, after excluding 4 sectors corresponding to government activities, covers $N = 61$ industries with growth rates over $T = 70$ years from 1947-2016. The data include nominal gross output ($Y_{j,t}$), nominal capital ($K_{j,t}$), nominal labor ($L_{j,t}$), and nominal intermediates ($M_{j,t}$). Similarly, they include constant dollar values of gross output ($y_{j,t}$), capital ($k_{j,t}$), labor ($\ell_{j,t}$), and intermediates ($m_{j,t}$).

3.1 Assembling the Sectoral Data

We construct the following additional series.

1. Nominal Value Added
$$V_{j,t} = Y_{j,t} - M_{j,t},$$

2. Capital Share
$$S^k_{j,t} = \frac{K_{j,t}}{Y_{j,t}},$$

3. Labor Share
$$S^\ell_{j,t} = \frac{L_{j,t}}{Y_{j,t}},$$

4. Intermediate Share
$$S^m_{j,t} = \frac{M_{j,t}}{Y_{j,t}},$$

5. Gross Output Growth Rates
$$100 \times \Delta \ln y_{j,t} = 100 \times \ln\left(\frac{y_{j,t}}{y_{j,t-1}}\right),$$

6. Capital Growth Rates
$$100 \times \Delta \ln k_{j,t} = 100 \times \ln\left(\frac{k_{j,t}}{k_{j,t-1}}\right),$$

7. Labor Growth Rates
$$100 \times \Delta \ln \ell_{j,t} = 100 \times \ln\left(\frac{\ell_{j,t}}{\ell_{j,t-1}}\right),$$

8. Intermediate Growth Rates
$$100 \times \Delta \ln m_{j,t} = 100 \times \ln\left(\frac{m_{j,t}}{m_{j,t-1}}\right),$$

9. TFP Growth Rates
$$100 \times \Delta \ln z_{j,t} = 100 \times \left[\begin{array}{c} \Delta \ln y_{j,t} - \frac{1}{2}\left(S^k_{j,t-1} + S^k_{j,t}\right)\Delta \ln k_{j,t} \\ -\frac{1}{2}\left(S^\ell_{j,t-1} + S^\ell_{j,t}\right)\Delta l_{j,t} - \frac{1}{2}\left(S^m_{j,t-1} + S^m_{j,t}\right)\Delta m_{j,t}, \end{array}\right],$$

10. Nominal Gross Output to Value Added Ratios

$$S^{YV}_{j,t} = \frac{Y_{j,t}}{V_{j,t}},$$

11. Value Added Growth Rates

$$100 \times \Delta \ln v_{j,t} = 100 \times \left[\frac{1}{2} \left(S^{YV}_{j,t-1} + S^{YV}_{j,t} \right) \Delta \ln y_{j,t} + \frac{1}{2} \left(2 - S^{YV}_{j,t-1} - S^{YV}_{j,t} \right) \Delta \ln m_{j,t} \right],$$

12. Nominal Gross Output in Value Added Shares (Hulten weights)

$$S^{VA}_{j,t} = \frac{Y_{j,t}}{\sum_{i=1}^{N} V_{i,t}},$$

13. Nominal Value Added in Total Value Added Shares

$$S^{V}_{j,t} = \frac{V_{j,t}}{\sum_{i=1}^{N} V_{i,t}}.$$

3.2 Aggregating KLEMS into Consolidated Sectors

We combine the disaggreated KLEMS sectors into broader consolidated sectors. For example, we might combine sectors $j \in \{1, ..., n\}$ into a single sector J. To that end, we incorporate data from the Input-Output (IO) tables, where $M^{IO}_{ij,t}$ denotes the nominal use of materials used by sector j purchased from sector i. Similarly, $V^{IO}_{j,t}$ denotes value-added in sector j in the IO tables.

We use the following formulas to create consolidated sectors.

1. Value Added Growth Rates

$$100 \times \Delta v_{J,t} = 100 \times \sum_{j \in J} \frac{1}{2} \left(\frac{V_{j,t}}{\sum_{i \in J} V_{i,t}} + \frac{V_{j,t-1}}{\sum_{i \in J} V_{i,t-1}} \right) \Delta \ln v_{j,t},$$

2. Capital Growth Rates

$$100 \times \Delta \ln k_{J,t} = 100 \times \sum_{j \in J} \frac{1}{2} \left(\frac{K_{j,t}}{\sum_{i \in J} K_{i,t}} + \frac{K_{j,t-1}}{\sum_{i \in J} K_{i,t-1}} \right) \Delta \ln k_{j,t},$$

3. Labor Growth Rates

$$100 \times \Delta \ln l_{J,t} = 100 \times \sum_{j \in J} \frac{1}{2} \left(\frac{L_{j,t}}{\sum_{i \in J} L_{i,t}} + \frac{L_{j,t-1}}{\sum_{i \in J} L_{i,t-1}} \right) \Delta \ln \ell_{j,t},$$

4. Nominal Intermediate Inputs. (IO Tables)

$$M_{J,t}^{IO} = \sum_{j \in J} \sum_{i \notin J} M_{ij,t}^{IO},$$

5. Nominal Value Added (IO Tables)

$$V_{J,t}^{IO} = \sum_{j \in J} V_{j,t}^{IO},$$

6. Nominal Gross Output (IO tables) in consolidated sector J,

$$Y_{J,t}^{IO} = V_{J,t}^{IO} + M_{J,t}^{IO},$$

7. TFP Growth Rates. In considering value-added TFP, $\Delta \ln z_{j,t}^{v}$, we have the following identities

$$\Delta \ln z_{j,t}^{v} = \frac{1}{2} \left(\frac{Y_{j,t}}{V_{j,t}} + \frac{Y_{j,t-1}}{V_{j,t-1}} \right) \Delta \ln z_{j,t},$$

$$\Delta \ln z_{J,t}^{v} = \sum_{j \in J} \frac{1}{2} \left(\frac{V_{j,t}}{\sum_{i \in J} V_{i,t}} + \frac{V_{j,t-1}}{\sum_{i \in J} V_{i,t-1}} \right) \Delta \ln z_{j,t}^{v},$$

$$\Delta \ln z_{J,t} = \frac{1}{2} \left(\frac{V_{J,t}^{IO}}{Y_{J,t}^{IO}} + \frac{V_{J,t-1}^{IO}}{Y_{J,t-1}^{IO}} \right) \Delta \ln z_{J,t}^{v}.$$

Combining these identities produces

$$\Delta \ln z_{J,t} = \frac{1}{2} \left(\frac{V_{J,t}^{IO}}{Y_{J,t}^{IO}} + \frac{V_{J,t-1}^{IO}}{Y_{J,t-1}^{IO}} \right) \sum_{j \in J} \frac{1}{2} \left(\frac{V_{j,t}}{\sum_{i \in J} V_{i,t}} + \frac{V_{j,t-1}}{\sum_{i \in J} V_{i,t-1}} \right) \frac{1}{2} \left(\frac{Y_{j,t}}{V_{j,t}} + \frac{Y_{j,t-1}}{V_{j,t-1}} \right) \Delta \ln z_{j,t}.$$

3.3 Aggregating the Consolidated Sectors

We aggregate TFP measures from the $N = 61$ original series, but also from the $N^{*} = 15$ consolidated series denoted by J above, using Hulten weights. Weights for the consolidated series follow from the IO Tables.

We use the following formulas to aggregate TFP growth rates.

1. Aggregate TFP Growth Rates from Original Industries

$$100 \times \Delta \ln z_t^{(N)} = 100 \times \sum_{j=1}^{N} \frac{1}{2} \left(S_{j,t}^{VA} + S_{j,t-1}^{VA} \right) \Delta \ln z_{j,t},$$

2. Hulten Weights for the Consolidated Industries

$$S_{J,t}^{VA} = \frac{Y_{J,t}^{IO}}{\sum_{i=1}^{N^*} V_{i,t}^{IO}},$$

3. Aggregate TFP Growth Rates from Consolidated Industries

$$100 \times \Delta \ln z_t^{(N^*)} = 100 \times \sum_{J=1}^{N^*} \frac{1}{2} \left(S_{J,t}^{VA} + S_{J,t-1}^{VA} \right) \Delta \ln z_{J,t}.$$

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