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Abstract

We develop a tractable bubbles model with financial frictions and downward wage rigidity. Competitive speculation in risky bubbles can result in excessive investment booms that precede inefficient busts, where post-bubble aggregate economic activities collapse below the pre-bubble trend. Risky bubbles can reduce ex-ante social welfare, and leaning-against-the-bubble policies that balance the boom-bust trade-off can be warranted. We further show that the collapse of a bubble can push the economy into a "secular stagnation" equilibrium, where the zero lower bound and the nominal wage rigidity constraint bind, leading to a persistent recession, such as the Japanese "lost decades."

1 Introduction

In the recent decades, many countries in the world, including Japan, the U.S., and several European economies, have experienced episodes of rapid speculative booms and busts in asset prices, followed by declines in economic activities and in some cases persistent recessions. More generally, throughout history, the collapse of large asset and credit booms tend to

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precede recessions and crises (e.g., Kindleberger and O'Keefe 2001; Jordà et al. 2015). These experiences have led policymakers to be increasingly aware of the potential risks of asset price bubbles, leading to discussions of macroprudential regulations such as "leaning-against-the-wind" policies – preventive measures to curb the booms in asset prices in order to mitigate the eventual busts.

However, despite the recent developments in the macroeconomic literature on asset bubbles, relatively little theoretical framework has analyzed the potential efficiency trade-off between the booms and busts of risky bubbly episodes and whether preventive policies are warranted. In particular, in most rational bubble models – the workhorse models to study the macroeconomic effects of bubbles in general equilibrium – private agents correctly perceive the risk of speculating in a bubbly asset and bubbles generally improve the efficiency of the financial system (e.g., see the literature surveys in Barlevy 2007, 2012, 2018; Miao 2014; Martin and Ventura 2017).

In this paper, we develop a tractable general equilibrium model to address the question of when and how risky rational bubbles can lead to inefficiencies and evaluate the welfare trade-off. We focus on the combination of financial frictions and downward wage rigidities during bubbly episodes. We posit an economy where entrepreneurial agents with heterogeneous productivity accumulate capital and, due to limited commitment, face financial frictions that constrain their ability to borrow from each other (e.g., Kiyotaki et al. 1997; Carlstrom and Fuerst 1997; Buera and Shin 2013). If the credit and capital markets cannot satisfy the demand for savings, speculative bubbles may arise. A rational bubble is an asset that is traded above its fundamental value; an agent purchases the overvalued asset because he or she expects to be able to sell it later. We assume bubbles are stochastic in the sense that in each period the price of the bubbly asset can collapse to the fundamental value with an exogenous probability (e.g., Blanchard and Watson 1982; Weil 1987).

The possibility of trading the bubbly asset facilitates the reallocation of resources across time, because the bubbly asset can act as a savings vehicle. Trading also facilitates reallocation across agents, because the bubbly asset increases entrepreneurs' net worth and hence their ability to borrow. Thus, the boom in the price of a bubbly asset leads to a boom in entrepreneurial net worth, credit, investment, output, wages, and consumption. When the boom finally turns into a bust, the economy simply converges back to the pre-bubble economy. Therefore, with financial frictions alone, the model so far implies that speculative bubbles help to crowd in productive investment and improve the overall efficiency of the economy, as implied in most existing expansionary bubble frameworks (e.g., Hirano et al. 2015; Miao and Wang 2018).

However, the implications change with downward wage rigidities. When an expansionary bubble collapses, the net worth of entrepreneurial agents also falls, leading to contractions in credit and investment. Thus, the demand for labor from firms also contracts. In a flexible labor market, wages will fall to clear the labor market. However, we assume that (real) wages are downwardly rigid. Then there will be rationing in the labor market, resulting in involuntary unemployment. An increase in unemployment can in turn lead to an endogenous and protracted recession by eroding the intertemporal allocation of resources. This is because the drop in employment reduces the return to capital investment, which then lowers entrepreneurs' net worth. This further leads to a contraction in capital investment, since entrepreneurs' ability to borrow and invest depends critically on their net worth. Therefore, the future capital stock will decline, causing further downward pressure on labor demand and wages, thus reducing future capital accumulation. The vicious cycle repeats and only stops when the capital stock has fallen enough, often *undershooting* the bubbleless steady-state level.

In short, our theory identifies the booms and busts of speculative bubbly episodes as an important source of shocks that can potentially trigger a deep and persistent recession, such as the lost decades in Japan or the Great Depression and Great Recession in the U.S.[1] We further show that when the bubble is sufficiently risky and the labor market is sufficiently rigid, society's welfare can be better off without bubbles. Our model thus provides a step toward bridging the views of policymakers and theoretical models of bubbles (Barlevy, 2018). In particular, our theory naturally implies that a "leaning-against-the-bubble" type of macroprudential policy intervention is warranted for excessively large bubbles. The source of inefficiencies is a form of "bubbly pecuniary externality," as individual investors do not internalize the effect of their portfolio choices in driving a large bubbly boom, which will lead to a large bust.

Finally, we extend the real model to an environment where *nominal* wages are downwardly rigid (Schmitt-Grohé and Uribe 2016; Schmitt-Grohé and Uribe 2017) and the central bank sets the nominal interest rate according to a Taylor rule that is subject to the zero lower bound (ZLB). We then show that the collapse of a large expansionary bubble triggers a sharp drop in the real interest rate, pushing the nominal interest rate against the ZLB. The intuition is as follows. By crowding in capital investment, the bubble leads to an investment boom. Thus, after the bubble collapses, the economy enters the post-bubble phase with a capital stock above the steady state, a situation that has been referred to as an "investment hangover" (Rognlie et al. 2014). The high capital stock implies a low marginal

[1] Other sources of shocks that have been highlighted in the literature, including but not limited to deleveraging shocks (e.g., Eggertsson and Krugman, 2012; Korinek and Simsek, 2016; Buera and Nicolini 2017), shocks to inflation expectations (e.g., Schmitt-Grohé and Uribe, 2017), or idiosyncratic risk shocks (e.g., Christiano et al., 2014; Acharya and Dogra, 2017).

product of capital and a low real interest rate. The collapse of a sufficiently large bubble can thus push the real interest rate so low that the ZLB binds. We show that, under certain conditions, the post-bubble economy may fall into a liquidity trap *steady state*, or "secular stagnation," where employment and investment are persistently and inefficiently low and inflation is below target. A vicious cycle can arise from the interaction between (i) a low interest rate environment, which constrains the monetary authority from raising inflation, exacerbating the nominal wage rigidity and unemployment problem and (ii) inefficient unemployment that lowers the marginal product of capital, which in turn lowers the interest rates. In the absence of other shocks, this cycle can keep the economy in a persistent slump.

Related literature. To the best of our knowledge, our paper is the first to construct a stochastic bubble model where the collapse of bubbles can trigger a collapse of aggregate economic activities below their bubbleless trend and hence provide a reason for why the economy may be better off without stochastic bubbles. Our paper is related to several strands of the literature. First, we help formalize the popular notion among policymakers that the collapse of risky bubbles can trigger inefficient recessions. A large number of papers emphasize the positive aspect of bubbles in reducing dynamic inefficiencies (e.g., Samuelson 1958; Diamond 1965; Tirole 1985) or reducing intratemporal inefficiencies in the allocation of resources (e.g., Farhi and Tirole 2011; Miao and Wang 2012, 2018; Martin and Ventura 2012; Graczyk and Phan 2016; Ikeda and Phan 2018). Other papers emphasize potential ex-ante inefficiencies of speculative bubble investment in diverting resources away from productive investment (e.g., Saint-Paul 1992; Grossman and Yanagawa 1993; King and Ferguson 1993; Hirano et al. 2015), generating excessive allocations of resources in certain sectors (e.g., Cahuc and Challe 2012; Miao et al. 2014), generating excessive volatility (Caballero and Krishnamurthy 2006; Ikeda and Phan 2016), or generating excessive default (Kocherlakota 2009; Barlevy 2014; Bengui and Phan 2018). Our paper complements this literature and highlights the ex-post inefficiency of bubbles by showing that their collapse can cause persistent involuntary unemployment.[2]

By embedding New Keynesian elements of rigidities into a rational bubbles framework, our paper is related to our earlier work, Hanson and Phan (2017). There, we developed a simple overlapping generations model based on Tirole (1985). A limitation of the overlapping generations framework in Hanson and Phan (2017) is that a period represents twenty or thirty years, making the model less appropriate for policy analyses at the business cycles frequency.

[2]For a complementary approach to modeling post-bubble unemployment using a search and matching model à-la Diamond-Mortensen-Pissarides, see Kocherlakota (2011) and Miao et al. (2016). Also see Domeij and Ellingsen (2018) and Illing et al. (2018).

Furthermore, the previous paper does not study the welfare effects of policies or the ZLB.

Our paper is also related to the overlapping-generations model of Asriyan et al. (2016) and provides a complementary approach to explaining post-bubble liquidity traps.[3] In their model, inefficiency arises in the liquidity trap as the holding of cash crowds out productive investment. In contrast, the inefficiency arises in our model because of the aforementioned bubbly pecuniary externality. Moreover, while they introduce a new form of nominal rigidity through the assumption that expectations about the future values of bubbly assets are set in nominal terms, we assume a nominal wage rigidity that is relatively standard in the recent New Keynesian literature.

Second, our paper is related to a large literature that investigates possible sources of shocks that trigger long recessions and liquidity traps in environments with New Keynesian frictions. Many papers have emphasized demand shocks driven by household deleveraging or tightening borrowing constraints (Eggertsson and Krugman 2012; Christiano et al. 2015; Korinek and Simsek 2016; Schmitt-Grohé and Uribe 2016), long-run factors such as aging demographics or safe asset shortages (Summers 2013; Eggertsson et al. 2016; Caballero and Farhi 2017), or overinvestment of capital (Rognlie et al. 2014). By highlighting the role of rational asset bubbles, our analysis offers a complementary narrative to those in the literature. Furthermore, in our model, the collapse of bubbles reduces the net worth of borrowers and an endogenous tightening of borrowing constraints in equilibrium, thus giving a possible microfoundation for the deleverage shocks in, e.g., Eggertsson and Krugman (2012) and Korinek and Simsek (2016). Similarly, in our model, expansionary bubbles lead to an endogenous boom in capital investment, thus giving a microfoundation to the investment overhang in Rognlie et al. (2014).

Finally, by providing a normative analysis with macroprudential policies on speculative bubble investment, our paper complements the literature on macroprudential policies in environments with financial frictions or aggregate demand externalities (e.g., Lorenzoni 2008; Olivier and Korinek 2010; He and Krishnamurthy 2011; Bianchi 2011; Eberly and Krishnamurthy 2014; Farhi and Werning 2016; Bianchi and Mendoza 2018).

The plan for the paper is as follows. Section 2 describes the main real model. Section 3 studies the bubbleless equilibrium and steady states, while Section 4 analyzes the bubbly equilibrium and steady states. Section 5 provides a welfare analysis. Section 6 provides an extension with nominal rigidity and the zero lower bound. Section 7 concludes. Detailed derivations and proofs are in the appendix.

[3] For a related and emerging body of literature that analyzes the effects of monetary policies on rational bubbles, see Gali (2014, 2016), Ikeda (2016), Dong et al. (2017), and Hirano et al. (2017).

2 Model

Consider an economy with two types of goods: a perishable consumption good and a capital good. Time is infinite and discrete. Firms are competitive, and there exist two types of agents, called entrepreneurs and workers, each with constant unit population. Entrepreneurs and workers have the same preferences over a consumption process $\{c_t\}_{t=0}^{\infty}$, given by

$$E_0\left(\sum_{t=0}^{\infty}\beta^t u(c_t)\right)$$

where the period utility function is $u(c) = \log(c)$, $\beta \in (0, 1)$ is the subjective discount factor, and $E_0(\cdot)$ is the expected value conditional on information in period 0.

2.1 Entrepreneurs

Entrepreneurs are the only producers of the capital good. They rent the capital produced to firms through a competitive capital rental market. Entrepreneurs face idiosyncratic productivity shocks: in each period, an entrepreneur receives a random productivity shock a, where a is independently and identically distributed (i.i.d.) according to a continuous distribution with a cumulative distribution function (CDF) denoted by F.[4] For tractability, we assume that the distribution is Pareto over $[1, \infty)$ with shape parameter $\sigma > 1$.

We denote the set of entrepreneurs by $J \equiv [0, 1]$. After knowing the idiosyncratic productivity shock the beginning of each period, an entrepreneur $j \in J$ produces the capital good according to the following technology:

$$k_{t+1}^j = a_t^j I_t^j,$$

where I_t^j is the investment in units of the consumption good in period t, k_{t+1}^j is the amount of the capital good produced in the subsequent period, and a_t^j is the productivity of the entrepreneur. For tractability, we assume capital depreciates completely after being used in each period (we relax this assumption in Appendix A.1.6 and in the numerical analysis).

Following the literature (e.g., Tirole 1985), we introduce a (pure) bubbly asset, which is a durable and perfectly divisible asset in fixed unit supply that does not generate any dividend but can be traded at positive equilibrium prices under some conditions. To model a stochastic bubble, we follow the literature (e.g., Weil 1987) and assume that in each period the bubble

[4] As is well known among models with heterogeneous productivity shocks (e.g., Carlstrom and Fuerst 1997; Bernanke et al. 1999; Kocherlakota 2009; Liu and Wang 2014), the i.i.d. assumption helps keep the model tractable. A model with persistent idiosyncratic shocks can only be solved numerically.

persists with a probability $\rho \in (0, 1)$ and permanently collapses[5] with the complementary probability $1 - \rho$, where a lower ρ means a riskier bubble. The two sources of uncertainty in the model are thus the idiosyncratic productivity shock and the aggregate bubble shock that leads to the collapse of the bubble.

Let b_t^j denote a share of a bubbly asset held by entrepreneur j and p_t^b be the price per unit of the bubbly asset. Then the entrepreneur's flow budget constraint is written as

$$c_t^j + I_t^j + p_t^b b_t^j = R_t^k k_t^j + d_t^j - R_{t-1,t} d_{t-1}^j + p_t^b b_{t-1}^j, \tag{2.1}$$

where $R_{t-1,t}$ is the gross interest rate between $t-1$ and t, d_t^i is the amount of net borrowing in period t, and R_t^k is the rental rate of capital in t. The left-hand side of this budget constraint consists of expenditure on consumption, capital investment, and the purchase of bubbly assets. The right-hand side is the available funds at date t, which consists of the return from capital investment in the previous period, new net borrowing minus the net debt repayment, and the return from selling bubbly assets. As is standard in the literature (e.g., Martin and Ventura 2012; Hirano and Yanagawa 2017; Miao and Wang 2018), we assume agents cannot invest a negative amount in the capital stock or the bubbly asset, i.e.,[6]

$$I_t^j, b_t^j \geq 0, \forall t. \tag{2.2}$$

The entrepreneur's net worth at the beginning of period t is:

$$e_t^j \equiv R_t^k a_{t-1}^j I_{t-1}^j - R_{t-1,t} d_{t-1}^j + p_t^b b_{t-1}^j. \tag{2.3}$$

Financial frictions: In a frictionless world, less productive entrepreneurs would like to lend and thus delegate investment to highly productive entrepreneurs. Agents can borrow and lend through one-period debt contracts. However, we assume there are frictions in the financial market so entrepreneurs face a leverage constraint:

$$d_t^j \leq \theta_t^j e_t^j, \tag{2.4}$$

which states that each entrepreneur's borrowing is limited by her net worth e_t^j. The limit $\theta_t^j \geq 0$ places a constraint on the entrepreneur's leverage ratio. In general, a larger θ can be interpreted as representing an environment with less financial friction.

This formulation of credit market friction is sufficiently general to encompass several

[5]That is, once collapsed, bubbles are not expected to remerge. As in Guerron-Quintana et al. (2018), the model can be extended to relax this assumption and allow for recurring bubbles.

[6]As otherwise, the ability to short sell would let agents borrow and bypass leverage constraint (2.4).

types of credit constraints considered in the financial friction literature. For example, if one assumes $\theta_t^j \equiv \frac{R_{t+1}^k \bar{\theta} a^j}{R_{t+1} - R_{t+1}^k \bar{\theta} a^j}$, where $\bar{\theta} \in [0, 1]$ is a constant, then the constraint maps to a standard collateral constraint: $R_{t,t+1} D_t(a) \leq \bar{\theta} R_{t+1}^k K_{t+1}(a)$, which can arise when entrepreneurs can only pledge to repay in the next period at most a fraction $\bar{\theta}$ of the value of their asset (e.g., Kiyotaki et al. 1997).

Alternatively, by assuming

$$\theta_t^j \equiv \theta,$$

where $\theta \geq 0$ is a constant, the constraint maps to an analytically convenient form of credit constraint that has been used extensively in the recent literature of general equilibrium with heterogeneous agents (e.g., Banerjee and Moll 2010; Buera and Shin 2013; Moll 2014; Ikeda and Phan forthcoming).[7] For tractability, we impose this assumption throughout the paper. While not essential for our results, the assumption of this simple leverage constraint, along with the assumption of a Pareto distribution, allows us to get analytical solutions to the model. These assumptions are relaxed in the general analysis in Appendix A.1.6.

The optimization problem of each entrepreneur j is as follows. In each period after knowing her productivity shock a_t^j, the entrepreneur chooses consumption c_t^j, capital investment I_t^j, net debt position d_t^j (where a negative d_t^j means lending), and net purchase of the bubbly asset $b_t^j - b_{t-1}^j$. Her objective is to maximize the lifetime expected utility $E_t \left(\sum_{s \geq 0} \beta^s \log c_{t+s}^j \right)$, subject to budget constraint (2.1), nonnegativity constraint (2.2), and leverage constraint (2.4).

2.2 Workers

Workers do not have access to capital production technologies. For simplicity, we assume workers are hand to mouth, i.e.,

$$c_t^w = w_t l_t, \tag{2.5}$$

where w_t is the wage rate and l_t is the employment level per worker.[8]

[7] For a microfoundation of this constraint, see, e.g., Korinek (2010).

[8] Alternatively, we can assume workers cannot borrow against their future labor income. Thus the optimization problem of workers is to maximize lifetime utility $E_0 \left(\sum_{t=0}^{\infty} \beta^t \ln c_t^w \right)$ subject to:

$$c_t^w + p_t^b b_t^w = w_t l_t + d_t^w - R_t d_{t-1}^w + p_t^b b_{t-1}^w,$$

and $d_t^w \leq 0$ and $b_t^w \geq 0$. In equilibrium, it can be shown that workers will be effectively hand to mouth, i.e., $c_t^w = w_t l_t$. Intuitively, due to financial friction, the interest rate (and the returns from bubble speculation) will be too low relative to the discount factor, and thereby it will be suboptimal for workers to save or to buy the bubbly asset (see Hirano et al. 2015 for more details).

2.3 Firms

In each period, there is a continuum of competitive firms that produce the consumption good from hiring labor from workers and renting capital from entrepreneurs. They employ a standard production function:

$$y_t^i = (k_t^i)^\alpha (l_t^i)^{1-\alpha},\ 0 < \alpha < 1,$$

where k_t^i and l_t^i are capital and labor inputs of a representative firm i. Competitive factor prices are given by the marginal products of capital and labor:

$$R_t^k = \alpha \left(\frac{L_t}{K_t}\right)^{1-\alpha} \tag{2.6}$$

$$w_t = (1-\alpha)\left(\frac{K_t}{L_t}\right)^{\alpha}, \tag{2.7}$$

where K_t and L_t are the aggregate capital stock and employment.

2.4 Downward wage rigidity (DWR)

We assume that real wages are downwardly rigid:

$$w_t \geq \gamma w_{t-1}, \forall t \geq 1, \tag{2.8}$$

where $\gamma \in [0,1]$ is a constant parameter that governs the degree of rigidity. The condition states that the real wage cannot fall below a certain fraction of the real wage in last period.[9]

The presence of rigid wages implies that the labor market does not necessarily clear. In each period, even though each worker inelastically supplies one unit of labor, the realized employment L_t per worker in equilibrium is determined by two conditions: feasibility constraint

$$L_t \leq 1, \tag{2.9}$$

and complementary-slackness condition

$$(1-L_t)(w_t - \gamma w_{t-1}) = 0. \tag{2.10}$$

These equations state that involuntary unemployment ($L_t < 1$) must be accompanied by a binding wage rigidity (2.8). Conversely, when (2.8) is slack, the economy must be in full

[9] For empirical evidence of real wage rigidity, see, e.g., Holden and Wulfsberg (2009) and Babecký et al. (2010).

employment ($L_t = 1$). For simplicity, we also assume that in the initial period $t = 0$ the legacy wage w_{-1} is sufficiently small so that the labor market clears in $t = 0$.

2.5 Equilibrium

Definition. Given initial $k_0^j = K_0$, $d_0^j = 0$, $b_0^j = 1$, p_0^b, a competitive equilibrium consists of prices $\{w_t, R_t^k, R_{t,t+1}, p_t^b\}_{t\geq 0}$ and quantities $\{\{I_t^j, k_{t+1}^j, c_t^j\}_{j\in J}, c_t^w, K_{t+1}, L_t\}_{t\geq 0}$ such that:

- Entrepreneurs and firms optimize,
- The consumption of a representative worker is given by (2.5),
- The credit market clears: $\int_0^1 d_t^j dj = 0$,
- The bubble market clears: $\int_0^1 b_t^j dj = 1$ if $p_t^b > 0$,
- The consumption good market clears: $\int_0^1 (c_t^j + I_t^j)dj + c_t^w = K_t^\alpha L_t^{1-\alpha}$,
- The capital market clears: $K_t = \int_0^1 k_t^j dj$,
- Labor market conditions (2.8), (2.9), and (2.10) hold

As usual, a steady state is an equilibrium where quantities, prices (in units of the consumption good), and inflation are time invariant.

3 Bubbleless benchmark

Let us first analyze the bubbleless equilibrium, where the price of the bubbly asset is equal to its fundamental value of zero throughout, i.e., $p_t^b = 0$ for all t. Omitted details of the derivations are relegated to Appendix A.1.1.

3.1 Equilibrium dynamics

3.1.1 Optimal decisions of individual entrepreneurs

In each period t, given the realization of her productivity shock a_t^j, each entrepreneur j chooses c_t^j, I_t^j, and d_t^j. Since the period utility function is logarithmic, the optimal action for the entrepreneur is to consume a fraction $1 - \beta$ of her net worth e_t^j:

$$c_t^j = (1 - \beta)e_t^j, \tag{3.1}$$

and invest/save the remaining fraction β:

$$I_t^j + (-d_t^j) = \beta e_t^j. \tag{3.2}$$

In the bubbleless benchmark, net worth (as defined in (2.3)) is simply capital income minus net debt repayment:

$$e_t^j = R_t^k a_{t-1}^j I_{t-1}^j - R_{t-1,t} d_{t-1}^j.$$

Both the savings options of investing in capital and lending in the credit market are riskless. Hence, the entrepreneur will simply choose the option that offers the highest rate of return. Lending yields a rate of return $R_{t,t+1}$, which is the same for all entrepreneurs. Capital investment yields a rate of return $a_t^j R_{t+1}^k$, which varies according to each entrepreneur's productivity a_t^j. Hence, in equilibrium, there is a *cutoff productivity threshold* $\bar{a}_t$ in each period such that: all entrepreneurs with $a_t^j < \bar{a}_t$ will only lend and not invest in capital (i.e., the constraint $I_t^j \geq 0$ binds), while those with $a_t^j > \bar{a}_t$ will only invest in capital and borrow as much as possible (i.e., leverage constraint (2.4) binds). Entrepreneurs with $a_t^j = \bar{a}_t$ (the "marginal investors") will be indifferent between lending and investing in capital, and their d_t^j and I_t^j are indeterminate. The indifference condition yields a mapping between the interest rate and the marginal product of capital:

$$R_{t,t+1} = \bar{a}_t R_{t+1}^k. \tag{3.3}$$

In summary, entrepreneurs' leverage ratio is given by:

$$\frac{d_t^j}{e_t^j} = \begin{cases} -\beta & \text{if } a_t^j < \bar{a}_t \\ \in [-\beta, \theta] & \text{if } a_t^j = \bar{a}_t \\ \theta & \text{if } a_t^j > \bar{a}_t \end{cases}. \tag{3.4}$$

Then, entrepreneurs' capital investment is given by:

$$I_t^j = \begin{cases} 0 & \text{if } a_t^j < \bar{a}_t \\ \in [0, (\beta+\theta)e_t^j] & \text{if } a_t^j = \bar{a}_t \\ (\beta+\theta)e_t^j & \text{if } a_t^j > \bar{a}_t \end{cases}, \tag{3.5}$$

and the amount of capital produced by each entrepreneur in $t+1$ is given by:

$$k_{t+1}^j = \begin{cases} 0 & \text{if } a_t^j < \bar{a}_t \\ \in [0, a_t^j(\beta+\theta)e_t^j] & \text{if } a_t^j = \bar{a}_t \\ a_t^j(\beta+\theta)e_t^j & \text{if } a_t^j > \bar{a}_t \end{cases} . \tag{3.6}$$

Finally, note that since the distribution of the productivity shocks is continuous, the measure of marginal investors is zero and thus their individual asset positions will not affect aggregation.

3.1.2 Aggregation

Given the decisions of individual entrepreneurs, we can characterize the aggregate variables of the economy. The aggregate net worth of entrepreneurs is equal to the aggregate capital income:

$$e_t = \int_0^1 e_t^j dj = \alpha K_t^\alpha L_t^{1-\alpha}. \tag{3.7}$$

The cutoff threshold $\bar{a}_t$ is determined by the credit market clearing condition $\int_0^1 d_t^j dj = 0$. By incorporating equations (3.3), (3.4), (3.5), and the assumption of i.i.d. productivity shocks, this condition can be rewritten as (detailed derivations in Appendix A.1.1):

$$\underbrace{F(\bar{a}_t) \cdot \beta e_t}_{\text{agg. credit supply}} = \underbrace{\theta \cdot (1 - F(\bar{a}_t)) \cdot e_t}_{\text{agg. credit demand}}, \tag{3.8}$$

where the left-hand side is the aggregate supply of credit (from less productive entrepreneurs, $a_t^j < \bar{a}_t$) and the right-hand side is the aggregate demand of credit (from more productive entrepreneurs, $a_t^j > \bar{a}_t$). By canceling the e_t term on both sides, we get a simple equation that determines $\bar{a}_t = \bar{a}_n$, which is the time-invariant solution to the following equation:

$$\beta F(\bar{a}_n) = \theta \cdot (1 - F(\bar{a}_n)). \tag{3.9}$$

Given that F is the CDF of a Pareto distribution over $[1, \infty)$ with shape parameter σ, this equation gives a closed-form solution for $\bar{a}_n$:

$$\bar{a}_n = \left(\frac{\beta+\theta}{\beta}\right)^{1/\sigma}. \tag{3.10}$$

The cutoff threshold is the key endogenous variable that characterizes the bubbleless equilibrium.

Given the cutoff threshold, the evolution of the aggregate capital stock can be derived from (3.6), (3.7), and (3.9) as:

$$K_{t+1} = \int_0^1 k_{t+1}^j dj = (\beta + \theta) \int_{\bar{a}_n} adF(a) \cdot \alpha K_t^\alpha L_t^{1-\alpha}. \tag{3.11}$$

The interest rate is then given by:

$$R_{t,t+1} = \bar{a}_n R_{t+1}^k = \bar{a}_n \alpha K_{t+1}^{\alpha-1}.$$

Finally, the aggregate employment and equilibrium wage are determined by labor market conditions (2.8), (2.9), and (2.10).

3.2 Bubbleless steady state

Given the equilibrium dynamics in Section 3.1 above, the steady state with no bubbles can be derived as follows. Because of the assumption that the rigidity parameter is a constant $\gamma \leq 1$, the downward wage rigidity condition (2.8) does not bind in steady state, leading to full employment:

$$L_n = 1. \tag{3.12}$$

Then, from (3.11) and (3.12), the aggregate capital stock can be expressed a function of $\bar{a}_n$:

$$K_n = (\mathcal{A}_n \alpha)^{\frac{1}{1-\alpha}}. \tag{3.13}$$

where for convenience we define:

$$\mathcal{A}_n \equiv (\beta + \theta) \int_{\bar{a}_n} adF(a). \tag{3.14}$$

From (3.3) and (3.13), the interest rate can also be expressed as a function of $\bar{a}_n$:

$$R_n = \frac{\bar{a}_n}{\mathcal{A}_n}, \tag{3.15}$$

In summary, equations (3.10), (3.12), (3.13), and (3.15) uniquely determine the bubbleless steady state.

4 Bubbly equilibrium

We now analyze a stochastic bubbly equilibrium, where the bubble price conditional on persistence p_t^b is positive for all t. We focus on the relevant parameter range in which the DWR is slack as long as the bubble persists. Detailed derivations are relegated to Appendix A.1.2.

4.1 Equilibrium dynamics while the bubble persists

4.1.1 Optimal decisions of individual entrepreneurs

Suppose the bubble persists in t, i.e., $p_t^b > 0$. Then, while the optimal consumption of each entrepreneur is still a fraction $1 - \beta$ of net worth as in equation (3.1), her portfolio optimization will include a new decision of speculating on the bubbly asset:

$$I_t^j + (-d_t^j) + p_t^b b_t^j = \beta e_t^j.$$

On the one hand, the savings options of investing in capital and lending yield riskless returns of $a_t^j R_{t+1}^k$ and $R_{t,t+1}$, respectively. As in the bubbleless benchmark, the bubbly equilibrium will feature a cutoff productivity threshold $\bar{a}_t$ in each period such that: all entrepreneurs with productivity shocks below this threshold will not invest in capital (i.e., the constraint $I_t^j \geq 0$ binds), and all those with productivity shocks above it will only invest in capital, sell all of their assets, and borrow as much as possible (i.e., the leverage constraint binds). Thus, the entrepreneurial capital investment decision and the amount of capital produced are given by equations (3.5) and (3.6), respectively, as in the bubbleless benchmark.

On the other hand, the speculative investment in the bubbly asset yields a risky return that is zero with probability $1 - \rho$. In the bubbly equilibrium, the less productive entrepreneurs must be willing to both lend and purchase the bubbly assets, and so must be indifferent between the two options. Thus, the net debt position of entrepreneurs is given by:

$$d_t^j = \begin{cases} -\beta e_t^j + p_t^b b_t^j & \text{if } a_t^j < \bar{a}_t \\ \in [-\beta e_t^j + p_t^b b_t^j, \theta e_t^j] & \text{if } a_t^j = \bar{a}_t \ , \\ \theta e_t^j & \text{if } a_t^j > \bar{a}_t \end{cases} \tag{4.1}$$

the bubbly investment is given by:

$$p_t^b b_t^j = \begin{cases} \beta e_t^j + d_t^j & \text{if } a_t^j < \bar{a}_t \\ \in [0, \beta e_t^j + d_t^j] & \text{if } a_t^j = \bar{a}_t \\ 0 & \text{if } a_t^j > \bar{a}_t \end{cases}, \tag{4.2}$$

and the indifference condition, which determines the growth rate of the price of the bubbly asset, is given by:

$$E_t\left[u'(c_{t+1}^j)\left(\frac{p_{t+1}^b}{p_t^b} - R_{t,t+1}\right)\right] = 0, \quad \text{if } a_t^j < \bar{a}_t. \tag{4.3}$$

In addition, the marginal investors are indifferent between lending and investing in capital:

$$E_t\left[u'(c_{t+1}^j)\left(a_t^j R_{t+1}^k - R_{t,t+1}\right)\right] = 0, \quad \text{if } a_t^j = \bar{a}_t. \tag{4.4}$$

The net worth of each entrepreneur now also contains the value of the bubbly assets purchased from last period:

$$e_t^j = R_t^k a_{t-1}^j I_{t-1}^j - R_{t-1,t} d_{t-1}^j + \underbrace{p_t^b b_{t-1}^j}_{\text{bubbly component}}.$$

Intuitively, the bubbly asset provides an additional investment vehicle for entrepreneurs. When they are less productive, they can invest in the bubbly asset. Then when they become more productive, they sell the asset in order to make more capital investment.

Note that at the individual level, because of their indifference, entrepreneurs may not see an advantage of having a bubble. However, at the aggregate level, the buying and selling of the bubbly asset allows for more resources to be transferred from less productive to more productive entrepreneurs in general equilibrium. In particular, more productive entrepreneurs can sell their holding of the bubbly asset to less productive ones. Thus, like money, the bubbly asset can be used as a storage of value that allows productive entrepreneurs to raise more resources for capital production. We now proceed to analyze the aggregate conditions in general equilibrium.

Remark 1. The optimal decisions of each entrepreneur necessarily satisfy the transversality condition $\lim_{t\to\infty} E_0 \beta^t u'(c_t^j) p_t^b b_t^j = 0$. In a representative agent model, this condition can be used to rule out the possibility of bubbles (e.g., Kamihigashi 2001). Intuitively, the condition imposes that the present discounted value of the individual investment in the bubbly asset $p_t^b b_t^j$ must be zero. In a representative model, because $b_t^j = b_t = 1$, this condition implies that the present discounted value of the total value of the bubbly asset p_t^b must be zero. However, with heterogeneous entrepreneurs and occasionally binding leverage constraints, the individual bubbly investment is *not* the same as the total value of the bubbly asset ($p_t^b b_t^j \neq p_t^b$), because entrepreneurs have heterogeneous bubbly investment positions (recall equation (4.2)). Therefore, in a heterogeneous agent model with incomplete markets like ours (or Kocherlakota 2009 and Hirano and Yanagawa 2017), the individual transversality condition does not rule out the possibility of bubbles in equilibrium (see Kocherlakota 1992 for a more general exposition of this point).

4.1.2 Aggregation

Aggregate variables of the bubbly economy evolve as follows. The aggregate net worth of entrepreneurs now consists of not only capital income, but also the value of the bubbly asset:

$$e_t = \int_0^1 e_t^j dj = \alpha K_t^\alpha L_t^{1-\alpha} + p_t^b. \tag{4.5}$$

The right-hand side of this equation highlights the bubble's crowd-in effect: the bubble resale value p_t^b helps increase the net worth of entrepreneurs in equilibrium.

The cutoff threshold $\bar{a}_t$ is determined by the credit market clearing condition $\int_0^1 d_t^j dj = 0$, or equivalently (see detailed derivations in Appendix A.1.2):

$$\underbrace{F(\bar{a}_t) \cdot \beta e_t - p_t^b}_{\text{agg. credit supply}} = \underbrace{\theta \cdot (1 - F(\bar{a}_t)) \cdot e_t}_{\text{agg. credit demand}}.$$

The left-hand side of this equation highlights the bubble's crowd-out effect: the aggregate speculative investment in the bubbly asset (p_t^b) crowds out the flow from aggregate savings ($G(\bar{a}_t)\beta e_t$) into the supply side of the credit market. By canceling e_t on both sides and defining the *bubble (over savings) ratio* as

$$\phi_t \equiv \frac{p_t^b}{\beta e_t},$$

the equation above can be rewritten as:

$$\beta \left(F(\bar{a}_t) - \phi_t \right) = \theta \cdot (1 - F(\bar{a}_t)). \tag{4.6}$$

Note that ϕ_t necessarily lies in $(0, 1)$.

From (4.5) and (4.6), the aggregate capital stock evolves according to:

$$K_{t+1} = \int_0^1 k_{t+1}^j dj = (\beta + \theta) \cdot \int_{\bar{a}_t} a dF(a) \cdot \left(\alpha K_t^\alpha L_t^{1-\alpha} + p_t^b \right). \tag{4.7}$$

Furthermore, indifference conditions (4.3) and (4.4) determine the interest rate and the growth of the bubbly asset, which are derived in Appendix A.1.2 as:

$$R_{t,t+1} = \bar{a}_t R^k_{t+1} = \bar{a}_t \alpha K^{\alpha-1}_{t+1}. \tag{4.8}$$

and

$$\frac{\phi_{t+1}}{\phi_t} = \frac{(1-\beta\phi_{t+1})\,\bar{a}_t}{(\beta+\theta)\int_{\bar{a}_t} adF(a)} \frac{F(\bar{a}_t)-\phi_t}{\rho F(\bar{a}_t)-\phi_t}. \tag{4.9}$$

Finally, the aggregate employment and equilibrium wage are determined by labor market conditions (2.8), (2.9), and (2.10).

4.2 Stochastic bubbly steady state

Given the equilibrium dynamics in the previous section, the stochastic bubbly steady state is characterized as follows. The steady-state version of credit-clearing condition (4.6) yields an equation that defines the bubble ratio ϕ as a function of $\bar{a}_b$:

$$\beta\,(F(\bar{a}_b)-\phi) = \theta \cdot (1-F(\bar{a}_b))\,. \tag{4.10}$$

The only difference between equation (4.10) and its counterpart (3.9) in the bubbleless benchmark is the presence of ϕ on the left-hand side, representing the fact that in the bubbly economy, relatively less productive entrepreneurs have the bubbly asset as an additional investment vehicle besides lending in the credit market. Since $\phi > 0$ (the bubble ratio is necessary positive in any bubbly equilibrium), these two equations also imply that $\bar{a}_b > \bar{a}_n$. Intuitively, as the bubbly asset provides a new investment opportunity, some entrepreneurs find it optimal to switch from investing in capital to investing in the bubbly asset, causing the productivity threshold to rise from $\bar{a}_n$ to $\bar{a}_b$.[10] Given that F is the CDF of a Pareto distribution over $[1,\infty)$ with shape parameter σ, from (4.10) we can get a closed-form expression for $\bar{a}_b$:

$$\bar{a}_b = \left(\frac{\beta+\theta}{\beta(1-\phi)}\right)^{1/\sigma} > \bar{a}_n = \left(\frac{\beta+\theta}{\beta}\right)^{1/\sigma}. \tag{4.11}$$

As in the bubbleless steady state, given the assumption that the rigidity parameter is a constant $\gamma \leq 1$, the downward wage rigidity condition (2.8) does not bind in steady state, leading to:

$$L_b = 1. \tag{4.12}$$

[10] As a consequence, the average entrepreneurial productivity is higher during a bubbly episode: it rises when the bubble arises and falls when the bubble collapses. Miao and Wang (2012) argue that this is consistent with empirical observations.

Then, from (4.7) and (4.12), the aggregate capital stock can be expressed as a function of $\bar{a}_b$ and ϕ:

$$K_b = \left(\frac{(\beta + \theta)\alpha}{1 - \beta\phi} \int_{\bar{a}_b} adF(a) \right)^{\frac{1}{1-\alpha}}. \tag{4.13}$$

From (4.4) and (4.13), the interest rate can also be expressed as a function of $\bar{a}_b$ and ϕ:

$$R_b = \frac{(1 - \beta\phi)\bar{a}_b}{(\beta + \theta) \int_{\bar{a}_b} adF(a)}. \tag{4.14}$$

Finally, from indifference condition (4.9) and from (4.14), the steady-state bubble ratio also has a closed-form solution:

$$\begin{aligned} \phi &= \frac{\rho - R_b}{1 - R_b} F(\bar{a}_b) \\ &= \frac{\theta}{\beta} \frac{1 - (1 - \beta\rho)\sigma}{\beta\sigma(1 - \rho) + \theta}. \end{aligned} \tag{4.15}$$

Equations (4.11) to (4.15) characterize the endogenous variables in the stochastic bubbly steady state. Given (4.15), the condition for the existence of a bubbly steady state can be characterized as follows:

Lemma 2. *A bubbly steady state exists if and only if:*

$$\frac{\sigma - 1}{\beta\sigma} < \rho, \tag{4.16}$$

Proof. Appendix A.2.1. □

The condition implies that for a stochastic bubble to exist, the probability that the bubble persists ρ has to be sufficiently high (as otherwise agents in the economy would deem the bubble to be too risky as an investment vehicle). Another direct corollary of (4.15) and (4.16) is that ϕ is strictly increasing in θ, implying that a more relaxed leverage constraint is associated with a larger bubble size in equilibrium.

For the rest of the paper, we will impose the bubble existence condition (4.16). Furthermore, as in the recent literature, we will focus on the relevant range of parameters in which the bubble is expansionary (the crowd-in effect dominates the crowd-out effect in steady state), that is,

$$K_b > K_n, \tag{4.17}$$

where the stochastic bubbly steady-state capital stock K_b is given by (4.13) and the bubbleless steady-state capital K_n is given by (3.13).[11]

4.3 Post-bubble dynamics

We now study the effect of the collapse of the bubble on the economy, which is the main focus of the paper. Suppose the bubble collapses at a certain period T, i.e., $p^b_{T+s} = 0,\ \forall s \geq 0$. As the expansionary effect of the bubble ends, the post-bubble capital stock and wage will decline toward the bubbleless steady state levels. However, if the downward wage rigidity constraint binds, then the wage cannot flexibly fall to clear the labor market. Instead, employment is determined by the demand of firms. The rigidly high wage thus leads to involuntary unemployment. The contraction in employment has two effects on the intertemporal equilibrium dynamics: it reduces the return from capital, and it reduces entrepreneurs' net worth. Both of these effects in turn reduce entrepreneurs' accumulation of capital. The wage rigidity thus amplifies and propagates the shock of bursting bubbles.

Let

$$s^* \equiv \min\{s \in \mathbb{N} | L_{T+s} = 1\},$$

where $\mathbb{N} \equiv \{0, 1, 2, \dots\}$. In other words, $T + s^*$ is the first post-bubble period when full employment is recovered. If $s^* > 0$, then we say the economy is in a *slump* between $T + 1$ and $T + s^* - 1$, as during this period there is *involuntary unemployment*: $L_t < 1$ for all $T + 1 \leq t \leq T + s^*$.

Given the tractability of the model, we can analytically characterize the post-bubble dynamics, including the depth and duration of the post-bubble unemployment episode. Intuitively, the economy escapes the slump when the equilibrium wage has fallen enough that the downward wage rigidity no longer binds, and the economy regains full employment and recovers toward the initial steady state.

Proposition 3. [Post-bubble slump] *Suppose the bubble collapses in period T. Then the economy enters a slump for s^* periods. The post-bubble equilibrium dynamics are given by:*

$$K_{T+s+1} = \left[\mathcal{A}_n \alpha K_T^{\alpha-1}\right]^{s+1} \gamma^{\frac{\alpha-1}{\alpha}\frac{s(s+1)}{2}} K_T \tag{4.18}$$

$$L_{T+s} = \left(\frac{1-\alpha}{w_{T+s}}\right)^{\frac{1}{\alpha}} K_{t+s}$$

$$w_{T+s} = \gamma^s w_T, \qquad \forall 0 \leq s < s^*,$$

[11] Written in exogenous parameters, this assumption is equivalent to $(1-\phi)^{\frac{\sigma-1}{\sigma}} > 1 - \beta\phi$, where ϕ is given by (4.15).

and the duration of the slump is given by:

$$s^* = \max\left\{0,\ \left\lceil 2\alpha \log_{\frac{1}{\gamma}}\left(\frac{K_T}{K_n}\right) - \frac{1+\alpha}{1-\alpha}\right\rceil\right\} \tag{4.19}$$

where the ceiling function $\lceil x \rceil$ denotes the least integer greater than or equal to x. The economy regains full employment and follows the dynamics of Section 3 for $t \geq T + s^*$.

Proof. Appendix A.2.2. □

4.3.1 Numerical illustration

In this section, we conduct a simple calibrated numerical exercise to illustrate the equilibrium dynamics. Since the model is intentionally designed to be stylized and parsimonious, this exercise should not be viewed as a full-fledged quantitative analysis but rather a suggestive quantitative illustration of the model's predictions. In this section, we also make two basic extensions to improve the mapping of the model to data: first, we assume the economy grows at an exogenous rate $g \geq 0$; second, we assume capital partially depreciates at rate $\delta \in [0, 1]$ (see Appendix A.1.6 for details).

We then calibrate the model to Japanese data as follows. We will choose parameters to match the pre-bubble phase (1970-1986) and the boom phase (the bubble period of 1987-1991) and let the model predict the bust phase (post-1991). There are two sets of parameters, the first of which can be set using relatively standard values from the literature. Specifically, we set a period to be a year, the capital share to be $\alpha = 0.33$, the discount factor to be $\beta = 0.96$, the capital depreciation rate to be $\delta = 0.09$, and the exogenous growth rate to be $g = 0.03$. Following Schmitt-Grohé and Uribe (2016), the downward wage rigidity parameter is set to be close to one: $\gamma = 0.97$. The second set of parameters, consisting of shape parameter σ of the productivity distribution, the financial friction parameter θ, and the bubble persistence parameter ρ, are less standard and will be calibrated. In particular, we choose σ, θ, and ρ to match R_n, $\frac{K_n}{Y_n}$, and $\frac{K_b+p_b}{Y_b}$ to three moments: the average real interest rate of 1.02 in Japan in the pre-bubble phase, the wealth over income ratios in the pre-bubble phase and in the bubble phase of 3.67 and 5.18, respectively.[12] The calibrated parameter values are $\sigma = 15.14$, $\theta = 0.05$, and $\rho = 0.998$.

Figure 1 illustrates a simulated equilibrium path for detrended aggregate variables under this parametrization. On this path, we set the economy at the stochastic bubbly steady state in the initial period, and then the bubble collapses in $t = 10$ (in the simulation, agents

[12] Data for the wealth over income ratios come from Piketty and Zucman (2014); data for GDP and real interest rate come from the World Bank; the dating of the bubble period is according to Shioji (2013).

rationally expect that the bubble is stochastic and can burst in any period). Equilibrium variables are plotted with the solid lines, and for comparison, the bubbleless steady state counterparts are plotted with dashed lines. As seen in the figure, as long as the bubble lasts, the economy experiences a boom in entrepreneurial net worth (relative to the bubbleless steady state), which leads to a boom in aggregate credit to entrepreneurs, consequently an increase in aggregated capital accumulation, output, wage, and consumption (both across entrepreneurs and workers).[13] Since the boom in the capital stock and bubble value is larger than that in output, the wealth over output ratio also increases during the bubbly episode.

Then, after the bubble collapses, the economy begins a contraction. Without nominal rigidities, the labor market would be flexible and the equilibrium wage, along with other aggregate variables in the post-bubble economy, would simply converge back to the bubbleless steady state with full employment. However, with downward wage rigidity, the post-bubble equilibrium wage may not flexibly fall to clear the labor market, leading to involuntary unemployment. The drop in employment not only reduces the economy's output, but also has important intertemporal effects. On the one hand, it reduces the net worth of entrepreneurs. On the other hand, it reduces the return rate on capital. Both of these effects depress capital accumulation, explaining the contractions of aggregate economic activities during the slump with involuntary unemployment.

As a consequence, aggregate output, net worth, capital, credit, and consumption can *undershoot* (i.e., drop below) the pre-bubble trend. The figure highlights the boom-bust trade-off: the bubble leads to a boom of about 3.4% in output (relative to the bubbleless steady state) as long as it persists, but its collapse leads to a recession, where the aggregate output drops as much as 2.1% below the bubbleless steady state. The economy experiences long "lost decades": about 20 years of declining output, which only recovers to its bubbleless trend after about 40 years.[14]

5 Welfare and policy analysis

We will now investigate the welfare effects of stochastic bubbles. We will focus on the expected welfare in steady state, which can be computed in closed forms. The welfare functions are defined as the lifetime expected utility in the corresponding steady state.

[13]Note that the boom in consumption is more pronounced for entrepreneurs, implying that entrepreneurs tend to gain more from the bubble than workers (as the increase in net worth allows entrepreneurs to increase their investment). This asymmetry could lead to interesting political economy implications, which are absent from this model and are left for future research.

[14]We reiterate that these numbers should be taken with caution, as the model is highly stylized and does not take into account other important phenomena for Japan, such as demographics.

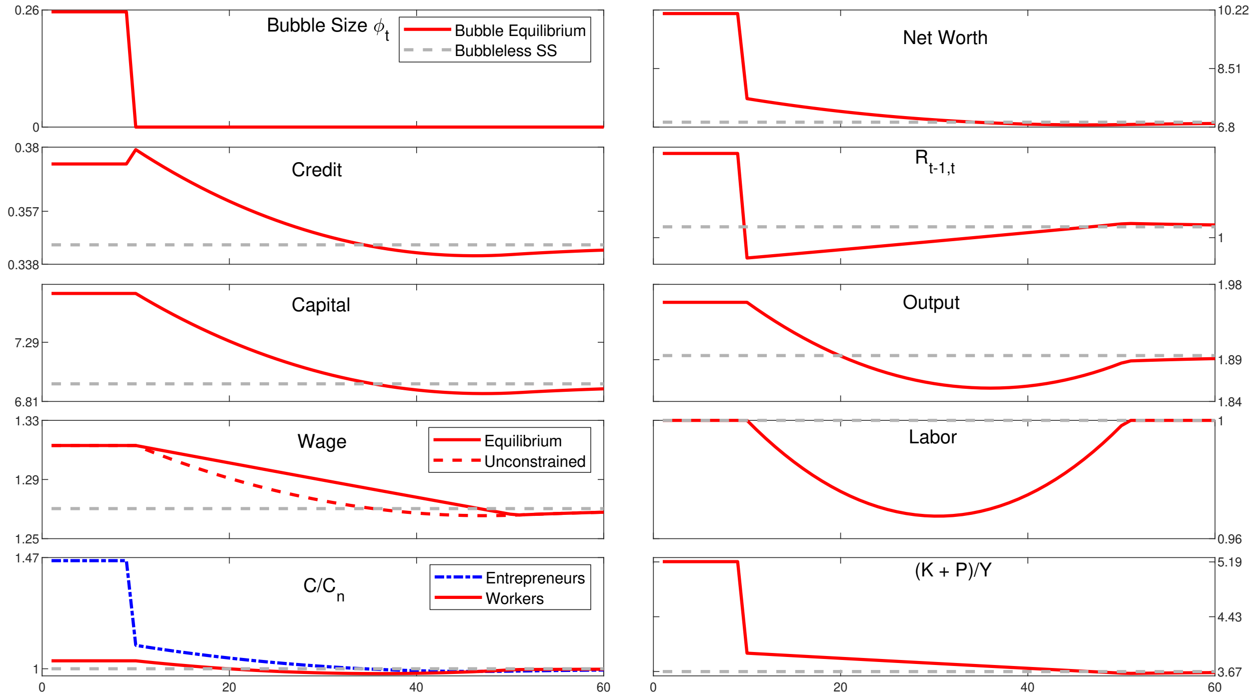

Figure 1: Equilibrium dynamics with bubble boom-bust. Solid lines represent detrended equilibrium variable values; gray dashed horizontal lines represent the corresponding bubbleless steady state values.

5.1 Workers

We start with the welfare of workers, which is more straightforward. The lifetime expected utility of a representative worker in the bubbleless steady state is simply $\frac{1}{1-\beta}\log(c_n^w)$, where $c_n^w = w_n = (1-\alpha)K_n^\alpha$. Thus, the welfare of workers in the bubbleless steady state, denoted by W_n, is given by:

$$W_n = \frac{\log\left[(1-\alpha)K_n^\alpha\right]}{1-\beta}, \tag{5.1}$$

where recall that K_n is given by (3.13).

The welfare of workers in the stochastic bubbly steady state features a boom-bust trade-off. As long as the bubble persists, their consumption is larger than that in the bubbleless steady state: $c_b^w = (1-\alpha)K_b^\alpha > c_n^w$. However, after the bubble collapses, the economy enters a slump for s^* periods, during which workers suffer from involuntary unemployment. The lifetime expected utility of a representative worker will be a weighted sum of the utility before and after the bubble collapses, with the weights depending on the bubble's risk of bursting $1-\rho$. In Appendix A.1.3, we show that the welfare of workers in the stochastic bubbly steady state, denoted by W_b, is given by:

$$W_b = \underbrace{\frac{\log\left[(1-\alpha)K_b^{\alpha}\right]}{1-\rho\beta}}_{\text{expected utility when bubble persists}} + \underbrace{\frac{\beta(1-\rho)}{1-\rho\beta}\left[\Gamma_0(s^*) + \Gamma_1(s^*)\log K_b\right]}_{\text{expected utility after bubble collapses}}, \tag{5.2}$$

where the expressions for Γ_0 and Γ_1 are provided in Appendix A.1.3.

It is clear that W_b depends on the bubble's risk of bursting $1-\rho$ and on the duration of the post-bubble slump s^*, which is itself a function of the degree of wage rigidity γ (recall (4.19)). It is straightforward to see that the slump length s^* is increasing in the degree of rigidity γ, and consequently, W_b is decreasing in γ. Similarly, W_b is increasing in the persistence probability ρ, i.e., a safer bubble yields a higher payoff.

5.2 Entrepreneurs

The welfare functions of entrepreneurs are more complex, due to their heterogeneity and portfolio optimization. In the bubbleless steady state, the lifetime expected utility of an entrepreneur j that starts the period with a net worth e^j, denoted by $V_n(e^j)$ satisfies the following equation:

$$V_n(e^j) = \underbrace{\log\left((1-\beta)e^j\right)}_{\text{current period utility}} + \beta \underbrace{F(\bar{a}_n)V_n(R_n\beta e^j)}_{\text{continuation value if } a^j \leq \bar{a}_n} + \beta \underbrace{\int_{\bar{a}_n} V_n\left(aR_n^k(\beta+\theta)e^j\right)dF(a)}_{\text{continuation value if } a^j > \bar{a}_n}. \tag{5.3}$$

Appendix A.1.3 provides an analytical solution to this equation. To streamline the analysis, let us assume that each entrepreneur starts the bubbleless steady state with an equal net worth, leading to $e^j = \alpha K_n^{\alpha}$. Then the bubbleless steady state entrepreneurial welfare is simply given by:

$$V_n \equiv V_n(\alpha K_n^{\alpha}).$$

Similarly, in the bubbly steady state, lifetime expected utility of an entrepreneur j that starts the period with a net worth e^j, denoted by $V_b(e^j)$ satisfies the following equation:

$$\begin{aligned} V_b(e^j) = & \underbrace{\log\left((1-\beta)e^j\right)}_{\text{current period utility}} \\ & +\rho\beta \underbrace{\left\{F(\bar{a}_b)V_b\left(\rho\frac{F(\bar{a}_b)-\phi}{\rho F(\bar{a}_b)-\phi}\bar{a}_b R_b^k \beta e^j\right) + \int_{\bar{a}_b} V_b\left(aR_b^k(\beta+\theta)e^j\right)dF(a)\right\}}_{\text{if bubble persists}} \\ & +(1-\rho)\beta \underbrace{\left\{F(\bar{a}_b)V_{burst}\left(\frac{F(\bar{a}_b)-\phi}{F(\bar{a}_b)}\bar{a}_b R_b^k \beta e^j\right) + \int_{\bar{a}_b} V_{burst}\left(aR_b^k(\beta+\theta)e^j\right)dF(a)\right\}}_{\text{if bubble bursts}}, \end{aligned} \tag{5.4}$$

where $V_{burst}(\cdot)$ denotes the continuation value after the bubble bursts. Appendix A.1.3 provide analytical solutions to $V_b(\cdot)$ and $V_{burst}(\cdot)$.

As in the bubbleless case, we assume for simplicity that each entrepreneur starts the bubbly steady state with an equal net worth, leading to $e^j = \frac{\alpha K_b^\alpha}{1-\beta\phi}$. Then the bubbly steady state entrepreneurial welfare is given by:

$$V_b \equiv V_b\left(\frac{\alpha K_b^{1-\alpha}}{1-\beta\phi}\right).$$

From the analyses in Sections 5.1 and 5.2, we can show the following proposition, which states that if the bubble is sufficiently risky and if there is sufficient wage rigidity, then agents in the economy are better off if there were no stochastically bursting bubbles.

Proposition 4. [Welfare-reducing stochastic bubble] *There exists $\bar{\gamma} \in (0,1)$ such that if there is sufficient wage rigidity ($\gamma > \bar{\gamma}$) and the bubble is sufficiently risky ($\rho < \bar{\rho} \equiv 1 - \frac{\alpha(1-\beta)^2}{\beta(\beta-\alpha)}$),[15] then the bubble reduces steady-state welfare for both workers and entrepreneurs:*

$$W_b < W_n, \ V_b < V_n.$$

Proof. Appendix A.2.3. □

5.3 Leaning-against-the-bubble policy

We have established that the boom and bust of a bubbly episode can push the economy into a recession with involuntary unemployment. The fundamental source of inefficiencies in this

[15] As a simple numerical illustration, assuming $\alpha = 0.33$, $\beta = 0.96$, then the proposition implies that if $\gamma = 1$ (wages cannot decline) and $\rho < \bar{\rho} = 0.999$, then agents in the economy will be better off without bubbles.

environment is a form of "bubbly pecuniary externality": individual entrepreneurs do not internalize the effect of their investment portfolio choices in driving a large bubbly boom, which will lead to a large bust due to the downward wage rigidity.

Under this context, policy responses are warranted. We will focus on a macroprudential policy of taxing bubble speculation, so that private agents internalize the pecuniary externality of the speculative bubble's boom and bust. As we will show, this policy has an effect of reducing the bubble size and is thus akin to the kind of "leaning-against-the-wind" policies that have been extensively discussed in the policy circle (e.g., Barlevy 2012, 2018) and is similar to the type of tax policies often considered in the macroprudential literature (e.g., Lorenzoni 2008; Gertler et al. 2012; Jeanne and Korinek 2013).[16]

Formally, consider a benevolent constrained policymaker who cares about the welfare of both workers and entrepreneurs. The policymaker is constrained in the sense that she cannot undo the frictions in the credit market (e.g., via redistribution) or frictions in the labor market. However, she can levy a macroprudential tax τ on the return from bubble speculation. As our focus is on steady-state welfare, we will assume for simplicity that the tax rate is constant. Then, under the policy, the budget constraint (2.1) becomes:

$$c_t^j + I_t^j + p_t^b b_t^j = R_t^k k_t^j + d_t^j - R_{t-1,t} d_{t-1}^j + (1-\tau) p_t^b b_{t-1}^j + T_t^j.$$

The after-tax return on bubble speculation for the entrepreneur is then $(1-\tau)p_{t+1}^b/p_t^b$. The policymaker rebates the tax revenue back to the entrepreneur through a lump-sum transfer:

$$T_t^j = \tau p_t^b b_{t-1}^j,$$

which the entrepreneur takes as given. Consistent with the aforementioned notion of constrained policymaking, this specification of tax and transfer implies that the policymaker cannot redistribute resources across entrepreneurs.

Appendix A.1.4 derives the bubbly equilibrium dynamics and steady state with the tax. A key result is that the steady-state bubble size is a decreasing function of the macroprudential tax τ:

$$\phi(\tau) = \frac{\theta(1-\sigma(1-\beta\rho(1-\tau)))}{\beta(\beta(1-\rho)(1-\tau)\sigma + \theta(1-\sigma\tau))} \leq \phi. \tag{5.5}$$

When $\tau = 0$, the bubble size collapses to $\phi(0) = \phi$, which is the laissez-faire size as derived

[16] As in most of the literature, we implicitly assume that policymakers can observe the bubble. Of course, this is a strong assumption. Alternatively, one can interpret the macroprudential policy as imposing a tax on speculative investments in broad classes of assets that are ex ante perceived to be likely to experience bubbles, such as real estate or stocks of certain types of companies.

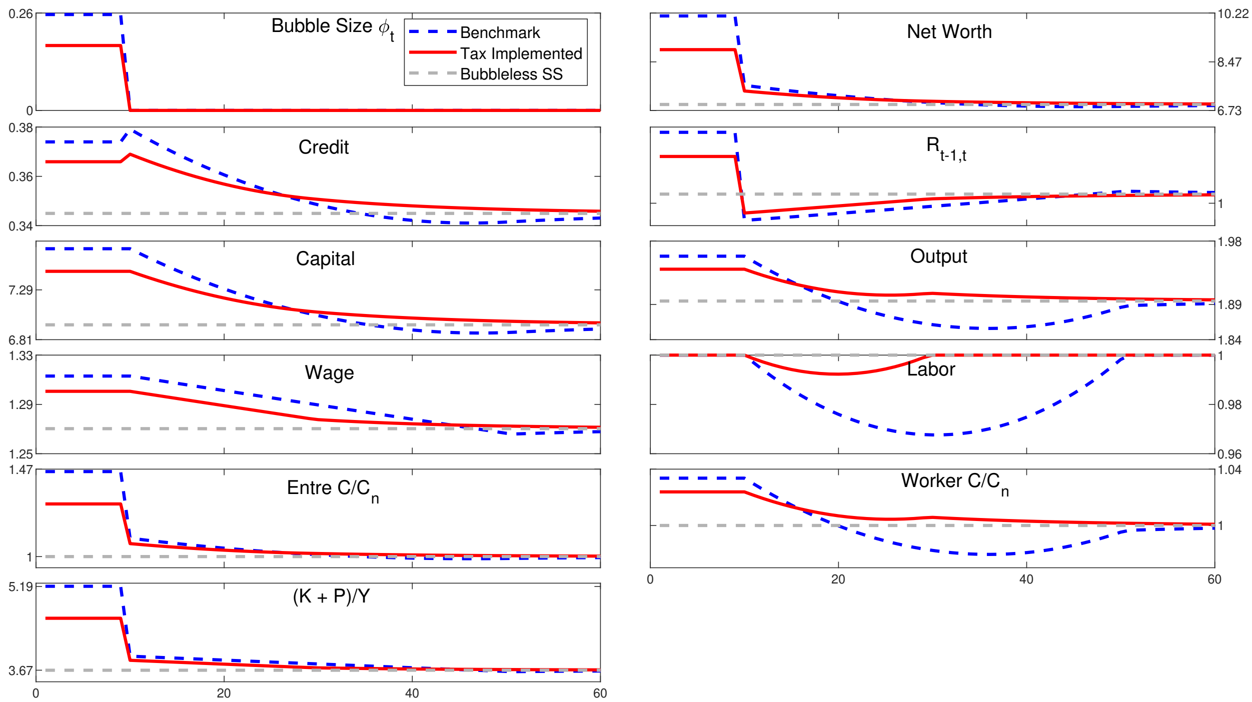

Figure 2: Equilibrium dynamics with bubble boom-bust. Solid and dashed lines represent detrended equilibrium variable values with and without tax, respectively; gray dashed horizontal lines represent the corresponding bubbleless steady state values.

in Section 4. Hence, the tax not only makes the bubble smaller, but also makes the bubble harder to arise. Specifically, under the tax, a bubbly steady state exists ($\phi(\tau) > 0$) if and only if

$$\frac{\sigma - 1}{\beta\sigma} < \rho(1 - \tau),$$

which is more stringent than the laissez-faire existence condition (4.16). Thus, by setting $\tau \geq \bar{\tau} \equiv 1 - \frac{\sigma-1}{\beta\sigma\rho}$, the policymaker can effectively rule out the possibility of a bubbly equilibrium.[17] Without loss of generality, we can thus focus on $\tau \leq \bar{\tau}$.

Figure 2 illustrates an equilibrium path with and without the tax. The dashed lines represent the laissez-faire equilibrium path (exactly as plotted in Figure 1), while the solid lines represent the economy under a macroprudential tax of $\tau = 1\%$. As shown in the figure, the tax effectively reduces the bubble size. There is a boom-bust trade-off: the policy mitigates the effects of a collapsing bubble (the slump is shorter and less severe), but it also reduces the boom in aggregate economic activities while the bubble lasts.

To evaluate the welfare effects of the tax, Appendix A.1.4 also derives the bubbly steady-state welfare expressions for both workers and entrepreneurs under the tax, denoted by

[17] For the parameter values used in Section 4.3.1, the associated value for $\bar{\tau}$ is 2.54%.

$W_b(\tau)$ and $V_b(\tau)$, respectively. Assume the policymaker assigns a Pareto weight $\lambda \in [0,1]$ on the welfare of workers (and $1-\lambda$ on the welfare of entrepreneurs). A *constrained optimal policy* is a macroprudential tax τ that maximizes the Pareto-weighted bubbly steady-state welfare:[18]

$$\max_{\tau \leq \bar{\tau}} \lambda W_b(\tau) + (1-\lambda) V_b(\tau).$$

Due to the highly nonlinear behaviors of W_b and V_b, in general the optimal tax can only be solved for with numerical methods. However, an interesting implication of our previous analysis is that when the conditions of Proposition 4 are met, an optimal policy is to rule out the possibility of the bubble altogether by setting $\tau = \bar{\tau}$. Formally:

Corollary 5. *There exists $\bar{\gamma} \in (0,1)$ such that if there is sufficient wage rigidity ($\gamma \geq \bar{\gamma}$) and the bubble is sufficiently risky ($\rho < \bar{\rho} \equiv 1 - \frac{\alpha(1-\beta)^2}{\beta(\beta-\alpha)}$), then a constrained-optimal macroprudential tax is to set*

$$\tau = \bar{\tau},$$

which effectively rules out the possibility of a bubble.

Proof. Appendix A.2.4. □

6 Zero lower bound

We now extend the real model by introducing downward *nominal* wage rigidity (DWNR) and a nominal interest rule that is constrained by the zero lower bound (ZLB). We will show that the collapse of a large bubble can push the economy into a "secular stagnation" equilibrium, where the ZLB on the nominal interest rate constrains the monetary authority from achieving the inflation target and the DNWR binds, leading to involuntary unemployment. Interestingly, under certain conditions, because of the interaction between the ZLB and the DNWR, the post-bubble economy may never exit from the liquidity trap and instead may converge to a bad bubbleless steady state with persistent unemployment.

DNWR: Formally, let P_t denote the price level of the consumption good in period t in unit of a currency, and let w_t continue to denote the real wage. Instead of the real wage rigidity condition (2.8), we impose the following assumption on nominal wages (à-la Schmitt-Grohé and Uribe, 2017):

$$P_t w_t \geq \gamma(L_t) P_{t-1} w_{t-1}, \forall t \geq 1,$$

[18] It is straightforward to show that a constrained-optimal policy implements a constrained-efficient allocation, where the notion of constrained efficiency is defined in Section A.1.5 of the Appendix, following the macroprudential literature.

where the degree of rigidity γ is now a function of L_t, which for simplicity is assumed to be:

$$\gamma(L) \equiv \gamma_0 L^{\gamma_1},\ \gamma_0, \gamma_1 > 0.$$

The fact that γ is increasing in L implies that nominal wages are more flexible as unemployment increases and more rigid as employment increases. Furthermore, as we will show, the assumption $\gamma_1 > 0$ implies that there could exist a "secular stagnation" bubbleless steady state that features involuntary unemployment. The nominal wage rigidity condition can be rewritten as:

$$w_t \geq \frac{\gamma(L_t)}{\Pi_{t-1,t}} w_{t-1}, \tag{6.1}$$

where $\Pi_{t-1,t} \equiv \frac{P_t}{P_{t-1}}$ is the gross inflation rate between $t-1$ and t.

ZLB: To close the model, we need to specify how price levels are determined. As is standard in the literature, we assume that the entrepreneurs can trade nominal government bonds, which yield an interest rate $1 + i_{t,t+1}$ and are available in net zero supply.[19] A monetary authority sets the nominal interest rate $1 + i_{t,t+1}$ between each period t and $t+1$ according to a Taylor rule subject to a ZLB on $i_{t,t+1}$:

$$1 + i_{t,t+1} = \max\left\{1,\ R^f_{t,t+1} \left(\Pi_{t-1,t}\right)^{\zeta} \left(\Pi^*\right)^{1-\zeta}\right\}, \tag{6.2}$$

where $R^f_{t,t+1}$ is the real interest rate that would prevail with full employment in $t+1$ (i.e., $L_{t+1} = 1$), $\Pi^* > 0$ is an inflation target, and $\zeta > 1$ is a constant. As is standard, the rule implies that if the monetary authority were not constrained by the ZLB, inflation would be stabilized at the target Π^*.[20]

The definition of an equilibrium is similar to before, except that we have an additional endogenous variable P_t in each period for the price level, the previous complementary-slackness condition for the labor market is now replaced with

$$(1 - L_t)\left(w_t - \frac{\gamma(L_t)}{\Pi_{t-1,t}} w_{t-1}\right) = 0,$$

and the monetary policy rule (6.2) holds.

[19] For algebraic simplicity, we have abstracted away from explicitly introducing the buying and selling of nominal bonds that are in net zero supply into the budget constraints of entrepreneurs.

[20] We do not model optimal monetary policy explicitly here. This is because in our model, an increase in the inflation rate always weakly improves welfare by mitigating the wage rigidity. Thus, setting a very high inflation target to avoid involuntary unemployment and the ZLB will be optimal. Realistically, there are costs of inflation, such as the costs associated with nominal price rigidities, that are not modeled explicitly here. Also, in practice, central banks tend to follow similar Taylor rules with inflation targets.

6.1 Bubbleless equilibrium and multiple steady states

Let us first characterize the bubbleless equilibrium. Detailed derivations are relegated to Appendix A.1. In the absence of bubbles, as in section3, the cutoff threshold and the evolution of the capital stock are again given by (3.9) and (3.11). The equilibrium wage w_t and employment L_t depend on whether the DNWR binds or not. Similarly, the inflation rate depends on whether the ZLB on the nominal interest rate binds or not. Formally,

$$w_t = \max\left\{(1-\alpha)K_t^{\alpha},\ \gamma(L_t)\frac{w_{t-1}}{\Pi_{t-1,t}}\right\}$$

and

$$\frac{\max\left\{1,\ R^f_{t,t+1}\left(\Pi_{t-1,t}\right)^{\zeta}\left(\Pi^*\right)^{1-\zeta}\right\}}{\Pi_{t,t+1}} = R_{t,t+1}.$$

For the rest of the paper, we assume:

$$\Pi^* > \gamma_0 > \frac{1}{R_n}, \tag{6.3}$$

where R_n is the bubbleless steady state interest rate, as given by (3.15). Under this assumption, because of the kink in the Taylor rule and the fact that the degree of wage rigidity is a function of employment, there are two possible bubbleless steady states. In the "good" steady state (which will continue to be denoted with a subscript n), there is full employment ($L_n = 1$), the ZLB is slack, the inflation is at the target Π^*, the capital stock is given by K_n as in (3.13), and the real interest rate is given by R_n as in (3.15).

There is another "bad" steady state, where the ZLB binds ($i = 0$) and inflation is below target, and there is involuntary unemployment ($\underline{L} < 1$), leading to a lower capital stock:

$$\underline{K} = K_n\underline{L} < K_n.$$

The real interest rate is given by the indifference condition of the marginal investor: $\underline{R} = \bar{a}_n\alpha\underline{K}^{\alpha-1}\underline{L}^{1-\alpha} = R_n$. The inflation rate is determined by the Fisher equation $\underline{R}\underline{\Pi} = 1$, or equivalently

$$\underline{\Pi} = \frac{(\beta+\theta)\int_{\bar{a}_n} a dF(a)}{\bar{a}_n},$$

which is smaller than the target Π^* under assumption (6.3). The employment level $\underline{L}$ is determined by the binding DNWR condition $1 = \frac{\gamma(\underline{L})}{\underline{\Pi}}$, which gives:

$$\underline{L} = \left(\underline{\Pi}/\gamma_0\right)^{\frac{1}{\gamma_1}}.$$

Assumption (6.3) guarantees that there is involuntary unemployment in this steady state: $\underline{L} < 1$, and the ZLB does indeed bind: $\underline{R}\Pi^{\zeta}(\Pi^*)^{1-\zeta} < 1$.

6.2 Bubbly equilibrium

We now analyze the bubbly economy. We focus on the relevant parameter range in which the DNWR and the ZLB are slack as long as the bubble persists.[21] Then as inflation is stabilized at the target, the bubbly equilibrium dynamics are as characterized in Section 4, and the steady state is as characterized in Section 4.2.

The post-bubble dynamics will however be different. Suppose the economy reaches the bubbly steady state and then bubble collapses at period T (i.e., $p^b_{T+s} = 0,\ \forall s \geq 0$). The collapse of the bubble *exerts downward pressure on the real interest rate* through two channels. First, after the bubble collapses, the productivity of the marginal investor decreases from $\bar{a}_b$ to $\bar{a}_n$. Thus, instead of the identity $R_{T,T+1} = \bar{a}_b R^k_{T+1}$ that would have prevailed if the bubble did not collapse in T, the real interest is given by $R_{T,T+1} = \bar{a}_n R^k_{T+1}$, with $\bar{a}_n < \bar{a}_b$. Second, as the bubble has an expansionary effect on capital accumulation, the post-bubble economy will follow the bubbleless dynamics as specified in the previous section but with an initial capital stock K_b, which is larger than that in the good steady state K_n. A high capital stock leads to a low marginal product of capital and thus a low interest rate. The combination of these two mechanisms exerts a downward pressure on the real interest rate and thus the nominal interest rate. If the bubble leads to sufficient large accumulation of capital stock, its collapse can push the interest rate against the ZLB. Formally:

Proposition 6. [Effect of bubble's collapse on real interest rate] *Suppose the economy has reached the steady state with a large expansionary bubble and then the bubble collapses in a period denoted by T. If the bubbly steady state K_b is sufficiently large such that*

$$K_b > \bar{K} \equiv (\bar{a}_n \mathcal{A}_n \Pi^*)^{\frac{1}{\alpha(1-\alpha)}} K_n,$$

then the Taylor rule (6.2) is constrained by the ZLB:

$$1 + i_{T,T+1} = 1 > R^f_{T,T+1} \left(\Pi_{T-1,T}\right)^{\zeta} \left(\Pi^*\right)^{1-\zeta}. \tag{6.4}$$

Proof. Appendix A.2.5. □

Remark 7. One could think of this as corresponding to a situation of "investment hangover," or capital overinvestment, at the end of an economic boom (Rognlie et al., 2014). The differ-

[21] This is the case when the initial capital stock K_0 and the initial bubble value p^b_0 are below the bubbly steady state levels, and $R_b > 1/\Pi^*$, where R_b is given by (4.14).

ence between our paper and Rognlie et al. (2014) is that the overinvestment is endogenous in our framework, while it is imposed exogenously in theirs.

The next result shows that in the post-bubble economy, whenever the ZLB binds, the the DNWR must also bind:

Lemma 8. [ZLB implies DNWR] *For any* $t \geq T+1$, *if* $i_{t-1,t} = 0$ *then* $L_t < 1$.

Proof. Appendix A.2.6. □

We say that the economy is in a *liquidity trap* in period t if the ZLB binds (implying $i_{t-1,t} = 0$) and the DNWR binds (implying $L_t < 1$). We now show a stark result that, under certain conditions, the post-bubble economy may *never* escape from the liquidity trap.[22] Specifically, we will construct a post-bubble equilibrium path where $L_t < 1$ and $i_{t-1,t} = 0$ for all $t \geq T+1$. The laws of motion of equilibrium quantities and prices K_t, L_t, and $\Pi_{t-1,t}$ are given by the bubbleless law of motion of capital (as derived in Section 6.1):

$$K_t = (\beta + \theta) \int_{\bar{a}_n} adF(a) \cdot \alpha K_{t-1}^{\alpha} L_{t-1}^{1-\alpha}, \tag{6.5}$$

the binding DNWR:

$$\frac{(K_t/L_t)^{\alpha}}{(K_{t-1}/L_{t-1})} = \frac{\gamma(L_t)}{\Pi_{t-1,t}}, \tag{6.6}$$

and a Fisher equation, which states that the marginal investor is indifferent between lending in the credit market and buying the nominal government bonds:

$$\underbrace{\bar{a}_n \alpha K_t^{\alpha-1} L_t^{1-\alpha}}_{R_{t-1,t}} \Pi_{t-1,t} = 1. \tag{6.7}$$

For the prices and quantities to indeed constitute an equilibrium, a necessary and sufficient condition is that the ZLB must bind, i.e., $R_{t-1,t}^{f} \left(\Pi_{t-2,t-1}\right)^{\zeta} \left(\Pi^{*}\right)^{1-\zeta} < 1$ for all t, where the real interest rate with full employment is given by $R_{t-1,t}^{f} = \bar{a}_n \alpha K_t^{\alpha-1}$. This inequality holds if and only if K_t is sufficiently large for all t, and the equilibrium dynamics above can be solved for in closed form.

The following proposition shows that, under certain conditions, the collapse of a bubble can push the economy into a *permanent* liquidity trap:

[22] In reality, there can be shocks (not modeled here) that pull the economy out of the liquidity trap, such as a good technology shock or another bubbly episode.

Proposition 9. [Post-bubble secular stagnation] *Let* $\{K_{T+t}, L_{T+t}, \Pi_{T+t-1,T+t}\}$ *be defined by the following closed-form expressions:*

$$\begin{aligned} K_{T+t} &= (\mathcal{A}_n\alpha)^{\frac{1-\alpha^t}{1-\alpha}} (K_T)^{\alpha^t} \left(\frac{\mathcal{A}_n}{\gamma_0\bar{a}_n}\right)^{\frac{1-\alpha}{\gamma_1}\left(\frac{1-\alpha^{t-1}}{1-\alpha} - \frac{1-((1+\gamma_1)\alpha)^{t-1}}{(1-(1+\gamma_1)\alpha)(1+\gamma_1)^{t-1}}\right)} \\ L_{T+t} &= \left(\frac{\mathcal{A}_n}{\gamma_0\bar{a}_n}\right)^{\frac{(1+\gamma_1)^t-1}{\gamma_1(1+\gamma_1)^t}} \\ \Pi_{T+t-1,T+t} &= \frac{1}{\alpha\bar{a}_n}\left(\frac{K_{T+t}}{L_{T+t}}\right)^{1-\alpha}. \end{aligned}$$

1. *These values constitute a post-bubble equilibrium path if and only if*

   $K_t > \left(\alpha\bar{a}_n \left(\frac{\Pi_{t-2,t-1}}{\Pi^*}\right)^\zeta \Pi^*\right)^{\frac{1}{1-\alpha}}$ *for all* $t \geq T+1$.

2. *On this equilibrium path, the economy experiences involuntary unemployment:* $L_{T+t} < 1$ *for all* $t > 0$*, and the economy converges to the bad bubbleless steady state with involuntary unemployment and below-target inflation described in Section 6.1.*

Proof. Appendix A.2.7. □

Figure 3 plots a simulated equilibrium path, in a manner similar to the simulation in Figure 1 (the dashed horizontal lines represent the good bubbleless steady state).[23] As seen in the figure, the collapse causes the real and nominal interest rate to fall sharply, and the nominal interest rate hits the ZLB. After the collapse, entrepreneurs cut down their investment, leading to a decline in the capital stock. The decline in the capital stock in turn causes a decline in the marginal product of labor. Wage would thus need to fall in order to clear the labor market. However, the wage floor creates a wedge that prevents labor market clearing, leading to involuntary unemployment. The economy gradually converges to the *bad* bubbleless steady state.

Why does the post-bubble economy converge to the bad bubbleless steady state? As explained previously, a large expansionary bubble can lead to a large overinvestment of capital (relative to the good bubbleless steady state). The bubble's eventual collapse will necessarily cause a sharp adjustment in market-clearing wages and a sharp decline in the real and nominal interest rates. A large drop in the interest rates can push the economy into a liquidity trap. The liquidity trap perpetuates as long as the monetary authority is constrained by the ZLB, leading to an inflation that is below the target. Low inflation in turn

[23] Again the simulation is for a model with exogenous TFP growth rate g and partial capital depreciation rate δ. Parameter values are $\gamma_0 = 0.99$, $\gamma_1 = 0.099$, $\Pi^* = 1.02$, and $\zeta = 1.5$ (following Schmitt-Grohé and Uribe 2017), while the rest are as in Section 4.3.1.

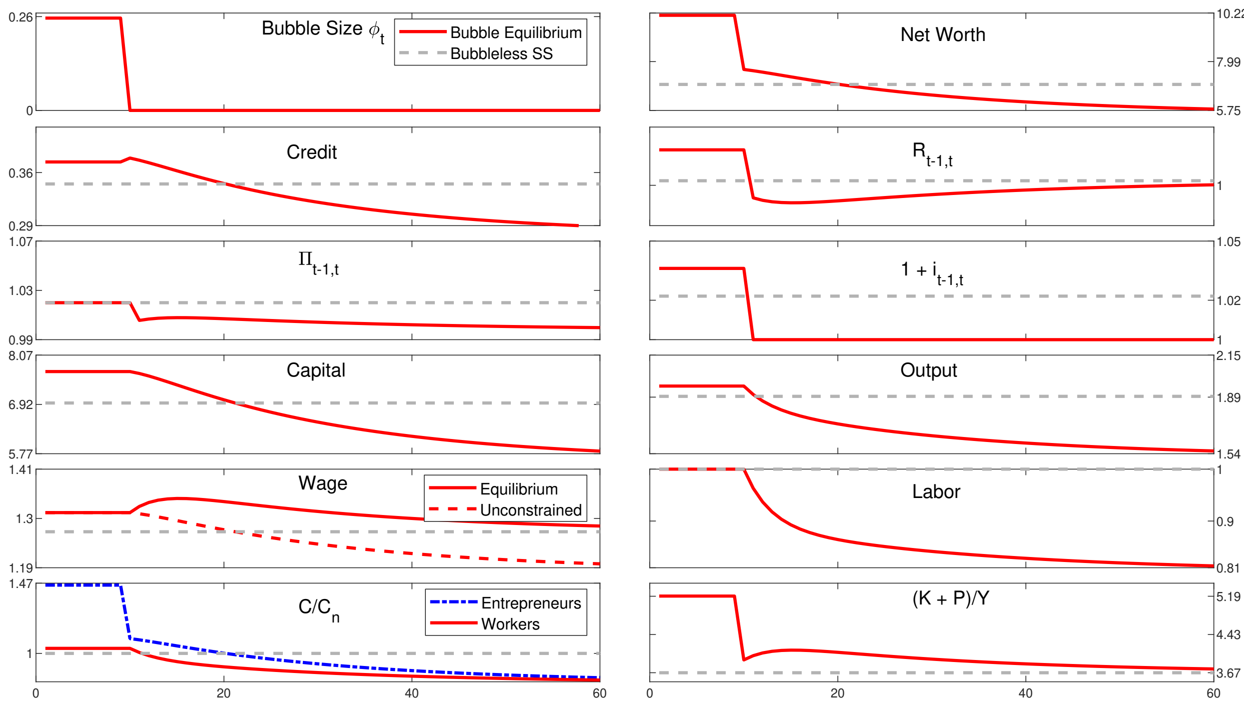

Figure 3: Persistent post-bubble liquidity trap. Solid lines represent detrended equilibrium variable values; gray dashed horizontal lines represent the corresponding good bubbleless steady state values.

exacerbates the DNWR, leading to lower employment. Finally, lower employment further reduces the marginal product of capital and the interest rates, creating a vicious cycle that perpetuates the liquidity trap.

7 Conclusion

We have developed a tractable rational bubbles model with downward wage rigidity. We show that expansionary bubbles could boost economic activities, but their collapse can push the economy into a persistent slump with involuntary unemployment, and investment, output, and consumption depressed below the pre-bubble levels. Under certain conditions, the economy is better off without stochastic bubbles altogether. The model's predictions are consistent with stylized features of recent bubbly episodes. The model highlights the trade-off between the economic gains during the boom due to the bubble and the loss from the bust. A macroprudential leaning-against-the-bubble policy of taxing speculative investment can help balance this boom-bust trade-off.

The model has several limitations. For instance, the model predicts that, even though stochastic bubbles can reduce welfare, a perfectly safe bubble (or a bubble without wage frictions) is desirable, as it helps mitigate financial frictions without any of the downside risk of an inefficient slump. Specifically, there is nothing in our model to inherently prevent a bubble from sustaining forever. Thus, the model cannot address the concern of policymakers that some rapid increases in asset prices are unsustainable. Incorporating elements from models with information frictions may help address this issue (see Brunnermeier and Oehmke 2013 or Barlevy 2018 for a survey). The model also features no equilibrium default and hence cannot address the fact that corporate and household bankruptcy rates rose sharply after the collapse of the Japanese or U.S. housing bubble. This drawback can potentially be addressed by incorporating an agency problem (e.g., Allen et al. 2017; Bengui and Phan 2018) into our framework. We leave these as potential avenues for future research.

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A Appendix

A.1 Derivations

A.1.1 Bubbleless equilibrium

The credit market clearing condition is $\int_0^1 d_t^j dj = 0$. By substituting (3.4), we get

$$F(\bar{a}_t)\beta e_t = \theta \int_0^1 1_{a^j > \bar{a}_t} e_t^j dj,$$

where 1 is the indicator function. Since net worth e_t^j is a function of past productivity a_{t-1}^j and the productivity shocks are i.i.d., the above equality can be rewritten as

$$F(\bar{a}_t)\beta e_t = \theta \int_0^1 1_{a^j > \bar{a}_t} dj \times \underbrace{\int e_t^j dj}_{e_t},$$

which yields (3.8), as stated in the main text. Recall that the CDF for the Pareto distribution over $[1, \infty)$ with shape parameter σ is $F(a) = 1 - a^{-\sigma}$. Thus, the solution to equation (3.9) for the bubbleless cutoff threshold is given by (3.10), i.e.:

$$\bar{a}_n = \left(\frac{\beta + \theta}{\beta} \right)^{1/\sigma}.$$

A closed-form expression for $\mathcal{A}_n$ is $\mathcal{A}_n = (\beta + \theta) \frac{\bar{a}_n^{1-\sigma} \sigma}{\sigma - 1}$. Thus, a closed-form expression for the interest rate in the bubbleless steady state is

$$R_n = \frac{\sigma - 1}{\beta \sigma}, \tag{A.1}$$

and for the capital stock is:

$$K_n = \left(\frac{\alpha \sigma \beta^{\frac{\sigma-1}{\sigma}} (\beta + \theta)^{1/\sigma}}{\sigma - 1} \right)^{\frac{1}{1-\alpha}}.$$

A.1.2 Bubbly equilibrium

Suppose the bubble persists in period t. By substituting (4.1) into credit market clearing condition $\int_0^1 d_t^j dj = 0$, we get:

$$\beta(F(\bar{a}_t) - \phi_t)e_t = \theta \int_0^1 1_{a^j > \bar{a}_t} e_t^j dj.$$

As the productivity shocks are i.i.d., the right-hand side again reduces to $\theta \int_0^1 1_{a^j > \bar{a}^t} dj \times \int e_t^j dj$. We thus arrive at (4.6). The closed-form solution to this equation is $\bar{a}_t = \left(\frac{\beta+\theta}{\beta(1-\phi_t)}\right)^{1/\sigma}$, leading to the steady-state value of $\bar{a}_b$ as in (4.10). The capital accumulation equation is then given by (4.7).

We now determine $R_{t,t+1}$ and R_{t+1}^k. From Proposition 3, we know that the slump only begins one period after the bubble collapses. That is, even if the bubble collapses in $t+1$, the slump only begins in $t+2$ and the labor market still clears in $t+1$, leading to $L_{t+1} = 1$. Hence, the rental rate of capital is given by $R_{t+1}^k = \alpha K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} = \alpha K_{t+1}^{\alpha-1}$ in period $t+1$, regardless of whether the bubble collapses or persists in $t+1$. Thus, from the perspective of the marginal investors, both the options of lending and investing in capital are safe. As a consequence, their indifference condition (4.4) reduces to

$$R_{t,t+1} = \bar{a}_t R_{t+1}^k = \bar{a}_t \alpha K_{t+1}^{\alpha-1}. \tag{A.2}$$

We now derive the bubble growth. Indifference condition (4.3) gives:

$$\rho \frac{1}{c_{t+1}^{j,\rho}} \frac{p_{t+1}^b}{p_t^b} = \left(\rho \frac{1}{c_{t+1}^{j,\rho}} + (1-\rho)\frac{1}{c_{t+1}^{j,1-\rho}}\right) R_{t,t+1}$$

for all j such that $a^j < \bar{a}_t$, where $c_{t+1}^{j,\rho}$ and $c_{t+1}^{j,1-\rho}$ denote the consumption of entrepreneur j when the bubble persists and when the bubble collapses in $t+1$, respectively. By substituting out the consumption values, this equation can be algebraically simplified to

$$\frac{\rho \frac{p_{t+1}^b}{p_t^b} - R_{t,t+1}}{1-\rho} = \frac{p_{t+1}^b b_t^j}{\beta e_t^j - p_t^b b_t^j}, \tag{A.3}$$

for all j such that $a^j < \bar{a}_t$. Furthermore, recall the following fact from algebra: if $\frac{x}{y} = \frac{x'}{y'}$, then $\frac{x}{y} = \frac{x'}{y'} = \frac{x+x'}{y+y'}$. Thus,

$$\frac{\rho\frac{p^b_{t+1}}{p^b_t} - R_{t,t+1}}{1-\rho} = \frac{p^b_{t+1}b^j_t}{\beta e^j_t - p^b_t b^j_t} = \frac{p^b_{t+1}\int 1_{a^j<\bar{a}_t} b^j_t dj}{\beta \int 1_{a^j<\bar{a}_t} e^j_t dj - p^b_t \int 1_{a^j<\bar{a}_t} b^j_t dj}.$$

Because of the bubble market clearing conditions, we then get:

$$\frac{\rho\frac{p^b_{t+1}}{p^b_t} - R_{t,t+1}}{1-\rho} = \frac{p^b_{t+1}}{\beta F(\bar{a}_t)e_t - p^b_t}.$$

By applying the definition that $\phi_t \equiv \frac{p^b_t}{\beta e_t}$, we get:

$$\frac{\rho\frac{\phi_{t+1}}{\phi_t}\frac{e_{t+1}}{e_t} - R_{t,t+1}}{1-\rho} = \frac{\phi_{t+1}}{F(\bar{a}_t) - \phi_t}. \tag{A.4}$$

The growth of net worth is given by:

$$\frac{e_{t+1}}{e_t} = \frac{R^k_{t+1}K_{t+1} + p^b_{t+1}}{e_t} = \frac{R^k_{t+1}K_{t+1}}{e_t} + \phi_{t+1}\beta\frac{e_{t+1}}{e_t}.$$

Combined with (4.7) and (A.2), the equation above yields:

$$\frac{e_{t+1}}{e_t} = \frac{R^k_{t+1}(\beta+\theta)\int_{\bar{a}_t} adF(a)}{1-\beta\phi_{t+1}}. \tag{A.5}$$

Combining (A.4) and (A.5) yields equation (4.9).

Finally, from the analysis above of the equilibrium dynamics, the bubbly steady state can be straightforwardly characterized as in the main text.

A.1.3 Welfare functions

The expected lifetime utility of a representative worker in the bubbly equilibrium in each period t is given by:

$$W_b(K_t, \phi_t) = \log(c^w_t) + \beta\rho W_b(K_{t+1}, \phi_{t+1}) + \beta(1-\rho)W_{burst}(K_{t+1})$$

where the first term is the instantaneous utility, with

$$c^w_t = w_t = (1-\alpha)K^\alpha_t,$$

and the second term is the continuation value conditional on the bubble persisting, and the last term is the continuation value conditional on the bubble collapsing in $t+1$. From Proposition 3, we can calculate this last term to be:

$$W_{burst}(K_{t+1}) = \Gamma_0(s^*) + \Gamma_1(s^*)\log K_{t+1},$$

where

$$\begin{aligned}
\Gamma_0(s^*) &= \frac{1}{1-\beta}\log(1-\alpha) \\
&\quad - \frac{1-\alpha}{\alpha}\left(\sum_{s=0}^{s^*-1}\frac{\beta^s s(s+1)}{2} + \frac{\alpha}{1-\beta}\frac{\beta^{s^*}s^*(s^*-1)}{2}\right)\log\gamma \\
&\quad + (1-\alpha)\left(\sum_{s=0}^{s^*}\beta^s s - \beta^{s^*}s^*\left(\frac{1-\alpha-\beta}{1-\beta}\right)\right)\log K_n \\
\Gamma_1(s^*) &= \sum_{s=0}^{s^*-1}\beta^s(\alpha-(1-\alpha)s) + \frac{\beta^{s^*}\alpha(1-(1-\alpha)s^*)}{1-\beta}.
\end{aligned}$$

Thus, the worker's welfare in the bubbly steady state is as given by (5.2).

We now calculate the welfare of entrepreneurs in the bubbleless steady state. Recall from the main text that in the bubbleless steady state, the lifetime expected utility of an entrepreneur j that starts the period with a net worth e^j, denoted by $V_n(e^j)$, satisfies equation (5.3). We solve $V_n(\cdot)$ by the guess and verify method. We conjecture that $V_n(e^j) = g_0 + g_1\log K_n + g_2\log e^j$. By plugging into the equation above and solving for g_0, g_1, g_2, we get:

$$\begin{aligned}
g_0 &= \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2}\left(\log(\bar{a}_n\beta\alpha) + \int_{\bar{a}_n}\log\frac{a(\beta+\theta)}{\bar{a}_n\beta}dF(a)\right) \\
&\quad - \left(\frac{1-\alpha}{1-\beta}\right)^2\frac{\beta\log K_n}{1-\beta\alpha} \\
g_1 &= -\frac{1-\alpha}{1-\beta}\frac{\beta\alpha}{1-\beta\alpha} \\
g_2 &= \frac{1}{1-\beta}.
\end{aligned}$$

Thus

$$V_n(e^j) = \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2}\left[\log(\alpha\bar{a}_n\beta) + \int_{\bar{a}_n} \log\left(\frac{a(\beta+\theta)}{\bar{a}_n\beta}\right) dF(a)\right]$$
$$- \frac{\beta(1-\alpha)}{(1-\beta)^2}\log K_n + \frac{1}{1-\beta}\log\left(e_t^j\right).$$

Under the definition $V_n \equiv V_n(\alpha K_n^{\alpha})$, we then get:

$$V_n = \frac{\log\alpha(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2}\left[\log(\alpha\bar{a}_n\beta) + \int_{\bar{a}_n} \log\left(\frac{a(\beta+\theta)}{\bar{a}_n\beta}\right) dF(a)\right] - \frac{\beta-\alpha}{(1-\beta)^2}\log K_n. \tag{A.6}$$

Finally, we compute the welfare of entrepreneurs in the bubbly steady state. Recall that the lifetime expected utility of an entrepreneur j that starts the period with a net worth e^j, denoted by $V_b(e^j)$, satisfies equation (5.4). From Proposition 3, we can calculate the post-bubble continuation value as given by:

$$V_{burst}(e^j) = \kappa_0(s^*, \bar{a}_n) - \kappa_1(s^*)\log K_b + \frac{\log((1-\beta)e^j)}{1-\beta}, \tag{A.7}$$

where:

$$\kappa_0(s^*, \bar{a}_n) = \frac{1-\beta^{s^*}}{1-\beta}\log(1-\beta) + \beta^{s^*}\left[\Gamma_n(\bar{a}_n) - \left(\frac{1-\alpha}{1-\beta}\right)^2 \frac{\beta}{1-\beta\alpha}\log K_n\right]$$
$$+ \frac{\beta-\beta^{s^*+1}}{(1-\beta)^2}\left[\log(\alpha\bar{a}_n\beta) + \int_{\bar{a}_n} \log\left(\frac{a(\beta+\theta)}{\bar{a}_n\beta}\right) dF(a)\right]$$
$$- \frac{\beta^{s^*+1}}{1-\beta}\frac{s^*\alpha(1-\alpha)}{1-\beta\alpha}\log\left[(\beta+\theta)\alpha\int_{\bar{a}_n} a dF(a)\right]$$
$$- \frac{(1-\alpha)}{2\alpha}\left[\left(\sum_{s=0}^{s^*-1}\beta^s s(s+1)\right) + \frac{\beta^{s^*}s^*}{(1-\beta)}\frac{1-\beta\alpha^2 + s^*(1-2\beta\alpha+\beta\alpha^2)}{1-\beta\alpha}\right]\log\gamma$$
$$\kappa_1(s^*) = (1-\alpha)\left[\sum_{s=0}^{s^*-1}\beta^s s + \frac{\beta^{s^*}}{1-\beta}\left(\frac{\beta\alpha + s^*(1-2\beta\alpha+\beta\alpha^2)}{1-\beta\alpha}\right)\right].$$

Again, by applying the guess and verify method to equation (5.4), we get the following solution for V_b:

$$V_b(e^j) = \Gamma_b(s^*) - \frac{\beta}{1-\beta\rho}\left(\frac{1-\alpha}{1-\beta} + (1-\rho)\kappa_1(s^*)\right)\log K_b + \frac{1}{1-\beta}\log\left(e^j\right), \tag{A.8}$$

where

$$
\begin{aligned}
\Gamma_b(s^*) = & \frac{1}{1-\beta\rho}\log(1-\beta) + \frac{\beta(1-\rho)}{(1-\beta\rho)}\kappa_0(s^*, \bar{a}_n) \\
& + \frac{\beta}{(1-\beta\rho)(1-\beta)}\left(\log(\alpha\bar{a}_b\beta) + \int_{\bar{a}_b}\log\frac{a^j(\beta+\theta)}{\bar{a}_b\beta}dF(a)\right) \\
& + \frac{\beta F(\bar{a}_b)}{(1-\beta\rho)(1-\beta)}\left(\rho\log\left(\frac{\theta(1-F(\bar{a}_b))}{\theta-(\theta+\beta(1-\rho))F(\bar{a}_b)}\rho\right) + (1-\rho)\log\left(\frac{\theta(1-F(\bar{a}_b))}{\beta F(\bar{a}_b)}\right)\right).
\end{aligned}
$$

Under the definition $V_b \equiv V_b(\frac{\alpha K_b^\alpha}{1-\beta\phi})$, we then get:

$$
V_b = \Gamma_b(s^*) - \frac{\beta}{1-\beta\rho}\left(\frac{1-\alpha}{1-\beta} + (1-\rho)\kappa_1(s^*)\right)\log K_b + \frac{1}{1-\beta}\log\left(\frac{\alpha K_b^\alpha}{1-\beta\phi}\right). \tag{A.9}
$$

A.1.4 Bubble dynamics and steady state with macroprudential tax

In the presence of a macroprudential tax, the equilibrium dynamics are similar to that of the bubble equilibrium, except that τ will affect the first-order condition with respect to bubbly investment of entrepreneurs. The indifference condition between investing in the bubbly asset and lending for entrepreneurs with productivity shock below $\bar{a}_t$ is now given by:

$$
E_t\left[u'(\bar{c}_{t+1})\frac{(1-\tau)p_{t+1}^b}{p_t^b}\right] = E_t\left[u'(\bar{c}_{t+1})R_{t,t+1}\right], \quad \text{if } a_t^j < \bar{a}_t,
$$

which can be reduced to:

$$
\frac{\rho\frac{(1-\tau)p_{t+1}^b}{p_t^b} - R_{t,t+1}}{1-\rho} = \frac{p_{t+1}^b b_t^j}{\beta e_t^j - p_t^b b_t^j}.
$$

By integrating across $a^j < \bar{a}_t$, and with more algebraic manipulations, we then get an aggregate expression:

$$
(1-\tau)\frac{\phi_{t+1}}{\phi_t} = \frac{(1-\beta\phi_{t+1})\bar{a}_t}{(\beta+\theta)\int_{\bar{a}_t} a dF(a)}\frac{F(\bar{a}_t)-\phi_t}{\rho F(\bar{a}_t)-\phi_t}.
$$

This equation gives a new expression that determines the bubble size in the bubbly steady state:

$$
1-\tau = \underbrace{\frac{(1-\beta\phi)\bar{a}_b}{(\beta+\theta)\int_{\bar{a}_b} a dF(a)}}_{R_b}\frac{F(\bar{a}_b)-\phi}{\rho F(\bar{a}_b)-\phi}.
$$

The closed-form expression for the bubble ratio is:

$$\phi(\tau) = \frac{\theta(1-\sigma(1-\beta\rho(1-\tau)))}{\beta(\beta(1-\rho)(1-\tau)\sigma+\theta(1-\sigma\tau))}$$

Note how ϕ is a decreasing function of τ. The bubble exists ($\phi > 0$) if and only if $\tau < \bar{\tau} \equiv 1 - \frac{\sigma-1}{\beta\sigma\rho}$. The capital stock in the bubbly steady state is then given by:

$$K_b(\tau) = \left(\frac{\beta+\theta}{1-\beta\phi(\tau)}\alpha \int_{\bar{a}_b(\tau)} a dF(a)\right)^{\frac{1}{1-\alpha}},$$

where the cutoff threshold is $\bar{a}_b(\tau) = \left(\frac{\beta+\theta}{\beta(1-\phi(\tau))}\right)^{1/\sigma}$.

Similar to the analysis in the main text and Section A.1.3, the duration of the post-bubble slump is given by (4.19) and the bubbly steady-state welfare expression for workers is given by (5.2), where we note that $K_b = K_b(\tau)$ is now a function of τ. The bubbly steady-state welfare expression for entrepreneurs is given by $V_b \equiv V_b(\frac{\alpha K_b^\alpha}{1-\beta\phi(\tau)})$, where $V_b(e^j)$ is as defined in (A.8), except that $\Gamma_b(s^*)$ is now defined as:

$$\begin{aligned}
&\Gamma_b\left(s^*\right)\\
&= \frac{1}{1-\beta\rho}\log\left(1-\beta\right) + \frac{\beta\left(1-\rho\right)}{\left(1-\beta\rho\right)}\kappa_0\left(s^*,\bar{a}_n\right)\\
&+ \frac{\beta}{\left(1-\beta\rho\right)\left(1-\beta\right)}\left(\log\left(\alpha\bar{a}_b\beta\right) + \int_{\bar{a}_b}\log\frac{a(\beta+\theta)}{\bar{a}_b\beta}dF\left(a\right)\right)\\
&+ \frac{\beta F\left(\bar{a}_b\right)}{\left(1-\beta\rho\right)\left(1-\beta\right)}\left(\rho\log\left(\frac{\left(1-\tau\right)\left(F\left(\bar{a}_b\right)-\phi\right)+\tau\phi\left(\frac{F(\bar{a}_b)-\phi}{F(\bar{a}_b)}\right)}{\left(1-\tau\right)\rho F\left(\bar{a}_b\right)-\left(1-\tau\rho\right)\phi}\rho\right) + \left(1-\rho\right)\log\left(\frac{F\left(\bar{a}_b\right)-\phi}{F\left(\bar{a}_b\right)}\right)\right).
\end{aligned}$$

A.1.5 Constrained efficiency

Following the macroprudential literature (e.g., Bianchi 2011; Bianchi and Mendoza 2018), we define constrained efficiency as follows. Consider a benevolent social planner who cares about both entrepreneurs and workers. The planner has restricted planning abilities: it chooses allocations subject to the resource, implementability, and leverage constraints, but allows the markets to clear competitively. Its policy instrument is limited to a constant tax on bubbly speculation. Since prices remain market-determined, the first-order conditions for private agents enter the planner's problem as implementability constraints. The key difference between the planner's problem and private agents' problems is that the *planner internalizes how its decisions affect prices.*

The implementability constraints for the planner will come from the first-order conditions of individual entrepreneurs. Recall that with the constant tax τ, the optimization problem of each entrepreneur j is:

$$\max_{\{c_t^j, d_t^j, b_t^j, I_t^j\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^j)$$

subject to

$$\begin{aligned} c_t^j + I_t^j + p_t^b b_t^j &= R_t^k a_{t-1}^j I_{t-1}^j + (1-\tau) p_t^b b_{t-1}^j - R_{t-1,t} d_t^j + d_t^j + T_t^j \\ I_t^j, b_t^j &\geq 0 \\ d_t^j &\leq \theta \cdot (R_t^k a_{t-1}^j I_{t-1}^j + p_t^b b_{t-1}^j - R_{t-1,t} d_t^j). \end{aligned}$$

It is more convenient to rewrite this problem in the following equivalent form: instead of choosing the amount of the bubbly asset b_t^j, each entrepreneur chooses the amount of (before-tax) bubbly investment $B_t^j \equiv p_t^b b_t^j$. Then the problem becomes:

$$\max_{\{c_t^j, d_t^j, B_t^j, I_t^j\}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^j)$$

subject to

$$\begin{aligned} c_t^j + I_t^j + B_t^j &= F_K(K_t, L_t) a_{t-1}^j I_{t-1}^j + (1-\tau) R_t^b B_{t-1}^j - R_{t-1,t} d_t^j + d_t^j + T_t^j \\ I_t^j, b_t^j &\geq 0 \\ d_t^j &\leq \theta \cdot (R_t^k a_{t-1}^j I_{t-1}^j + p_t^b b_{t-1}^j - R_{t-1,t} d_t^j), \end{aligned}$$

where $R_t^b \equiv \frac{p_t^b}{p_{t-1}^b}$ denotes the return on the bubbly asset. In the budget constraint above, we have also replaced R_t^k with the marginal product of capital.

The entrepreneurs' first-order conditions are:

$$u'(c_t^j) = \beta a_t^j F_K(K_{t+1}, L_{t+1}) E_t u'(c_{t+1}^j) + \lambda_{I,t}^j \beta \theta a_t^j R_{t,t+1}^k E_t \lambda_{d,t+1}^j \tag{A.10}$$

$$u'(c_t^j) = \beta \rho (1-\tau) R_{t+1}^{b,\rho} u'(c_{t+1}^{j,\rho}) + \lambda_{b,t}^j / p_t^b + \beta \theta E_t \left[R_{t,t+1} \lambda_{d,t+1}^j \right] \tag{A.11}$$

$$u'(c_t^j) = \beta R_{t,t+1} \cdot E_t u'(c_{t+1}^j) + \lambda_{d,t}^j + \beta \theta R_{t,t+1} E_t \lambda_{d,t+1}^j, \tag{A.12}$$

$$\mu_{I,t}^j I_t^j = \mu_{b,t}^j b_t^j = \mu_{d,t}^j \left(\theta (R_t^k a_{t-1}^j I_{t-1}^j + (1-\tau) p_t^b b_{t-1}^j - R_{t-1,t} d_t^j) - d_t^j \right) = 0, \tag{A.13}$$

where $R_t^{b,\rho}$ and $c_t^{j,\rho}$ are the return on the bubbly asset and consumption conditional on the state that the bubble persisting in t, $\mu_{I,t}^j$, $\mu_{b,t}^j$, $\mu_{d,t}^j$ are the Lagrange multipliers associated with the nonnegativity constraints on I_t^j and b_t^j and the leverage constraint, respectively.

We can now define the planner's problem:

Definition 10. The constrained central planner's problem is as follows:

$$\max_{\{c_t^j, B_t^j \geq 0, I_t^j \geq 0, d_t^j, R_{t-1,t}, R_t^{b+}, \mu_{I,t}^j, \mu_{b,t}^j, \mu_{d,t}^j, \tau\}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\lambda \cdot u(c_t^w) + (1-\lambda) \cdot \int_{j \in J} u(c_t^j) \right),$$

where $\lambda \in [0,1]$ is the Pareto weight the planner assigns to the representative worker and $1-\lambda$ is that for entrepreneurs, subject to the following individual budget constraints:

$$\begin{aligned} c_t^w &= F_L(K_t, L_t) L_t \\ c_t^j &= F_K(K_t, L_t) a_{t-1}^j I_{t-1}^j + ((1-\tau) R_t^b B_{t-1}^j - B_t^j) - R_{t-1,t} d_{t-1}^j + d_t^j - I_t^j + T_t^j, \end{aligned}$$

the planner's transfer being given by:

$$T_t^j = \tau B_t^j, \forall j \in J,$$

leverage constraint:

$$d_t^j \leq \theta \cdot \underbrace{(F_K(K_t, L_t) a_{t-1}^j I_{t-1}^j + R_t^b B_{t-1}^j - R_{t-1,t} d_{t-1}^j)}_{e_t^j},$$

implementability constraints (A.10-A.13), and the labor market conditions:

$$\begin{aligned} F_L(K_t, L_t) &\geq \gamma F_L(K_{t-1}, L_{t-1}) \qquad \text{(A.14)} \\ L_t &\leq 1 \\ (1-L_t)(F_L(K_t, L_t) - \gamma F_L(K_{t-1}, L_{t-1})) &= 0. \end{aligned}$$

As in the main text, we focus on the planner's problem in the bubbly steady state and assume that each entrepreneur begins the steady state with the same net worth.

Note that the planner takes as given the exogenous sunspot process of bubble bursting. The key thing to notice is that unlike individual agents, *the planner internalizes the downward wage rigidity condition (A.14) in its optimization problem.* As a consequence, the first-order conditions of the planner will contain a Lagrange multiplier associated with this constraint, which would be otherwise absent in the laissez-faire first-order conditions of individual entrepreneurs. This formalizes the notion that the competitive equilibrium allocations are constrained inefficient.

A.1.6 Extension: a generalized model

We now extend the model in several directions. First, we allow for exogenous growth. Specifically, assume that the production technology is given by:

$$Y_t = K_t^{\alpha} \cdot (A_t L_t)^{1-\alpha},$$

where the technology term A_t grows at an exogenous growth rate $g \geq 0$:

$$A_t \equiv (1+g)^t.$$

Throughout, we will use a lower case letter to denote a detrended variable, for example:

$$y_t \equiv \frac{Y_t}{(1+g)^t},\ k_t \equiv \frac{K_t}{(1+g)^t}, w_t \equiv \frac{W_t}{(1+g)^t} \dots$$

The downward wage rigidity condition is growth-adjusted, meaning:

$$w_t \geq (1+g)\gamma w_{t-1}.$$

Second, we allow for partial capital depreciation. Specifically, we assume that the capital stock depreciates at a rate $\delta \in [0,1]$. As in Kocherlakota (2009), we assume that the capital good can be converted back one-to-one to the consumption good.

Third, we allow for a more general continuous productivity distribution F over a subset of $(0,\infty)$. Fourth, we allow for a more flexible leverage constraint:

$$d_t^j \leq \theta_t(a_t^j) \cdot e_t^j,$$

where $\theta_t(\cdot)$ is a (possibly time-varying) measurable function of entrepreneur j's productivity shock a_t^j.

Remark 11. The main model in Section 2 is a special case of this extended model (where $g=0$, $\delta=1$, $\theta_t(a)=\theta$, and F is Pareto over $[1,\infty)$). Note further that the calibration exercise in Section 4.3.1 uses only the first two extended assumptions (i.e., $g>0$ and $\delta<1$).

The changes compared to the main model are as follows. We first consider the bubbleless equilibrium. The time-invariant cutoff threshold is implicitly determined by the credit-market clearing condition:

$$\beta F(\bar{a}_n) = \int_{\bar{a}_n} \theta(a) dF(a).$$

The detrended capital stock evolves according to the following new law of motion:

$$(1+g)\,k_{t+1} = \int_{\bar{a}_n} (\beta + \theta_t(a))a \; dF(a) \left(\alpha k_t^{\alpha} L_t^{1-\alpha} + (1-\delta)\,k_t\right).$$

The bubbleless steady-state capital stock is thus:

$$k_n = \left(\frac{\alpha \int_{\bar{a}_n} (\beta + \theta(a))a \; dF(a)}{(1+g) - (1-\delta) \int_{\bar{a}_n} (\beta + \theta(a))a \; dF(a)} \right)^{\frac{1}{1-\alpha}}.$$

The bubbleless steady-state interest rate is:

$$\begin{aligned} R_n &= \bar{a}_n \left(\alpha k_n^{\alpha-1} + 1 - \delta\right) \\ &= \frac{(1+g)\bar{a}_n}{\int_{\bar{a}_n} (\beta + \theta(a))a \; dF(a)}. \end{aligned}$$

We now consider the bubbly equilibrium. The cutoff threshold is implicitly determined by the credit-market clearing condition:

$$\beta \left(F(\bar{a}_t) - \phi_t\right) = \int_{\bar{a}_t} \theta(a) dF(a).$$

The detrended capital stock evolves according to the following new law of motion:

$$(1+g)\,k_{t+1} = \int_{\bar{a}_t} (\beta + \theta(a))a \; dF(a) \cdot \left(\frac{\alpha k_t^{\alpha} L_t^{1-\alpha} + (1-\delta)\,k_t}{1 - \beta\phi} \right).$$

The new law of motion of the bubble price is given by:

$$(1+g)\frac{p_{t+1}^b}{p_t^b} = \frac{F\left(\bar{a}_t\right) - \phi_t}{\rho F\left(\bar{a}_t\right) - \phi_t} R_{t,t+1}.$$

The bubbly steady state is characterized by the following new expression for the (detrended) capital stock:

$$k_b = \left(\frac{\alpha \int_{\bar{a}_b} (\beta + \theta(a))a \; dF(a)}{(1+g)\left(1 - \beta\phi\right) - (1-\delta) \int_{\bar{a}_b} (\beta + \theta(a))a \; dF(a)} \right)^{\frac{1}{1-\alpha}},$$

and the following new expression for the interest rate:

$$R_b = \frac{(1+g)\left(1 - \beta\phi\right)\bar{a}_b}{\int_{\bar{a}_b} (\beta + \theta(a))a \; dF(a)},$$

where the cutoff threshold $\bar{a}_b$ as given in the main model, and the following new equation that determines the bubble ratio:

$$\phi = \frac{(1+g)\rho - R_b}{(1+g) - R_b} F(\bar{a}_b).$$

After the bubble collapses, the economy evolves according to the bubbleless equilibrium dynamics.

A.2 Proofs

A.2.1 Proof of Lemma 2

Proof. For the variables characterized by equations (4.11) to (4.15) to constitute a bubbly steady state, a necessary and sufficient condition is $\phi \in (0,1)$. From (4.15), $\phi > 0$ equivalent to $1 - (1-\beta\rho)\sigma > 0$, i.e., (4.16). And once (4.16) is satisfied, it is immediately true that $\phi < 1$.

Finally, note that from (A.1), condition (4.16) is equivalent to $R_n < \rho$. □

A.2.2 Proof of Proposition 3

Proof. By the labor market clearing conditions during the slump, the wage rigidity must bind in all periods for which $L_{T+s} < 1$. Thus, defining $s \in (0, s^*)$ as a period for which $L_{T+s} < 1$, the wage must follow:

$$w_{T+s} = \gamma w_{T+s-1} = \gamma^s w_T. \tag{A.15}$$

Recall from the first-order conditions of firms that for all t:

$$L_t = \left(\frac{1-\alpha}{w_t}\right)^{\frac{1}{\alpha}} K_t. \tag{A.16}$$

By (3.11), (A.15) and (A.16), the capital law of motion after the burst is:

$$\begin{aligned} K_{T+s+1} &= \mathcal{A}_n \alpha \left(\frac{K_{T+s}}{L_{t+s}}\right)^{\alpha-1} K_{T+s} \\ &= \mathcal{A}_n \alpha \left(\frac{w_{T+s}}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} K_{T+s} \\ &= \mathcal{A}_n \alpha \left(\frac{\gamma^s w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}} K_{T+s}. \end{aligned}$$

Thus, by recursion, we get:

$$K_{T+s+1} = \left[\mathcal{A}_n \alpha \left(\frac{w_T}{1-\alpha}\right)^{\frac{\alpha-1}{\alpha}}\right]^{s+1} \gamma^{\frac{\alpha-1}{\alpha}\frac{s(s+1)}{2}} K_T.$$

Finally, by substituting in (A.16) and $L_T = 1$ for w_T, we get:

$$K_{T+s+1} = \left[\mathcal{A}_n \alpha K_T^{\alpha-1}\right]^{s+1} \gamma^{\frac{\alpha-1}{\alpha}\frac{s(s+1)}{2}} K_T,$$

as desired.

Finally, we determine the duration s^* of the slump. Recall:

$$\begin{aligned} s^* &\equiv \min\{s \in \mathbb{N} \mid L_{T+s} = 1\} \\ &= \min\left\{s \in \mathbb{N} \mid w^f_{T+s} \geq \gamma w_{T+s-1}\right\}, \end{aligned}$$

where $w^f_{T+s} = (1-\alpha) K^\alpha_{T+s}$ represents the wage level consistent with full employment. Then we can rewrite s^* as:

$$\begin{aligned} s^* &= \min\left\{s \in \mathbb{N} \mid (1-\alpha) K^\alpha_{T+s} \geq \gamma^s w_T\right\} \\ &= \min\left\{s \in \mathbb{N} \mid \left[\left(\mathcal{A}_n \alpha K_T^{\alpha-1}\right)^s \gamma^{\frac{\alpha-1}{\alpha}\frac{s(s-1)}{2}} K_T\right]^\alpha \geq \gamma^s K^\alpha_T\right\}. \end{aligned}$$

Algebraic manipulation yields (4.19). □

A.2.3 Proof of Proposition 4

Proof. We first focus on workers. Define $\Delta W \equiv W_b - W_n$. Then from (5.1) and (5.2):

$$\begin{aligned} \Delta W &= \frac{\log\left((1-\alpha)K_b^\alpha\right)}{1-\rho\beta} - \frac{\log\left((1-\alpha)K_n^\alpha\right)}{1-\beta} \\ &\quad + \frac{\beta(1-\rho)}{1-\rho\beta}\left(\Gamma_0(s^*) + \Gamma_1(s^*)\log K_b\right) \end{aligned}$$

Recall from (4.19) that $\lim_{\gamma\to 1} s^* = \infty$. Thus

$$\begin{aligned} \lim_{\gamma\to 1} \Gamma_0(s^*) &= \frac{\log(1-\alpha)}{1-\beta} + \frac{\beta(1-\alpha)}{(1-\beta)^2}\log K_n \\ \lim_{\gamma\to 1} \Gamma_1(s^*) &= \frac{\alpha-\beta}{(1-\beta)^2} \end{aligned}$$

and:

$$\begin{aligned}\lim_{\gamma\to 1}\Delta W &= \frac{\log\left((1-\alpha)K_b^{\alpha}\right)}{1-\rho\beta}-\frac{\log\left((1-\alpha)K_n^{\alpha}\right)}{1-\beta}\\ &\quad+\frac{\beta\left(1-\rho\right)}{1-\rho\beta}\left(\frac{1}{1-\beta}\log\left(1-\alpha\right)+\frac{\beta\left(1-\alpha\right)}{\left(1-\beta\right)^2}\log K_n+\frac{\alpha-\beta}{\left(1-\beta\right)^2}\log K_b\right)\\ &= \frac{\alpha\left(1-\beta\right)^2+\beta\left(1-\rho\right)\left(\alpha-\beta\right)}{\left(1-\rho\beta\right)\left(1-\beta\right)^2}\left(\log K_b-\log K_n\right).\end{aligned}$$

Recall that the assume bubble is expansionary ($K_b > K_n$). Therefore, if the bubble is sufficiently risky so that

$$\rho < \bar{\rho} \equiv 1-\frac{\alpha\left(1-\beta\right)^2}{\beta\left(\beta-\alpha\right)},$$

then the last expression is negative, implying $\lim_{\gamma\to 1}\Delta W < 0$. Thus, there exists $\gamma_w < 1$ such that if $\gamma > \gamma_1$ and $\rho < \bar{\rho}$, then $W_b < W_n$.

We now focus on entrepreneurs. Similarly, we define $\Delta V \equiv V_b - V_n$. Then from (A.6) and (A.9) and by taking $\gamma \to 1$, after algebraic manipulations, we can derive the following limit:

$$\begin{aligned}\lim_{\gamma\to 1}\Delta V =&G(\phi)\\ \equiv&\frac{\beta(\beta\phi+\theta)\left((1-\rho)\log\left(\frac{\theta(1-\phi)}{\beta\phi+\theta}\right)+\rho\log\left(\frac{\theta\rho(1-\phi)}{\theta\rho-\phi(\beta(1-\rho)+\theta)}\right)\right)}{(1-\beta)(1-\beta\rho)(\beta+\theta)}\\ &-\frac{\beta^2\phi\left(\log\left(\frac{\beta+\theta}{\beta}\right)+\frac{1}{\sigma}\right)}{(1-\beta)(1-\beta\rho)(\beta+\theta)}\\ &-\frac{((2-\alpha)\beta-1)\log\left(\frac{1}{1-\beta\phi}\right)}{(1-\alpha)(\beta-1)^2}+\frac{\frac{(\sigma-1)(\alpha-\beta)}{\alpha-1}+\frac{(\beta-1)\beta}{\beta\rho-1}}{(\beta-1)^2\sigma}\log\left(\frac{1}{1-\phi}\right).\end{aligned} \tag{A.17}$$

Note that G is a decreasing function of ϕ and furthermore $G(0) = 0$. Because in equilibrium $\phi > 0$, it follows that $\lim_{\gamma\to 1}\Delta V < 0$. Thus, there exists $\gamma_e < 1$ such that if $\gamma > \gamma_e$, then $V_b < V_n$. The proof is complete by letting $\bar{\gamma} = \max\{\gamma_w, \gamma_e\}$. □

A.2.4 Proof of Corollary 5

Proof. First, we show that there exists $\gamma_e < 1$ such that if $\gamma > \gamma_e$, then $V_b(\tau) \leq V_n$ for all $\tau \leq \bar{\tau}$, with equality if and only if $\tau = \bar{\tau}$. Fix any $\tau < \bar{\tau}$. By applying the same algebraic manipulations as in Section A.2.3, we have $\lim_{\gamma\to 1} V_b(\tau) - V_n = \tilde{G}(\phi(\tau))$, where $\tilde{G}$ is defined as

$$
\begin{aligned}
\tilde{G}(\phi(\tau)) \equiv & \frac{\beta(\beta\phi(\tau)+\theta)\left((1-\rho)\log\left(\frac{\theta(1-\phi(\tau))}{\beta\phi(\tau)+\theta}\right)+\rho\log\left(\frac{(1-\tau)\theta(1-\phi(\tau))+\tau\phi(\tau)(\beta+\theta)\frac{\theta(1-\phi(\tau))}{\theta+\beta\phi(\tau)}}{(1-\tau)\rho\theta-(1-\tau\rho)\phi(\tau)(\beta(1-(1-\tau)\rho)+\theta)}\rho\right)\right)}{(1-\beta)(1-\beta\rho)(\beta+\theta)} \\
& -\frac{\beta^2\phi\left(\log\left(\frac{\beta+\theta}{\beta}\right)+\frac{1}{\sigma}\right)}{(1-\beta)(1-\beta\rho)(\beta+\theta)} \\
& -\frac{((2-\alpha)\beta-1)\log\left(\frac{1}{1-\beta\phi(\tau)}\right)}{(1-\alpha)(\beta-1)^2}+\frac{\frac{(\sigma-1)(\alpha-\beta)}{\alpha-1}+\frac{(\beta-1)\beta}{\beta\rho-1}}{(\beta-1)^2\sigma}\log\left(\frac{1}{1-\phi(\tau)}\right).
\end{aligned}
$$

Note that $\tilde{G}$ is a decreasing function of $\phi(\tau)$ and $\tilde{G}(0)=0$. Because in equilibrium $\phi(\tau)>0$, it follows that $\lim_{\gamma\to 1} V_b(\tau)-V_n=\tilde{G}(\phi(\tau))<0$. Thus, for all $\epsilon>0$, there exists $\gamma(\epsilon)<1$ such that $V_b(\tau)-V_n<\tilde{G}(\phi(\tau))+\epsilon$ whenever $\gamma>\gamma(\epsilon)$. By letting $\epsilon=-\tilde{G}(\phi(\tau))$ and $\gamma_e=\gamma(-\tilde{G}(\phi(\tau)))$, we then get $V_b(\tau)<V_n$ for all $\gamma>\gamma_e$, as desired. Finally, note that when $\tau=\bar{\tau}$, the bubble disappears, i.e., $\phi(\bar{\tau})=0$ and thus $V_b(\bar{\tau})=V_n$.

Similarly, we show that there exists $\gamma_w<1$ such that if $\gamma>\gamma_w$ and $\rho<\bar{\rho}$, then $W_b(\tau)\le W_n$ for all $\tau\le\bar{\tau}$, with equality if and only if $\tau=\bar{\tau}$. Fix any $\tau<\bar{\tau}$. If $K_b(\tau)\le K_n$, i.e., the taxed bubble is contractionary, then it is obvious that the welfare of workers cannot be better off with the bubble than without, as the equilibrium wage in the bubbly equilibrium path would always be smaller than w_n – the wage in the bubbleless steady state. Thus, we can focus on the expansionary case of $K_b(\tau)>K_n$. By applying the same algebraic manipulations as in Section A.2.3, we get

$$
\lim_{\gamma\to 1} W_b(\tau)-W_n=H(\tau)\equiv\frac{\alpha\left(1-\beta\right)^2+\beta\left(1-\rho\right)\left(\alpha-\beta\right)}{\left(1-\rho\beta\right)\left(1-\beta\right)^2}\left(\log K_b(\tau)-\log K_n\right),
$$

where the right-hand side $H(\tau)$ is strictly negative whenever $\rho<\bar{\rho}$. Thus, for all $\epsilon>0$, there exists $\gamma(\epsilon)$ such that $W_b(\tau)-W_n<H(\tau)+\epsilon$. By letting $\epsilon=-H(\tau)$ and $\gamma_w=\gamma(-H(\tau))$, we then get $W_b(\tau)<W_n$ for all $\gamma>\gamma_w$, as desired. Finally, when $\tau=\bar{\tau}$, the bubble disappears and thus trivially $W_b(\bar{\tau})=W_n$.

The proof is complete by letting $\bar{\gamma}=\max\{\gamma_w,\gamma_e\}$, as it is immediate from the results above that $\arg\max_{\tau\le\bar{\tau}}\lambda W_b(\tau)+(1-\lambda)V_b(\tau)=\bar{\tau}$ when $\gamma>\bar{\gamma}$ and $\rho<\bar{\rho}$. □

A.2.5 Proof of Proposition 6

First, we show that the inflation is at the target and there is full employment in the immediate aftermath of the bubble's collapse:

Lemma 12. $\Pi_{T-1,T}=\Pi^*$ *and* $L_T=1$.

Proof. To see this, recall that, by assumption, the economy is still in the bubbly steady state in $T-1$, and therefore the nominal interest rate $i_{T-1,T}$ is determined by the unconstrained Taylor rule $1+i_{T-1,T} = R_b\Pi^*$.

Furthermore, recall the Fisher equation that equates the expected return from nominal bond holding and real lending for entrepreneurs below the threshold $\bar{a}_b$ between period $T-1$ and T:

$$1+i_{T-1,T} = \frac{\rho u'(c_b^L)R_b\Pi^* + (1-\rho)u'(c_T^L)R_{T-1,T}\Pi_{T-1,T}}{\rho u'(c_b^L) + (1-\rho)u'(c_T^L)},$$

where the superscript L denotes entrepreneurs with productivity below $\bar{a}_b$. Here, we have used the fact that in the good state that the bubble persists in period T (which happens with probability ρ from the information set at $T-1$), the economy continues to be in the bubbly steady state with consumption level c_b^L for the L-type, the real interest rate is R_b and inflation is Π^*. Thus the indifference condition above can be rewritten as:

$$R_b\Pi^* = \frac{\rho u'(c_b^L)R_b\Pi^* + (1-\rho)u'(c_T^L)R_{T-1,T}\Pi_{T-1,T}}{\rho u'(c_b^L) + (1-\rho)u'(c_T^L)},$$

or equivalently

$$R_b\Pi^* = R_{T-1,T}\Pi_{T-1,T}.$$

In addition, recall that the real interest rate between $T-1$ and T is given by:

$$R_{T-1,T} = \bar{a}_b\alpha\left(\frac{L_T}{K_T}\right)^{1-\alpha} = R_b L_T^{1-\alpha}.$$

Thus the equation above reduces to:

$$\Pi^* = \Pi_{T-1,T}L_T^{1-\alpha}. \tag{A.18}$$

Now suppose on the contrary that $L_T < 1$. Then the DNWR must bind at T, or

$$\frac{w_T}{w_{T-1}} = \frac{\gamma(L_T)}{\Pi_{T-1,T}}.$$

By substituting the first-order condition of firms (2.7), we then get:

$$L_T^{-\alpha} = \frac{\gamma(L_T)}{\Pi_{T-1,T}}. \tag{A.19}$$

Equations (A.18) and (A.19) then imply

$$\Pi^* = \gamma_0 L_T^{1+\gamma_1} < \gamma_0.$$

However, this violates assumption (6.3).

Therefore, it must be that $L_T = 1$. Equation (A.18) then implies $\Pi_{T-1,T} = 1$. □

Now the proof for the proposition follows straightforwardly:

Proof. Given $\Pi_{T-1,T} = \Pi^*$, it is immediate that (6.4) is equivalent to

$$R^f_{T,T+1}\Pi^* < 1.$$

As the bubble has collapsed in $T+1$, the real interest rate with full employment is simply given by:

$$R^f_{T,T+1} = \bar{a}_n \alpha K_{T+1}^{\alpha-1},$$

as the post-bubble economy follows the bubbleless dynamics. In addition, from the law of motion of capital, we have $K_{T+1} = \alpha \mathcal{A}_n K_T^\alpha L_T^{1-\alpha} = \alpha \mathcal{A}_n K_b^\alpha$. Therefore, $R^f_{T,T+1}\Pi^* < 1$ if and only if $\bar{a}_n \alpha \left(\alpha \mathcal{A}_n K_b^\alpha\right)^{\alpha-1} < \frac{1}{\Pi^*}$, which is equivalent to (6.4). □

A.2.6 Proof of Lemma 8

Proof. Suppose on the contrary that $i_{t-1,t} = 0$ but $L_t = 1$. The DNWR constraint is slack, implying that $\frac{w_t^f}{w_{t-1}} \geq \frac{\gamma_0}{\Pi_{t-1,t}}$, or equivalently, inflation must be sufficiently high:

$$\frac{K_t^\alpha}{(K_{t-1}/L_{t-1})^\alpha} \geq \frac{\gamma_0}{\Pi_{t-1,t}}. \tag{A.20}$$

However, the inflation rate is determined by the Fisher equation $1 + i_{t-1,t} = R_{t-1,t}\Pi_{t-1,t}$, or

$$1 = \underbrace{\bar{a}_n \alpha K_t^{\alpha-1}}_{R_{t-1,t} \text{ with } L_t = 1} \Pi_{t-1,t}. \tag{A.21}$$

Substitute (A.21) into (A.20), we get a condition that the real interest rate and thus the marginal product of capital must be sufficiently low:

$$\gamma_0 \bar{a}_n \alpha K_t^{\alpha-1} \leq \frac{K_t^\alpha}{(K_{t-1}/L_{t-1})^\alpha},$$

or equivalently, the capital stock must be sufficiently high:

$$K_t \geq \gamma_0 \bar{a}_n \alpha (K_{t-1}/L_{t-1})^{\alpha}.$$

Substituting the law of motion of capital: $K_t = \mathcal{A}_n \alpha K_{t-1}^{\alpha} L_{t-1}^{1-\alpha}$ into the inequality above yields

$$L_{t-1} \geq \frac{\gamma_0 a^L}{\mathcal{A}_n}$$

However, as $1 \geq L_{t-1}$, it then follows that $1 \geq \frac{\gamma_0 a^L}{\mathcal{A}_n}$, which contradicts assumption (6.3). □

A.2.7 Proof of Proposition 9

Proof. For notation simplicity, let us normalize the period when the bubble bursts to be period 0; that is, $T = 0$. Then, on the unemployment path $L_1, L_2, \cdots < 1$ and $i_{0,1} = i_{1,2} = \cdots = 0$. The unemployment path can be characterized as follows. The flow of capital is given by

$$K_t = \mathcal{A}_n \alpha K_{t-1}^{\alpha} L_{t-1}^{1-\alpha}.$$

Binding DNWR and ZLB provide the following two equations, respectively:

$$\Pi_{t-1,t} \frac{(K_t/L_t)^{\alpha}}{(K_{t-1}/L_{t-1})^{\alpha}} = \gamma(1 - L_t)$$
$$\underbrace{\bar{a}_n \alpha K_t^{\alpha-1} L_t^{1-\alpha}}_{R_{t-1,t}} \Pi_{t-1,t} = 1.$$

Combining the two above equations yields

$$\frac{(K_t/L_t)^{\alpha}}{(K_{t-1}/L_{t-1})^{\alpha}} = \bar{a}_n \alpha K_t^{\alpha-1} L_t^{1-\alpha} \gamma(1 - L_t).$$

Rewriting the above equation by utilizing the parameterization of $\gamma(\cdot)$,

$$K_t \left(\frac{L_{t-1}}{K_{t-1}} \right)^{\alpha} = \bar{a}_n \alpha \gamma_0 L_t^{1+\gamma_1}.$$

By substituting in the flow of capital, we find a recursive form for labor:

$$L_t = \left(\frac{\mathcal{A}_n}{\gamma_0 \bar{a}_n} L_{t-1} \right)^{\frac{1}{1+\gamma_1}}.$$

Similarly, inflation can be expressed as a function of last period's labor and capital:

$$\begin{aligned}\Pi_{t-1,t} &= \frac{1}{R_{t-1,t}} = \frac{1}{\bar{a}_n \alpha} \left(\frac{K_t}{L_t} \right)^{1-\alpha} \\ &= \frac{1}{a^L \alpha} \left(\gamma_0 \bar{a}_n \alpha K_{t-1}^{\alpha} L_{t-1}^{\frac{\gamma_1 - \alpha(1+\gamma_1)}{1+\gamma_1}} \left(\frac{\mathcal{A}_n}{\gamma_0 \bar{a}_n} \right)^{\frac{\gamma_1}{1+\gamma_1}} \right)^{1-\alpha}.\end{aligned}$$

These expressions can be further simplified by recursively plugging in for $L_{t-1}, L_{t-2}, \ldots, L_1$. Therefore, labor, L_t, can be written as a function of $L_0 = 1$ (as shown in Appendix A.2.5, there is full employment in the period the bubble bursts) and t:

$$\begin{aligned}L_t &= \left(\frac{\mathcal{A}_n}{\gamma_0 \bar{a}_n} L_{t-1} \right)^{\frac{1}{1+\gamma_1}} \\ &= \left(\left(\frac{\mathcal{A}_n}{\gamma_0 \bar{a}_n} \right)^{\sum_{s=0}^{t-1} \left(\frac{1}{1+\gamma_1} \right)^s} \underbrace{L_0^{\left(\frac{1}{1+\gamma_1} \right)^{t-1}}}_{=1} \right)^{\frac{1}{1+\gamma_1}} \\ &= \left(\frac{\mathcal{A}_n}{\gamma_0 \bar{a}_n} \right)^{\frac{(1+\gamma_1)^t - 1}{\gamma_1 (1+\gamma_1)^t}}.\end{aligned}$$

Similarly, using the flow of capital equation and working backward, K_t can be written as a function of K_0, t, and all past L_t:

$$\begin{aligned}K_t &= \mathcal{A}_n \alpha K_{t-1}^{\alpha} L_{t-1}^{1-\alpha} \\ &= (\mathcal{A}_n \alpha)^{\sum_{s=0}^{t-1} \alpha^s} K_0^{\alpha^t} \left(\prod_{s=1}^{t-1} L_{t-s}^{\alpha^{s-1}} \right)^{1-\alpha} \\ &= (\mathcal{A}_n \alpha)^{\frac{1-\alpha^t}{1-\alpha}} K_0^{\alpha^t} \left(\frac{\mathcal{A}_n}{\gamma_0 \bar{a}_n} \right)^{\frac{1-\alpha}{\gamma_1} \left(\frac{1-\alpha^{t-1}}{(1-\alpha)} - \frac{1-(\alpha(1+\gamma_1))^{t-1}}{(1-\alpha(1+\gamma_1))(1+\gamma_1)^{t-1}} \right)}.\end{aligned}$$

For these values to constitute an equilibrium path after the collapse of the bubble in period T, the necessary and sufficient conditions are that the DNWR and the ZLB do indeed bind. From Proposition 8, we know it is sufficient to show that the ZLB binds, i.e., $R_{t-1,t}^f (\Pi_{t-2,t-1})^{\zeta} (\Pi^*)^{1-\zeta} < 1$ for all t, where the real interest rate with full employment is given by $R^f_{t-1,t} = \bar{a}_n \alpha K_t^{\alpha-1}$. This inequality holds if and only if $K_t > \left(\bar{a}_n \alpha \left(\frac{\Pi_{t-2,t-1}}{\Pi^*} \right)^{\zeta} \Pi^* \right)^{\frac{1}{1-\alpha}}$ for all t, as stated in the proposition.

Finally, it is algebraically straightforward to show that $\lim_{t\to\infty} K_{T+t} = K$, $\lim_{t\to\infty} L_{T+t} = L$ and $\lim_{t\to\infty} \Pi_{T+t-1,T+t} = \Pi$, where K, L, and Π are the capital, labor, and inflation in the bad bubbleless steady state as established in Section 3. □