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Optimal Incentive Contracts with Job Destruction Risk

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Optimal incentive contracts with job destruction risk*

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Abstract

We study the implications of job destruction risk for optimal incentives in a long-term contract with moral hazard. We extend the dynamic principal-agent model of Sannikov (2008) by adding an exogenous Poisson shock that makes the match between the firm and the agent permanently unproductive. In modeling job destruction as an exogenous Poisson shock, we follow the Diamond-Mortensen-Pissarides search-and-matching literature. The optimal contract shows how job destruction risk is shared between the firm and the agent. Arrival of the job-destruction shock is always bad news for the firm but can be good news for the agent. In particular, under weak conditions, the optimal contract has exactly two regions. If the agent's continuation value is below a threshold, the agent's continuation value experiences a negative jump upon arrival of the job-destruction shock. If the agent's value is above this threshold, however, the jump in the agent's continuation value is positive, i.e., the agent gets rewarded when the match becomes unproductive. This pattern of adjustment of the agent's value at job destruction allows the firm to reduce the costs of effort incentives while the match is productive. In particular, it allows the firm to adjust the drift of the agent's continuation value process so as to decrease the risk of reaching either of the two inefficient agent retirement points. Further, we study the sensitivity of the optimal contract to the arrival rate of job destruction.

Keywords: dynamic moral hazard, job destruction, jump risk
JEL codes: D86

1 Introduction

Understanding ex-post income heterogeneity of ex-ante identical workers is an important question in economics. Two extensively studied explanations for this heterogeneity are search

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frictions and information frictions. Although these two sources of heterogeneity were traditionally studied in separation from each other, the literature has been moving recently toward studying search and information frictions jointly in order to obtain models that can better account for the data. This paper's goal is to contribute to this effort by studying the implications of the risk of job destruction, which is commonly used in the search and matching literature to generate separations, on the optimal long-term contract in a dynamic private-action (i.e., moral hazard) environment.

To this end, we study the implications of job destruction risk in the dynamic moral hazard model of Sannikov (2008). Like most papers in dynamic contracting literature, Sannikov (2008) assumes that productivity of the match between the firm and the worker/agent is time-invariant, i.e., although output is subject to transitory idiosyncratic shocks, the match remains productive indefinitely into the future. This assumption makes it difficult to integrate dynamic contracting models with the search and matching theory, e.g., Mortensen and Pissarides (1994), where persistent match productivity shocks are a basic source of heterogeneity. We take a step toward removing this limitation of the dynamic contracting theory by allowing for persistent shocks to the productivity of the match between the firm and the agent. In particular, we study the implications of a job-destruction shock that makes the match permanently unproductive.

We follow Pissarides (1985) and Mortensen and Pissarides (1994) in modeling job destruction as an exogenous, observable Poisson shock that arrives with a known intensity λ. Prior to the arrival of this shock, our model is identical to Sannikov (2008): The agent chooses privately a costly action a_t at all t, where more-costly actions have a larger positive impact on the firm's expected flow of revenue. At a random time θ, the match becomes unproductive: no further revenue will be generated regardless of what actions the agent takes.

The job-destruction shock can be naturally interpreted in two ways. First, it can be viewed as a productivity shock that affects the quantity of output produced inside the match. Second, as in Mortensen and Pissarides (1994), it can be interpreted as a shock to the market price of the differentiated good produced by the firm. Complete job destruction at shock arrival means that either the physical productivity of the match or the market price of output become permanently zero at date θ.

Our main results show how job destruction risk impacts the optimal contract. The contract has the same qualitative features as in Sannikov (2008). The agent's promised utility W_t is a state variable sufficient for recursive characterization of the solution. The agent's effort is positive everywhere except at two absorbing boundaries of the support of W_t: the low retirement point at $W = 0$ and the high retirement point $W_{gp} > 0$. The firm's optimal profit function $F(W)$ is hump-shaped with a unique maximum at $W^* \in (0, W_{gp})$. The agent receives no compensation when $W_t \leq W^*$ and positive compensation when $W_t > W^*$. At θ, the contract becomes static:

the agent is asked for no effort and is provided a constant retirement/severance payment flow c'.

However, the agent is not fully insured against the job-destruction shock. At θ, both the agent's continuation utility and the firm's continuation profit jump, i.e., job destruction risk is shared between the firm and the agent. We show that the arrival of a job-destruction shock is always bad news for the firm but can be good news for the agent. In particular, under weak conditions, the optimal contract has exactly two regions. If the agent's continuation value is below a threshold, denoted by W_{nj}, the agent's continuation value experiences a negative jump at θ. If the agent's value is above this threshold, however, the jump in the agent's continuation value is positive, i.e., the agent gets rewarded when the match becomes unproductive.

This pattern of adjustment of the agent's value at job destruction is optimal because it allows the firm to reduce the costs of effort incentives before the job-destruction shock arrives, i.e, while the match is productive and the agent exerts effort. In particular, the level of continuation utility promised to the agent conditional on job destruction is inversely related to the growth (drift) of the agent's promised utility conditional on no job destruction. If the firm promises more after job destruction, drift of W_t is lower prior to that event and vice versa. In the optimal contract, the promise of utility after job destruction is therefore used to manipulate the dynamics of W_t so as to maximize the firm's profit. There are two effects. First, there is the wealth effect: the agent with higher W is more expensive to elicit effort from, which eventually leads to retirement of the agent at W_{gp}. This effect calls for using the post-job-destruction utility promise, W', to lower the drift of W_t. Second, there is the "poverty trap" effect: if W_t hits 0, incentives no longer can be provided to the agent, as he has no further "skin in the game." This effect calls for using W' to increase the drift of W_t. In the optimal contract, the poverty trap effect dominates at low W and the wealth effect dominates at high W, with the unique threshold being W_{nj}.

In this paper, we study job destruction risk in the contracting problem between a single firm and a single agent/worker without considering equilibrium in a broader labor market. We do not explicitly model separations and transitions of agents from one job to another after a job-destruction shock. Similar to Sannikov (2008), we use the simplifying assumption that the agent retires after job destruction. However, our analysis provides a building block for solving labor market equilibrium models with long-term contracts under moral hazard and subject to job-destruction shocks. Any such model, e.g., one that would integrate moral hazard and an explicit search friction a la Diamond-Mortensen-Pissarides, will need to solve a contracting problem similar to ours. Our contract characterization results will continue to apply as long as the equilibrium post-separation continuation value function for the firm satisfies the sufficient condition on the post-retirement profit function that we identify in this paper.

Related literature Several studies explore the impact of jump risk on the optimal provision of incentives in risk-neutral environments without the consumption smoothing motive, e.g., Hoffmann and Pfeil (2010), Piskorski and Tchistyi (2010), DeMarzo et al. (2014). We share with these studies the optimality condition that equates, whenever possible, the firm's marginal value before and after the jump shock. Our model provides additional implications for optimal compensation, which, due to risk aversion, should remain continuous at job destruction.

Li (2017) allows risk aversion, a recurrent match-productivity shock, and provides a recursive procedure for computing the optimal contract numerically. We study a permanent job-destruction shock and provide analytical characterization of the optimal contract. Our analytical results can be extended to allow temporary, recurrent spells of zero productivity.

Tsuyuhara (2016) studies long-term contracts with moral hazard and job destruction embedded in a labor market with directed search. That paper, however, does not allow for payments from the firm to the agent after job destruction. In our paper, we allow for such payments (severance or retirement benefits) and show that they are important for the agent's incentives inside the match. A similar model with long-term contracts, moral hazard, and job-destruction shocks is solved in Lamadon (2016). There, however, output in the match does not depend on the agent's effort. The moral hazard problem applies to the probability of a job-destruction shock. Similar to Tsuyuhara (2016), Lamadon (2016) does not allow for compensation conditional on job destruction.

Organization The rest of this paper is organized as follows. Section 2 lays out the model and conducts preliminary analysis of the HJB equation. Section 3 provides our main results on the jumps in the firm's and the agent's continuation values at job destruction. Section 4 examines contract dynamics and exit probabilities. Sections 5 and 6 study the sensitivity of various contract features to the severity of the risk of job destruction, as measured by the rate of arrival λ. Section 7 concludes.

2 The principal-agent problem

The principal-agent contracting problem is the same as in Sannikov (2008) except that at a Poisson time θ the productivity of the match ends, i.e., the job is destroyed. Before θ, the cumulative output X_t produced by the agent up to date t follows

$$dX_t = A_t dt + \sigma dZ_t,$$

where $A_t \in \mathcal{A}$ is the agent's action (effort), Z_t is a standard Brownian motion on $(\Omega, \mathcal{F}, P)$. After θ, the cumulative output follows

$$dX_t = 0, \tag{1}$$

i.e., no further output is produced inside the match. Time θ arrives with Poison intensity λ and is independent of Z_t.

The set of feasible actions $\mathcal{A} \subset \mathbb{R}$, as in Sannikov, is compact with the smallest element 0. The contract is a pair of progressively measurable processes $\{(C_t, A_t), 0 \leq t < \infty\}$, where A_t is the action recommended for the agent to take at t and C_t is his compensation. The agent and the principal evaluate the contract according to, respectively,

$$\mathbb{E}\left[r \int_0^\infty e^{-rt} \left(u(C_t) - h(A_t)\right) dt\right],$$

and

$$\mathbb{E}\left[r \int_0^\infty e^{-rt}(A_t - C_t) dt\right],$$

where $r > 0$. The agent's utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is C^2 with $u' > 0$, $u'' < 0$, $\lim_{c\to 0} u'(c) = 0$, and $u(0) = 0$. The function $h : \mathcal{A} \to \mathbb{R}_+$ representing the agent's disutility from effort is increasing and convex with $h(0) = 0$. In addition, we follow Sannikov (2008) in assuming that there exists $\gamma_0 > 0$ such that $h(a) \geq \gamma_0 a$ for all $a \in \mathcal{A}$.

Under a given contract (C, A), the agent's continuation value process is

$$W_t := \mathbb{E}_t\left[r \int_t^\infty e^{-r(s-t)} \left(u(C_s) - h(A_s)\right) ds\right].$$

The Sannikov model is a special case of this specification with $\lambda = 0$, i.e., the match remains productive indefinitely.

Clearly, after the match productivity termination shock hits, the optimal action is $A_t = 0$ forever and the profit function for the firm is

$$F_0(W') = -c' \text{ such that } u(c') = W', \tag{2}$$

with the agent receiving a constant payment c' forever after the arrival of the shock, and W' represents the agent's continuation utility *after* the arrival of the shock, i.e., after any jump-at-arrival. Here, c' can be interpreted as the flow of compensation to the agent in retirement or as a severance benefit paid out upon termination of the job. The firm's after-shock profit function, F_0, is negative, strictly decreasing and, by strict concavity of u, strictly concave.

Before the shock, the dynamics of the continuation value are

$$dW_t = r(W_t - u(C_t) + h(A_t))dt + rY_t(dX_t - A_t dt) + \Delta_t(dN_t - \lambda dt), \tag{3}$$

where $dX_t - A_t dt$ is the agent's observed performance relative to the benchmark $A_t dt$, rY_t represents the sensitivity of the agent's continuation value to his performance, Δ_t is the sensitivity to the Poisson shock, and N_t is the counting process stopped at 1. As in Sannikov (2008), the contract is incentive compatible (IC) at t if

$$A_t \in \underset{a\in\mathcal{A}}{\operatorname{argmax}}\, Y_t a - h(a). \tag{4}$$

In equation (3), we can see how the risk of job destruction affects the drift of the agent's continuation value while the job is active, i.e., before the arrival of the job-destruction shock. With $dN_t = 0$, equation (3) reduces to

$$dW_t = (r(W_t - u(C_t) + h(A_t)) - \Delta_t \lambda)\, dt + rY_t(dX_t - A_t dt)dt.$$

As we see, larger Δ_t implies, ceteris paribus, a smaller rate of increase in W_t that is needed to deliver W_t to the agent over time as the job continues to survive.

The same observation can be made using the agent's post-job-destruction promised continuation value. The sensitivity process Δ_t shows by how much the agent's continuation value changes on arrival of the job-destruction shock in any date and state. Therefore, we can express it as

$$\Delta_t = W_t' - W_t = u(c_t') - W_t, \tag{5}$$

where W_t' is the agent's continuation value at t in the event $\theta = t$, c_t' is the (constant) level of the retirement (or severance) benefit grated to the agent in the same event. Using W_t' or c_t' instead of Δ_t, we can express the expected change in W_t conditional on job survival, i.e., the drift term in (3) before job destruction, as

$$\begin{aligned} & (r+\lambda)W_t - r(u(C_t) - h(A_t)) - \lambda W_t' \\ = \ & (r+\lambda)W_t - r(u(C_t) - h(A_t)) - \lambda u(c_t'). \end{aligned}$$

Since W_t is replaced with $W_t' = u(c_t')$ in the event of job destruction at t, a higher promise W_t', or, equivalently, a higher severance promise c_t', decreases the drift in W_t before job destruction.

In the recursive form, the firm's problem is to maximize the profit $F(W)$ that it can attain in the relationship with the agent when the agent is owed the continuation value W. The HJB equation for this problem is

$$(r+\lambda)F(W) = \max_{c,a,Y,\Delta} ra - rc + F'(W)r\left(W - u(c) + h(a) - \frac{\lambda}{r}\Delta\right) + \frac{1}{2}F''(W)r^2\sigma^2Y^2 + \lambda F_0(W+\Delta), \tag{6}$$

where W and $W + \Delta$ must remain nonnegative because 0 is the agent's minimax payoff.

The first order (FO) condition over Δ is

$$-F'(W)\lambda + \lambda F_0'(W+\Delta) \leq 0$$

with strict equality if $W + \Delta > 0$.

Because $F_0'(W) \leq 0$ for all W, this FO condition leads to two cases. If $F'(W) > 0$, then $W + \Delta = 0$, so $\Delta(W) = -W$. If $F'(W) \leq 0$, then $F_0'(W + \Delta) = F'(W)$, so $\Delta(W) = -W + (F_0')^{-1}(F'(W))$.

The interpretation of the optimal adjustment Δ is as follows. By increasing the promise of the continuation value to be delivered to the agent after job destruction, the firm gains additional (negative) profit $F_0'(W + \Delta)$, which it discounts at the rate of the shock arrival λ. That same increase in the agent's value post-arrival decreases the drift of the agent's continuation value conditional on the job's survival, at the rate λ. This lower drift increases the firm's profit at the rate $F'(W)\lambda$. The optimal Δ is set where the marginal cost is equal to its marginal benefit.

Remark In the first best, the agent would get constant consumption forever and work until the arrival of the job-destruction shock. Thus, his continuation utility would jump upward at the moment of shock arrival, i.e., $\Delta > 0$ in the first best. The first-best profit function is

$$F_{\text{fb}}(W) := \max_{c,a}\{ \frac{r}{r+\lambda}a - c \ : \ u(c) - \frac{r}{r+\lambda}h(a) = W\}. \tag{7}$$

With moral hazard, we will have $\Delta < 0$ at least for small W. But consumption will be continuous over the jump moment both with moral hazard and in first best.

2.1 Equivalent expressions of the HJB equation

Since the static contract is optimal after the shock, we can equivalently use the after-shock agent's continuation value $W' = W + \Delta$, or his constant retirement/severance flow of compensation, c', where $u(c') = W'$, as controls in the HJB equation, instead of Δ.

With $W' = W + \Delta$, the HJB can be written as

$$(r+\lambda)F(W) = \max_{c,a,Y,W'\geq 0} ra - rc + F'(W)r\left(W - u(c) + h(a) - \frac{\lambda}{r}(W' - W)\right) + \frac{1}{2}F''(W)r^2\sigma^2Y^2 + \lambda F_0(W').$$

The maximization of terms that involve W', i.e., $\max_{W'\geq 0}\{-F'(W)W' + F_0(W')\}$, implies that $W' = 0$ if $F'(W) \geq 0$, and W' solves

$$-F'(W) + F_0'(W') = 0$$

if $F'(W) < 0$. This says that, when possible, the slope of the firm's profit function should match before and after the shock. This is the same as the earlier discussion of the FO condition with respect to Δ.

Further, if we use retirement/severance flow c', we can write the HJB as

$$(r+\lambda)F(W) = \max_{c,a,Y,c'\geq 0} ra - rc + F'(W)r\left(W - u(c) + h(a) - \frac{\lambda}{r}(u(c') - W)\right) + \frac{1}{2}F''(W)r^2\sigma^2Y^2 - \lambda c'.$$

The FO condition with respect to c', $F'(W)u'(c') = -1$, is the same as the FO condition for c, the agent's consumption before the shock. This shows that at the time of shock arrival consumption does not change, although the continuation value typically will. So, the arrival of the shock "freezes" the current c and makes it fixed forever after.

Because $c' = c$, it will be convenient for us to eliminate c' and just use c as a single control variable (representing consumption now and forever if the shock hits now). The HJB is

$$(r+\lambda)F(W) = \max_{c,a,Y} ra - (r+\lambda)c + F'(W)\left((r+\lambda)(W-u(c)) + rh(a)\right) + \frac{1}{2}F''(W)r^2\sigma^2Y^2$$

or

$$F(W) = \max_{c,a,Y} \frac{r}{r+\lambda}a - c + F'(W)(W-u(c)) + \frac{r}{r+\lambda}F'(W)h(a) + \frac{r}{r+\lambda}\frac{1}{2}F''(W)r\sigma^2Y^2, \quad (8)$$

where, up to the time of arrival of the job-destruction shock, W_t follows

$$dW_t = ((r+\lambda)(W_t - u(c)) + rh(a))dt + rY\sigma dZ_t. \quad (9)$$

Collecting terms, we can write the HJB in the following form

$$\frac{r+\lambda}{r}\left(F(W) - F'(W)W - \max_{c\geq 0}\left\{-c - F'(W)u(c)\right\}\right) = \max_{a,Y}\left\{a + F'(W)h(a) + \frac{1}{2}F''(W)r\sigma^2Y^2\right\}.$$

Note that this equation reduces to the HJB equation of Sannikov (2008) when $\lambda = 0$.

2.2 Monotonicity of terms in the HJB equation

The left hand side of the HJB equation is monotone in λ. Whether the left hand side is increasing or decreasing in λ depends on the sign of $F(W) - F'(W)W - \max_{c\geq 0}\left\{-c - F'(W)u(c)\right\}$. But this sign is always positive as long as we are solving for a curve F in the region $F(W) \geq F_0(W)$ at all W, and F is concave.

To see this, change c to W' again using $u(c) = W'$. We have

$$\begin{aligned} F(W) - F'(W)W - \max_{c\geq 0}\left\{-c - F'(W)u(c)\right\} &= F(W) - F'(W)W - \max_{W'\geq 0}\left\{F_0(W') - F'(W)W'\right\} \\ &= \min_{W'\geq 0}\left\{F(W) - F'(W)W - F_0(W') + F'(W)W'\right\} \\ &= \min_{W'\geq 0}\left\{F(W) + F'(W)\left(W' - W\right) - F_0(W')\right\}. \end{aligned}$$

This quantity is the minimal vertical distance between the tangent to F at W and F_0. By concavity of F, the tangent is always above F, i.e., $F(W) + F'(W)(W' - W) \geq F(W')$ for all W and W'. So

$$\min_{W'\geq 0}\left\{F(W) + F'(W)\left(W' - W\right) - F_0(W')\right\} \geq \min_{W'\geq 0}\left\{F(W') - F_0(W')\right\} \geq 0,$$

where the last inequality uses $F \geq F_0$.

Thus, the left hand side of the HJB is increasing in λ as long as we are solving for a concave F above F_0. The optimal solution will satisfy these conditions.

2.3 Option value of the agent's effort

Let us denote this distance by S, i.e., let

$$\begin{aligned} S(W) &:= F(W) - \max_{W' \geq 0} \left\{ F'(W)(W - W') + F_0(W') \right\} \\ &= F(W) - \max_{c' \geq 0} \left\{ F'(W)(W - u(c')) - c' \right\}. \end{aligned}$$

Note that S is a function of $F(W)$, $F'(W)$, and W. With this notation, the HJB reads

$$(1 + \frac{\lambda}{r})S(W) = \max_{a,Y} \left\{ a + F'(W)h(a) + \frac{1}{2}F''(W)r\sigma^2 Y^2 \right\},$$

or, using the IC condition $Y = h'(a)$,

$$S(W) = \frac{r}{r+\lambda} \max_a \left\{ a + F'(W)h(a) + \frac{1}{2}F''(W)r\sigma^2 h'(a)^2 \right\},$$

which has an intuitive interpretation: $S(W) = F(W) - \max_{W' \geq 0} \left\{ F'(W)(W - W') + F_0(W') \right\}$ is the firm's surplus from being able to induce positive effort from the agent.[1] The right hand side of the HJB shows where this surplus is coming from. It is equal to expected output from effort, a, less the cost of compensating the agent for his disutility of effort $h(a)$, less the firm's cost of having to induce volatility $Y = h'(a)$ in the state variable (it is a cost because $F'' < 0$). The factor $\frac{r}{r+\lambda}$ represents the risk of productivity termination, i.e., job destruction.

Note that the same variable represents the surplus from the agent's effort at the first best contract. Indeed, the FOC in

$$S_{\text{fb}}(W) := F_{\text{fb}}(W) - \max_{c' \geq 0} \left\{ F'_{\text{fb}}(W)(W - u(c')) - c' \right\}$$

implies $c' = c_{\text{fb}}(W)$. Using this and (7), we have

$$\begin{aligned} S_{\text{fb}}(W) &= \frac{r}{r+\lambda} a_{\text{fb}} - c_{\text{fb}} - \left(F'_{\text{fb}}(W)(W - u(c_{\text{fb}})) - c_{\text{fb}} \right) \\ &= \frac{r}{r+\lambda} a_{\text{fb}} - F'_{\text{fb}}(W)(W - u(c_{\text{fb}})) \\ &= \frac{r}{r+\lambda} a_{\text{fb}} + F'_{\text{fb}}(W) \frac{r}{r+\lambda} h(a_{\text{fb}}) \\ &= \frac{r}{r+\lambda} \max_a \left\{ a + F'_{\text{fb}}(W)h(a) \right\}, \end{aligned}$$

[1] If effort is no longer an option, then the firm's profit is $F_0(W')$ after incurring the cost $F'(W)(W - W')$ of optimally adjusting the agent's value from W to W'.

where a_{fb} and c_{fb} at evaluated at W.[2] This shows that in the first best the calculation of the surplus from the agent's effort is the same except for the cost of volatility does not show up in the formula. Also note that $S_{\text{fb}}(W) > 0$ at all $W < W^*_{gp}$ and equals zero at W^*_{gp}.

2.4 Existence, regularity, and computation

Solving the HJB for F'', equivalently, we have

$$F''(W) = \min_{a \geq 0} \frac{(1 + \frac{\lambda}{r})S(W) - a - F'(W)h(a)}{\frac{1}{2}r\sigma^2(h'(a))^2}, \tag{10}$$

which can be solved forward from $W = 0$ using boundary conditions $(F(0), F'(0)) = (0, x)$ for any $x \geq 0$.

As in Sannikov (2008), the optimal profit curve is obtained by looking for the initial slope $F'(0)$ such that the solution curve stays above F_0 and touches it at a point, which is denoted by W_{gp}. Sannikov (2008) shows that the contract constructed from the policy functions that attain this solution in the HJB equation is optimal. These results, proved in Lemmas 1, 2, 3, Proposition 3, Lemma 4, and Proposition 4 of Sannikov (2008), also hold in our model. It is easy to see that nothing changes in the proofs of these results when a constant factor $(1 + \frac{\lambda}{r})$ multiplying terms $F(W) - F'(W)(W - u(c)) + c$ is inserted into the HJB equation.

Solving (10) for the optimal F results with a profit function qualitatively similar to the solution in Sannikov (2008): F is strictly concave with a unique maximum at $W^* := \operatorname{argmax} F(W)$, where $0 < W^* < W_{gp}$.

3 The jump at job destruction

In this section, we provide our main results that show how the optimal contract is affected by the risk of job destruction and what happens at the arrival of a job-destruction shock.

3.1 The jump in the agent's continuation value

Proposition 1 *In the optimal contract, $W'(W) = 0$ at all $W \leq W^*$, and W' is strictly increasing in W at all $W^* < W < W_{gp}$.*

The first part follows from $F'(W) > 0$ below W^*. The second part follows simply from strict concavity of F_0, which is a direct implication of strict concavity of u.

[2] Note that the third line uses the promise-keeping constraint in (7) and the last line uses the FOC for a_{fb}.

Figure 1 illustrates this proposition in a computed example. The jump in the agent's continuation value at job destruction is represented by the vertical distance between the $W'(W)$ curve and the 45 degrees line. For all $W \leq W^*$, where $F'(W) \geq 0$ and $c = 0$, we have $W' = 0$, i.e., the agent loses all of his continuation value with $c' = 0$. Above W^*, the retirement/severance value W' is increasing in W, i.e., agents with larger continuation value at arrival receive a higher retirement/severance value as well.

Further, Figure 1 suggests that the sign of $\Delta = W' - W$ only changes once. That is, in the region above W^*, agents with relatively small W are hurt by job destruction. However, agents with high W gain from it. We prove this feature of the optimal contract under an additional assumption.

Proposition 2 *Assume F_0' is weakly concave. Then there exists a unique W_{nj} such that $\Delta(W) < 0$ for all $0 < W < W_{nj}$; $\Delta(W_{nj}) = 0$; and $\Delta(W) > 0$ for all $0 < W_{nj} < W_{gp}$. Also, $W_{nj} > W^*$.*

Proposition 2 shows that it is optimal to widen the spread of the agent's value at the arrival of the job-destruction shock: the agents with high W (higher than W_{nj}) see their value increased, $\Delta(W) > 0$, while the agents with low W (lower than W_{nj}) experience a drop in their continuation value at job destruction, $\Delta(W) < 0$. To see why doing so is profitable, recall that $\Delta(W)$ has an inverse impact on the drift of W_t prior to job destruction: positive Δ decreases the drift of W_t and negative Δ increases it. By suppressing the growth of W_t when W_t is high and increasing it when W_t is low, the optimal policy $\Delta(W)$ decreases the chance of hitting either of the two inefficient agent retirement points, 0 and W_{gp}, while the match remains productive. This decrease in the chance of early contract termination improves efficiency.

The assumption of weak concavity of F_0' is a convenient sufficient condition that can be relaxed. If u is trice differentiable, this assumption is equivalent to $u'''(c)u'(c) \leq 3u''(c)^2$ at all c, which is very simple condition to verify.

Proposition 2 also implies that $S(W)$ is single peaked with a unique maximum at W_{nj}. Indeed, differentiating $S(W)$ we have

$$\begin{aligned}\frac{dS}{dW} &= F'(W) - F''(W)(W - W'(W)) - F'(W) \\ &= F''(W)\left(W'(W) - W\right) \\ &= F''(W)\Delta(W)\end{aligned}$$

so Δ and S' are of opposite sign. Proposition 2, by pinning down the sign of $\Delta(W)$, implies the following:

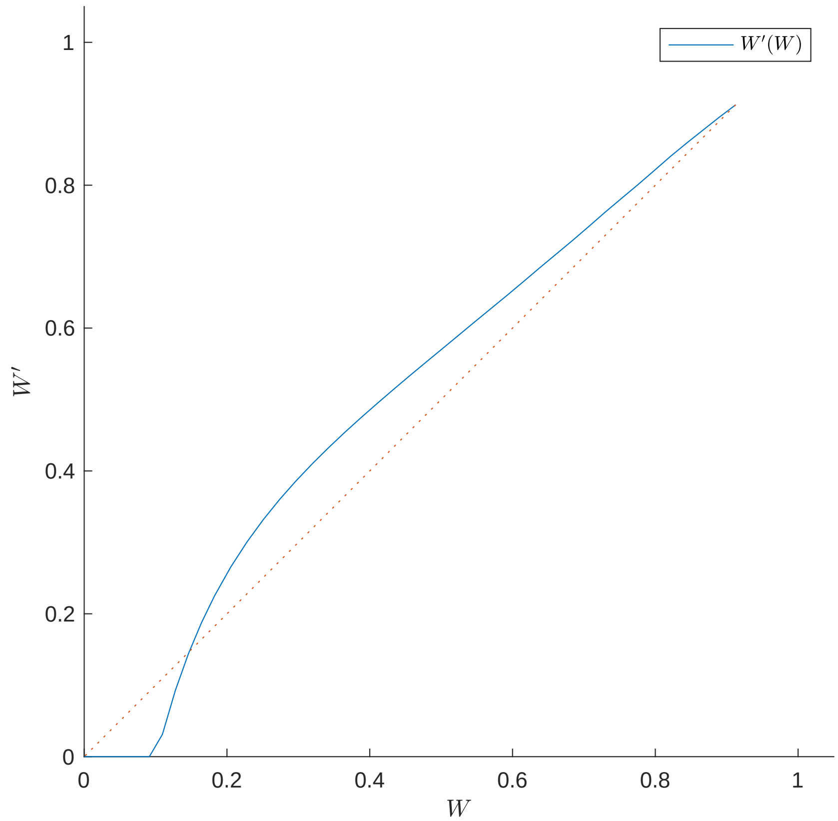

Figure 1: Optimal jump of the agent's continuation value at job destruction.

Corollary 1 *$S'(W) > 0$ at all $W < W_{nj}$, $S'(W_{nj}) = 0$, and $S'(W) < 0$ for all $0 < W_{nj} < W_{gp}$. Thus, W_{nj} is the unique peak point of $S(W)$.*

This means that the jump in the agent's continuation value at job destruction is zero only at the single W at which the firm's option value of the agent's effort is maximal.[3]

3.2 The jump in the firm's value

From the definition of S, we have $S(W) = F(W) + F'(W)\Delta(W) - F_0(W'(W))$, which gives us

$$F(W) - F_0(W'(W)) = S(W) - F'(W)\Delta(W).$$

This says that the firm's loss of value at job destruction, $F(W) - F_0(W'(W))$, equals the loss of the option on the agent's effort, $S(W)$, and the cost of the agent's gain in utility, $\Delta(W)$, valued at the marginal price of utility, $-F'(W)$. The loss of productivity option value is always

[3]Note that $S(W)$ does not have to be concave.

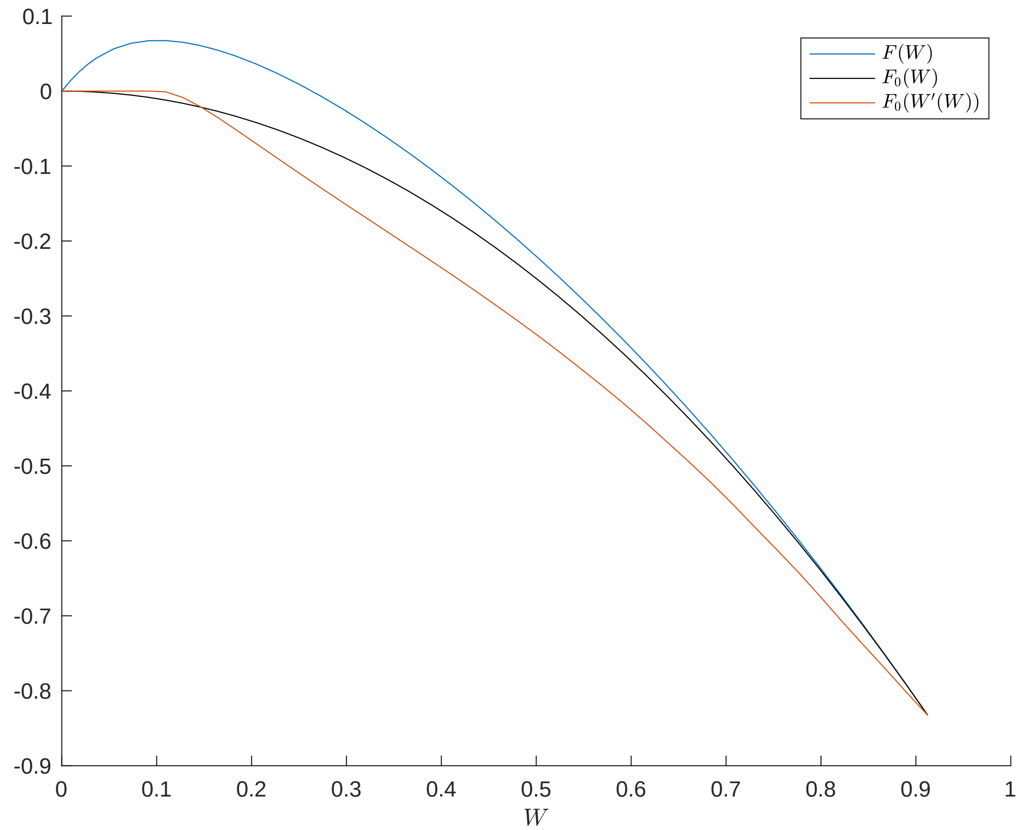

Figure 2: The firm's loss of profit after the optimal jump in W at arrival of job destruction.

positive, as $S \geq 0$. The second term may be positive or negative, as both Δ and F' change sings.

Proposition 3 *Assume F_0' is weakly concave. Then the firm always loses value at arrival of the job-destruction shock: $F(W) \geq F_0(W'(W))$, strictly at all $0 < W < W_{gp}$.*

The above result is very intuitive at W equal 0 or W_{gp}. There, there is no loss of profits due to job destruction because the job is dissolved endogenously at these two points anyway (i.e., both S and Δ are zero there).

At all $0 < W < W_{gp}$, the effort option value $S(W)$ is strictly positive. The firm's value of the agent's utility jump depends on W. Above W_{nj}, $\Delta > 0$ and $F' < 0$, i.e., in addition to the loss of productivity, the firm is hurt at job destruction by a positive jump in the value it owes to the agent. Below W^*, the firm is again hurt by the adjustment to the agent's value, but for the opposite reason, as signs of both Δ and F' are switched. With $\Delta < 0$ and $F' > 0$, the agent's loss of value actually hurts the firm's value. In fact, the agent loses all promised value as $W' = 0$ in this region, but this does not help the firm as all of the agent's value in this region comes from his future expected compensation. Finally, in the region between W^* and W_{nj} we have $\Delta < 0$ and $F' < 0$, so the loss of the agent's utility value does offset the firm's loss of productivity to an extent. However, it turns out that that this offset is insufficient.

Figure 2 provides a computed example. Note the small region in which $F_0(W'(W)) > F_0(W)$. There, the negative jump in the agent's continuation value at job destruction partially compensates the firm for the loss of the option on the agent's effort.

4 Contract dynamics and exit probabilities

In this section, we discuss contract dynamics with particular attention to exits. We will characterize these features of the optimal contract by finding an associated ODE and solving it numerically using the policies from the optimal contract.

We will denote the drift of W_t under the optimal contract by $\mu(W)$ and its volatility by $\nu(W)$. We have from (9) that

$$\begin{aligned} \mu(W) &= (r+\lambda)\big(W - u(c(W))\big) + rh(a(W)), \\ \nu(W) &= r\sigma Y(W), \end{aligned}$$

where $c(W)$, $a(W)$, and $Y(W)$ are the policy functions from the optimal contract.

Figure 3 shows the drift and volatility functions in a computed example, where λ gives a realistic contract duration. Following the labor literature, we target in our parametrization an average duration of a job to be 10 calendar quarters. We approximate job duration here as the expected time to arrival of a job-destruction shock, $1/\lambda$. As we see, the contract has interesting dynamics. At small W, the drift of W is positive and high with high volatility. The contract likely moves out of this region toward the middle region of W. There, the contract "slows down," i.e., has drift close to zero and moderate volatility, which means the contract will likely spend a lot of time in the middle region once it reaches it. Thus, the job-destruction shock is likely to arrive while W_t is in that region.

4.1 Time remaining and exit probability

Let us denote by $T(W)$ the expected time until job end (including both exogenous job destruction and endogenous agent retirement). The job can "end" in three ways: the agent may be retired at 0, retired at W_{gp}, or the job is ended by a job-destruction shock.

The probability of end/exit at 0 will be denoted by $P_0(W)$, and the probability of exit at W_{gp} by $P_{gp}(W)$. The probability that the contract ends with the arrival of the job-destruction shock will be denoted by $P_{JD}(W)$. Clearly, $P_0(W) + P_{gp}(W) + P_{JD}(W) = 1$ for all W.

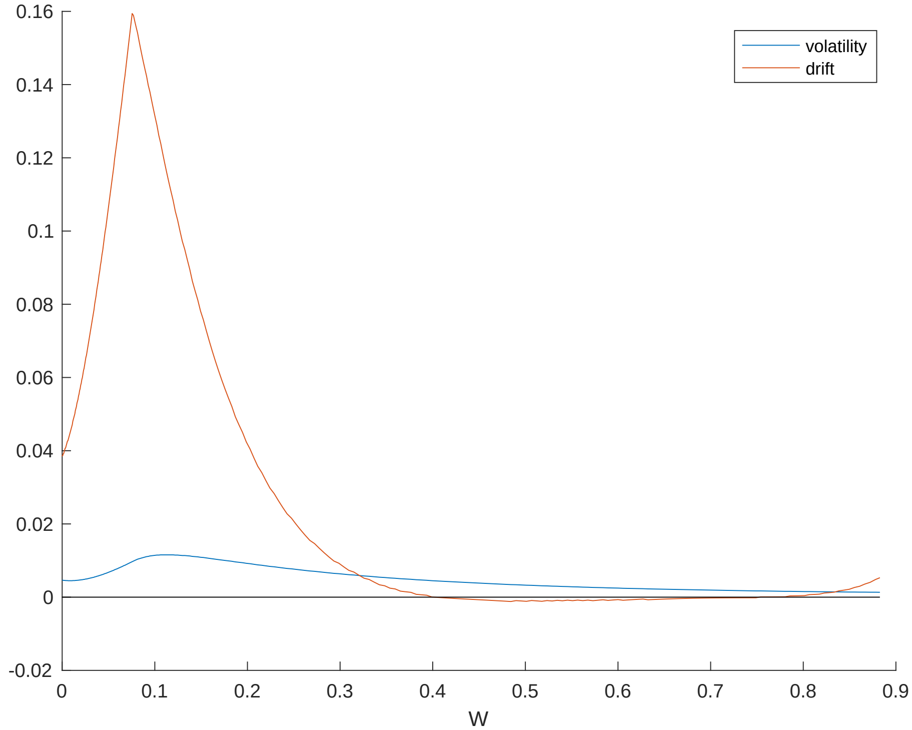

Figure 3: Drift $\mu(W)$ and volatility $\nu(W)$ at the optimal contract under a parametrization with average time till job destruction of 10 quarters.

We compute T and P by finding an ODE for each of them. These ODEs can then be easily solved numerically using policy functions from the optimal contract.

Lemma 1 *T and P satisfy the following ODEs*

$$\lambda T(W) = 1 + T'(W)\mu(W) + \frac{1}{2}T''(W)\nu(W)^2,$$
$$(r+\lambda)P(W) = P'(W)\mu(W) + \frac{1}{2}P''(W)\nu(W)^2,$$

with boundary conditions $T(0) = T(W_{gp}) = 0$, $P_0(0) = 1$, $P_0(W_{gp}) = 0$, $P_{gp}(0) = 0$, $P_{gp}(W_{gp}) = 1$, *and* $P_{JD}(0) = P_{JD}(1) = 0$.

Figure 4 shows T and the three P functions in our parametrized example. The probability of exit at either end of the support of W drops very quickly in the distance between W and this boundary. In the middle, the contract "slows down" very dramatically, as we saw in Figure 3. It is therefore extremely unlikely that W_t reaches either retirement point before the arrival of a job-destruction shock. Thus, in this region, $T(W)$ is very close to $1/\lambda$ and $P_{JD}(W)$ is very close to 1.

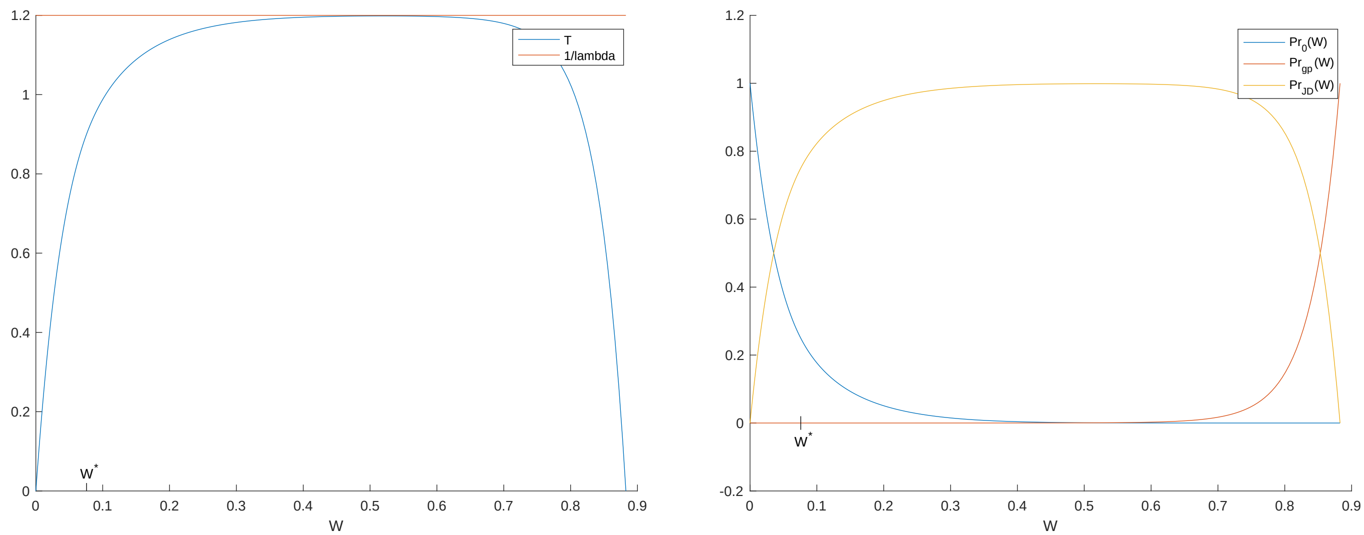

Figure 4: T and P under parametrization with $r/\lambda = 0.12$, i.e., average job duration of 10 quarters.

4.2 Other contract features

This method, i.e., finding a function by numerically solving an ODE, can be used to compute other features of the optimal contract and the value function. In this section, we illustrate this point by computing a decomposition of F into its cost and revenue parts.

The expected, discounted remaining revenue will be denoted by $R(W)$, and the expected discounted remaining wage bill by $B(W)$. That is:

$$R(W_t) = \mathbb{E}_t \left[r \int_0^\infty e^{-rs} a(W_{t+s}) ds \right] \quad \text{and} \quad B(W_t) = \mathbb{E}_t \left[r \int_0^\infty e^{-rs} c(W_{t+s}) ds \right],$$

so we have $F(W) = R(W) - B(W)$ for all W.

Lemma 2 *R and B satisfy the following ODEs*

$$\begin{aligned} (r+\lambda)R(W) &= ra + R'(W)\mu(W) + \frac{1}{2}R''(W)\nu(W)^2, \\ (r+\lambda)B(W) &= (r+\lambda)c + B'(W)\mu(W) + \frac{1}{2}B''(W)\nu(W)^2, \end{aligned}$$

with boundary conditions $R(0) = R(W_{gp}) = 0$, *and* $B(0) = 0$, $B(W_{gp}) = u^{-1}(W_{gp}) = -F_0(W_{gp})$.

Figure 5 shows the solutions of the ODEs for R and B, along with the profit function F, obtained with λ such that $1/\lambda$ is 10 quarters, as before. As we see, the wage bill by $B(W)$ accounts for most of the variation in $F(W)$ with expected revenue $R(W)$ being relatively flat. However, it is the revenue part of F that gives it its hump shape, as B is monotone in W.

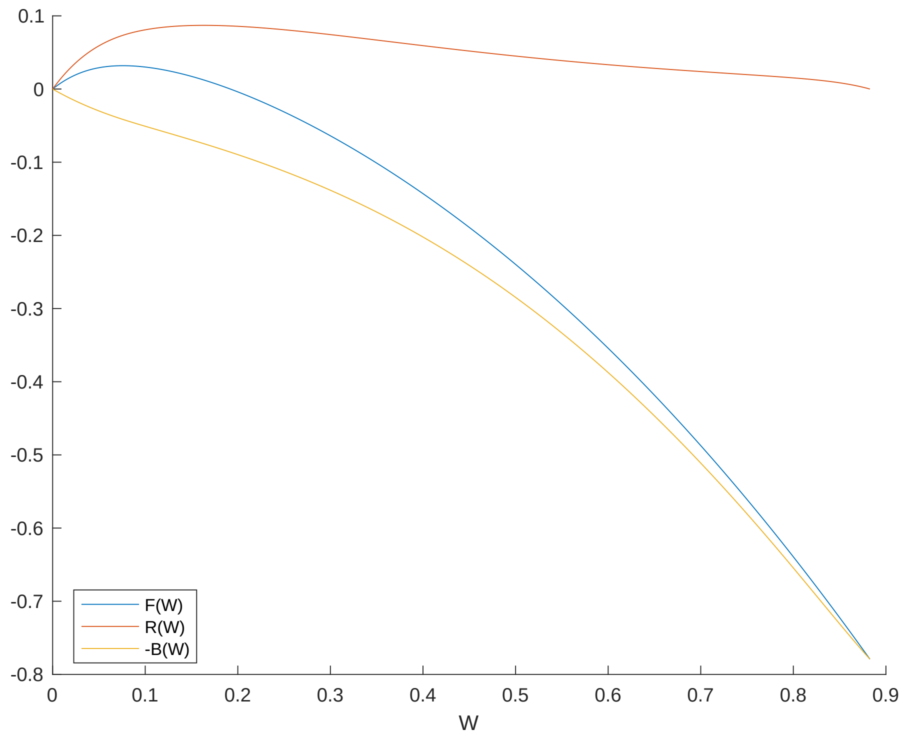

Figure 5: Profit, revenue and the wage bill.

5 Sensitivity of profit to job destruction risk

In this section, we study how the firm's expected discounted profit depends on the expected duration of the relationship, $1/\lambda$, or, equivalently, on λ measuring job-destruction risk. For comparisons with respect to the level of λ, let us always write $\tilde{\lambda} > \lambda \geq 0$ and denote the solution under $\tilde{\lambda}$ as $\tilde{F}$.

Lemma 3 *Take two solutions F and $\tilde{F}$ of the HJB, where the first is with λ and the second is with $\tilde{\lambda} > \lambda$, such that $F(W^0) \leq \tilde{F}(W^0)$ and $F'(W^0) < \tilde{F}'(W^0)$ at some $W^0 \geq 0$. Then $F'(W) < \tilde{F}'(W)$ at all $W > W^0$.*

From here on, by F and $\tilde{F}$ we mean the optimal solution curves under λ and $\tilde{\lambda}$, respectively.

Proposition 4 *$F(W) > \tilde{F}(W)$ at all $0 < W \leq \tilde{W}_{gp}$, and $W_{gp} > \tilde{W}_{gp}$.*

Intuitively, the above result shows that principal-agent relationships with lower expected duration, i.e., faster rate of arrival of the job-destruction shock, are less profitable to the firm for any fixed value W the firm might owe to the agent. Further, the upper agent retirement point W_{gp} is always lower in the relationships with lower expected duration. That is, the firm has a weaker incentive to invest in the agent's incentives in relationships with a higher risk of exogenous job destruction.

5.1 Application of the Feynman-Kac formula

Here, we use the approach of Lemma 4 of DeMarzo and Sannikov (2006) to derive the following formula.

Lemma 4

$$\frac{\partial F(W_0)}{\partial \lambda} = -\mathbb{E}\left[\int_0^\tau e^{-(r+\lambda)t} S(W_t)dt\right] < 0, \tag{11}$$

where τ denotes the time of fist exit of W_t from $(0, W_{gp})$.

Further, since $S(0) = S(W_{gp}) = 0$ and the dynamics of W_t stop if either of these points is reached, we can write the above equation as

$$\frac{\partial F(W_0)}{\partial \lambda} = -\mathbb{E}\left[\int_0^\infty e^{-(r+\lambda)t} S(W_t)dt\right].$$

Proposition 4 was obtained by a different argument. The above formula provides additional information of the magnitude of $\frac{\partial F(W_0)}{\partial \lambda}$ by relating it to the firm's option value on the agent's effort, $S(W)$. This expression will allow us to show, numerically, that the firm's profit is less sensitive to the job destruction risk with moral hazard than it would be without moral hazard (i.e., in the first-best contract).

5.2 Profit sensitivity relative to the first best

Differentiating the first-best profit function F_{fb} given in (7) with respect to λ and using the envelope condition, we obtain

$$\begin{aligned}\frac{\partial F_{\text{fb}}(W_0)}{\partial \lambda} &= -\frac{r}{(r+\lambda)^2}(a_{\text{fb}} + F'_{\text{fb}}(W_0)h(a_{\text{fb}})) \\ &= -\frac{1}{r+\lambda}S_{\text{fb}}(W_0) \\ &= -\int_0^\infty e^{-(r+\lambda)t} S_{\text{fb}}(W_0)dt.\end{aligned}$$

In Section 2.3, we show that $S_{\text{fb}}(W_0) > 0$ for any $W_0 < W^*_{gp}$. The above formula thus implies that $\frac{\partial F_{\text{fb}}(W_0)}{\partial \lambda} < 0$, as in the case with moral hazard shown above.

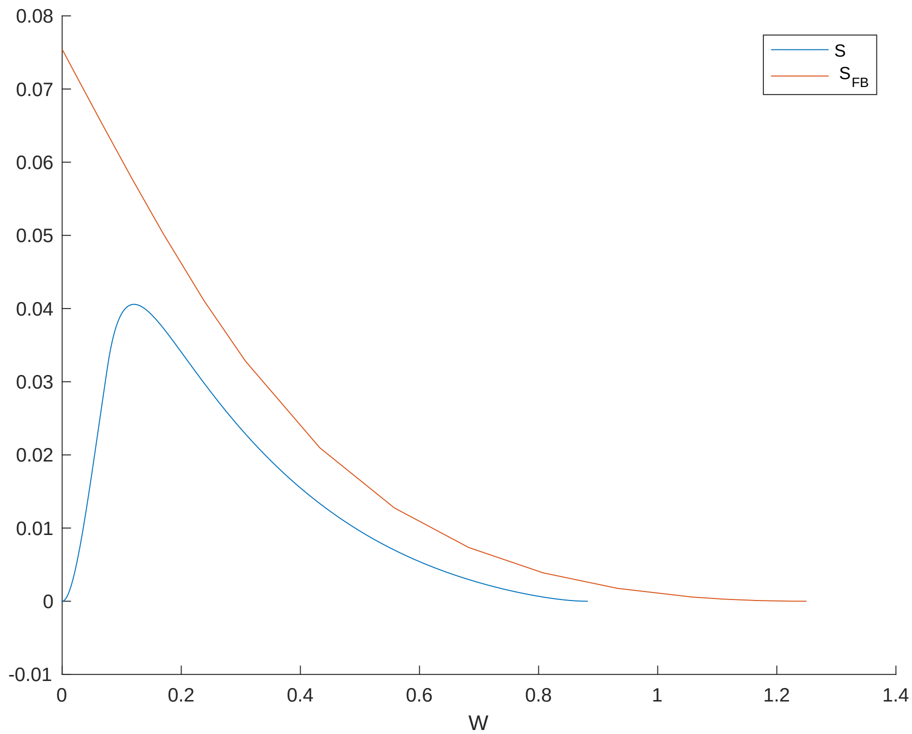

Figure 6: The function S in the optimal contract with moral hazard and in the first best.

The comparison between $\frac{\partial F(W_0)}{\partial \lambda}$ and $\frac{\partial F_{\text{fb}}(W_0)}{\partial \lambda}$ is difficult to obtain analytically because the first-best contract is static. This means that S takes into account two components of the costs of higher effort: F', and F'' representing the cost of volatility. In contrast, S_{fb} only needs to account for one cost, F'_{fb}. However, F' will generally be less negative than F'_{fb}, so it is unclear if two smaller costs are larger than one bigger cost. For this reason, we compare these measures of sensitivity numerically.

Figure 6 plots S and S_{fb} in our parametrized example. Since $S < S_{\text{fb}}$, we have $\frac{\partial F(W)}{\partial \lambda} > \frac{\partial F_{\text{fb}}(W)}{\partial \lambda}$ for all $0 \leq W \leq W_{gp}$, regardless of the fact that the agent's continuation value process has complicated dynamics under moral hazard but is reduced to a constant under first best.

6 Sensitivity of compensation front-loading to job destruction risk

In this section, we briefly discuss the impact of the risk of job destruction on the amount of compensation front-loading in the optimal contract. We maintain the assumption that F'_0 is weakly concave.

Proposition 5 *There exists a unique* W^s *such that for all* $W \in [0, \tilde{W}_{gp}]$, $c(W) \leq \tilde{c}(W)$ *if and only if* $W \leq W^s$. *That is, the contract associated with higher* λ *involves more front-loaded payments.*

Following the terminology of Sannikov (2008), a contract involves more front-loaded payment when it pays higher wage now rather than later. Proposition 5 concludes that a more likely job-destruction shock (higher λ) induces front-loading of wages when the agent's continuation utility is low. When the continuation utility is high, the optimal contract front-loads less as the job-destruction risk increases.

7 Conclusion

In this paper, we study the impact of exogenous job destruction risk on the optimal long-term contract in an dynamic moral hazard environment. We show that post-job-destitution payments to the agent are an important incentive device. In particular, they help the firm control the drift of the agent's continuation value inside the contract. In the optimal contract, these payments are used to keep the agent's continuation value in the region where the firm's option value of using the agent's effort is highest. The contract promises a positive jump in the agent's value at job destruction whenever the firm's option value on the agent's effort is decreasing in the agent's value. Likewise, a negative jump is promised whenever the firm's option value is increasing. These promised jumps help keep the state variable near the peak of the firm's option value.

In the model without job destruction risk, the optimal contract has two exit points: the low and high agent retirement points 0 and W_{gp}, respectively. Job destruction adds the third contract exit possibility. Further, our analysis suggests that with job destruction the high retirement exit point becomes unreachable, as the contract dynamics become very slow before the contract can get there. Only in relationships with very long expected duration, the high retirement point may remain reachable. This conjecture can be further explored by studying the limiting contract as the job destruction arrival rate goes to infinity.

Our model can be extended in several interesting directions. Job destruction can be endogenized in a way similar to Mortensen and Pissarides (1994): the job-destruction shock can be partial, i.e., reducing the productivity of the match to a low but positive level. It is natural to conjecture that the contract would terminate endogenously (retire the agent at 0 or W_{gp}) shortly after the arrival of such a shock. The model can be embedded in a broader labor market with search, as in Lamadon (2016) and Tsuyuhara (2016), to study the impact of severance payments on the agent's search behavior.

Appendix

Proof of Proposition 1

Above W^*, the FO condition for W' is $F'(W) = F_0'(W'(W))$. Differentiation yields

$$F''(W) = F_0''(W')\frac{dW'}{dW}.$$

That W' is strictly increasing in this region follows from strict concavity of F and F_0.

QED

Proof of Proposition 2

Step 1. Differentiating the HJB and canceling out like terms, we get

$$\begin{aligned} 0 &= F''r\left(W - u + h - \frac{\lambda}{r}(W' - W)\right) + \frac{1}{2}F'''r^2\sigma^2Y^2 \\ &= F''r\left(h - (1 + \frac{\lambda}{r})(W' - W)\right) + \frac{1}{2}F'''r^2\sigma^2Y^2. \end{aligned}$$

From here we get that $W' - W < 0$ implies $F'''(W) > 0$ (i.e., F' must be strictly convex when $\Delta < 0$).

Step 2. We know $F'(0) > 0 = F_0'(0)$. We also know that F approaches F_0 from above as W becomes close to W_{gp}, which means that for $\varepsilon > 0$ small enough, $F(W) > F_0(W)$ for all $W \in (W_{gp} - \varepsilon, W_{gp})$ and $F(W_{gp}) = F_0(W_{gp})$. This implies that $F'(W) < F_0'(W)$ for all $W < W_{gp}$ close enough to W_{gp}. Therefore, the number of crossings between F' and F_0' on $(0, W_{gp})$ is odd. We will show that this number is one. If this number is three or more, then there must exist a point $\tilde{W}$ somewhere between the second and the third crossing at which F' is more concave than F_0'. (If this was not true, the third crossing of F' and F_0' would not exist.) Thus, $F'''(\tilde{W}) < F_0'''(\tilde{W}) \leq 0$, where the weak inequality uses the assumption of weak concavity of F_0'. I.e., F' is strictly concave at $\tilde{W}$. Since $\tilde{W}$ is between the second and the third crossing of F' and F_0', we have $F'(\tilde{W}) > F_0'(\tilde{W})$, which implies that $W'(\tilde{W}) < \tilde{W}$. We obtain a contradiction because we showed in Step 1 that F' must be convex when $W' < W$, i.e., $W'(\tilde{W}) < \tilde{W}$ implies $F'''(\tilde{W}) > 0$.

Step 3. Denote the unique crossing point between F' and F_0' by W_{nj}. The sign of $\Delta(W)$ is the same as the sign of $F_0' - F'$, which is negative for $W < W_{nj}$ and positive $W > W_{nj}$ by the single crossing property shown in Step 2. Also, $W_{nj} > W^*$ follows from $F'(W^*) = 0$, $F'(W_{nj}) = F_0'(W_{nj}) < 0$, and F' continuous and decreasing.

QED

Proof of Proposition 3

If the contract reaches either of the two agent retirement points before the shock arrives, then there is no jump.

For any fixed $0 < W < W_{gp}$, we have

$$F(W) > F_0(W). \tag{12}$$

At W_{nj}, which satisfies $0 < W_{nj} < W_{gp}$, we have $W'(W_{nj}) = W_{nj}$, so if the shock arrives when $W_t = W_{nj}$, the firm loses simply because of the loss of productivity, i.e., because of equation (12). Indeed, we have $F(W_{nj}) > F_0(W_{nj}) = F_0(W'(W_{nj}))$.

At all $W_{nj} < W < W_{gp}$, the firm loses at shock arrival for two reasons. First, as in the previous case, because of the productivity loss, i.e., equation (12). Second, the firm loses because the agent gains and F is strictly decreasing in this region (recall $W^* < W_{nj}$). Indeed $W > W_{nj}$ implies $W'(W) > W$ and F strictly decreasing implies $F(W) > F(W'(W))$. Together, we have $F(W) > F(W'(W)) > F_0(W'(W))$.

At all $0 < W \leq W^*$, the adjustment from W to W' still hurts the firm, but its two components switch signs. The agent gives up value, but this hurts the firm because F is increasing in this region. In fact, $W'(W) = 0 < W$ for all $0 < W \leq W^*$, so we have $F(W) > 0 = F_0(0) = F_0(W'(W))$.

For $W^* < W < W_{nj}$, we have $W'(W) < W$ and F is decreasing, so the jump in W benefits the firm. Thus, the loss of productivity effect and the jump in W effect work in opposite directions. We will show that the productivity loss effect is stronger, i.e., the fact that the agent gives up value does not make up for the loss of productivity.

Because $F(W^*) > 0 = F_0(W'(W^*))$, showing that $F(W) - F_0(W'(W))$ is increasing at all $W^* < W < W_{nj}$ will be sufficient for $F(W) > F_0(W'(W))$ at all $W^* < W < W_{nj}$. We will show that this is the case. Taking the derivative of $F(W) - F_0(W'(W))$ and using the FO condition $F'(W) = F_0'(W'(W))$ we have

$$F'(W) - F_0'(W')\frac{dW'}{dW} = F'(W)\left(1 - \frac{dW'}{dW}\right),$$

which means that it is sufficient to show that $\frac{dW'}{dW} > 1$ at all $W^* < W < W_{nj}$. We know that $W'(W^*) = 0$ and $W'(W_{nj}) = W_{nj}$, so W' starts below the 45 degrees line and catches up to it over the interval $W^* < W < W_{nj}$, but we need to show that the catching up has no gaps.

Differentiating the FO condition $F'(W) = F_0'(W'(W))$ yields

$$F''(W) = F_0''(W')\frac{dW'}{dW}.$$

We have $F''(W) < F''(W_{nj}) \leq F_0''(W_{nj}) \leq F_0''(W')$. The first inequality follows from the strict convexity of F' at all $W < W_{nj}$ by Step 1 in the proof Proposition 2. The second inequality follows from the fact that F' crosses F_0' from above at W_{nj}. The third inequality follows from the (weak) concavity of F_0' and $W' < W < W_{nj}$. Finally, from $F''(W) < F_0''(W') < 0$ it follows that $F''(W)/F_0''(W') > 1$, which implies

$$\frac{dW'}{dW} = F''(W)/F_0''(W') > 1.$$

QED

Proof of Lemma 1

For the expected time T, define $H = \int_0^\infty 1_{s<\theta} 1_{s<\tau} ds$, where θ is the arrival time of the job-destruction shock, and τ is the time when W_t hits 0 or W_{gp}. Define a martingale H_t as $H_t = \mathbb{E}_t[H] = \int_0^t \mathbb{E}_t[1_{s<\theta}] 1_{s<\tau} ds + \mathbb{E}_t[1_{t<\theta}] \mathbb{E}[\int_t^\infty 1_{s<\theta} 1_{s<\tau} ds | \mathcal{F}_t, t < \theta] = \int_0^t e^{-\lambda s} 1_{s<\tau} ds + e^{-\lambda t} 1_{t<\tau} T(W_t)$. If $t < \tau$, then its drift is

$$e^{-\lambda t} \left(1 + T'(W)((r+\lambda)(W-u) + rh) + \frac{1}{2} T''(W)(r\sigma Y)^2 - \lambda T(W) \right),$$

which must be zero.

For the exit probability functions P, the argument is similar.

Proof of Lemma 2

Let $\mathcal{F}_t$ be the information set containing the sample path of the diffusion process of W_t, and let $\mathcal{G}_t$ be the information set containing the realization of the job-destruction shock. $\mathcal{F}_t$ and $\mathcal{G}_t$ are independent. We denote $\mathbb{E}[\cdot|\mathcal{F}_t]$ as $\mathbb{E}_t[\cdot]$ to simplify notation.

For the revenue function R, define $H = \int_0^\infty 1_{s<\theta} 1_{s<\tau} r e^{-rs} a_s ds$, and a martingale $H_t = \mathbb{E}_t[H] = \int_0^t \mathbb{E}_t[1_{s<\theta}] 1_{s<\tau} r e^{-rs} a_s ds + \mathbb{E}_t[1_{t<\theta}] 1_{t<\tau} e^{-rt} R(W_t) = \int_0^t r e^{-(r+\lambda)s} 1_{s<\tau} a_s ds + e^{-(r+\lambda)t} 1_{t<\tau} R(W_t)$. If $t < \tau$, then its drift is

$$e^{-(r+\lambda)t} \left(r a_t + R'(W)((r+\lambda)(W-u) + rh) + \frac{1}{2} R''(W)(r\sigma Y)^2 - (r+\lambda) R(W) \right),$$

which must be zero because H_t is a martingale.

For the wage bill B, define $\tilde{c}_t = c_{t\wedge\tau}$ and $H = \int_0^\infty r e^{-rs} (1_{s<\theta} \tilde{c}_s + 1_{s\geq\theta} \tilde{c}_\theta) ds = \int_0^\theta r e^{-rs} \tilde{c}_s ds +$

$e^{-r\theta}\tilde{c}_\theta$. Define the martingale as

$$\begin{aligned}
H_t &= \mathbb{E}_t[H] \\
&= \mathbb{E}_t\left[\int_0^{\theta\wedge t} re^{-rs}\tilde{c}_s ds + 1_{\theta\geq t}\int_t^\theta re^{-rs}\tilde{c}_s ds\right] + \mathbb{E}_t[1_{\theta<t}e^{-r\theta}\tilde{c}_\theta + 1_{\theta\geq t}e^{-r\theta}\tilde{c}_\theta] \\
&= \mathbb{E}_t\left[\int_0^{\theta\wedge t} re^{-rs}\tilde{c}_s ds + 1_{\theta<t}e^{-r\theta}\tilde{c}_\theta\right] + \mathbb{E}_t\left[1_{\theta\geq t}\left(\int_t^\theta re^{-rs}\tilde{c}_s ds + e^{-r\theta}\tilde{c}_\theta\right)\right] \\
&= \int_0^t \mathbb{E}_t[1_{\theta\geq s}]re^{-rs}\tilde{c}_s ds + \mathbb{E}_t[1_{\theta<t}e^{-r\theta}\tilde{c}_\theta] + \mathbb{E}_t[1_{\theta\geq t}]E\left[\left(\int_t^\theta re^{-rs}\tilde{c}_s ds + e^{-r\theta}\tilde{c}_\theta\right)|\mathcal{F}_t, \theta\geq t\right] \\
&= \int_0^t re^{-(r+\lambda)s}\tilde{c}_s ds + \int_0^t \lambda e^{-\lambda s}e^{-rs}\tilde{c}_s ds + e^{-\lambda t}e^{-rt}B(W_t) \\
&= \int_0^t (r+\lambda)e^{-(r+\lambda)s}\tilde{c}_s ds + e^{-(r+\lambda)t}B(W_t).
\end{aligned}$$

If $t<\tau$, then its drift is

$$e^{-(r+\lambda)t}\left((r+\lambda)c_t + B'(W)((r+\lambda)(W-u)+rh) + \frac{1}{2}B''(W)(r\sigma Y)^2 - (r+\lambda)B(W)\right) = 0.$$

QED

Proof of Lemma 3

Let us use the following notation

$$H_{a,Y,c;\lambda}(W,F,F') := \frac{(1+\frac{\lambda}{r})(F - F'(W-u(c)) + c) - a - F'h(a)}{\frac{1}{2}r\sigma^2 Y^2}.$$

By contradiction, let's define W^1 as the smallest point at which $F'(W^1) = \tilde{F}'(W^1)$. Because $F(W^0) \leq \tilde{F}(W^0)$ and $F'(W) < \tilde{F}'(W)$ at all $W \in [W^0, W^1)$, we have $F(W^1) < \tilde{F}(W^1)$, and

$$F''(W^1) \leq H_{a,Y,c;\lambda}(W^1, F(W^1), F'(W^1)) < H_{a,Y,c;\tilde{\lambda}}(W^1, \tilde{F}(W^1), \tilde{F}'(W^1)) = \tilde{F}''(W^1),$$

where (a,Y,c) are controls that attain $\tilde{F}''(W^1)$. The strict inequality is true because $H_{a,Y,c;\lambda}(W,F,F')$ is strictly increasing in F and increasing in λ. This implies $F'(W^1-\varepsilon) > \tilde{F}'(W^1-\varepsilon)$ for a sufficiently small $\varepsilon > 0$, which contradicts the definition of W^1.

QED

Proof of Proposition 4

1. Fix F and take a candidate solution for $\tilde{F}$. Because $\tilde{\lambda} > \lambda$, $F'(0) = \tilde{F}'(0)$ implies $F'(W) < \tilde{F}'(W)$ for all $W > 0$, which shows that the solution $\tilde{F}$ never returns to F_0 so it is not a feasible

candidate for an optimal contract. Lemma 3 now implies that $F'(0) > \tilde{F}'(0)$ at the optimal solution $\tilde{F}$. Thus, $F(\varepsilon) > \tilde{F}(\varepsilon)$ for all sufficiently small $\varepsilon > 0$.

2. We show that F and $\tilde{F}$ must meet on $(0, W_{gp}]$, i.e., it cannot be that $F(W) > \tilde{F}(W)$ for all $(0, W_{gp}]$. Indeed, we'd have $F_0(W_{gp}) = F(W_{gp}) > \tilde{F}(W_{gp})$, which contradicts $F_0 \leq \tilde{F}$.

3. Let $\hat{W} \leq W_{gp}$ be the smallest $W > 0$ where F and $\tilde{F}$ have the same value. We show that $\tilde{W}_{gp} \leq \hat{W}$. If not, then Lemma 3 implies that $\tilde{F}$ never returns to F_0, so $\tilde{W}_{gp}$ does not exist, which is a contradiction.

4. There are two possibilities: $\tilde{W}_{gp} = \hat{W} = W_{gp}$ and $\tilde{W}_{gp} < \hat{W} < W_{gp}$. We show that the first is not the case. Indeed, with $\tilde{W}_{gp} = W_{gp}$ the value matching and smooth pasting conditions

$$F(W_{gp}) = \tilde{F}(W_{gp}) = F_0(W_{gp}),$$
$$F'(W_{gp}) = \tilde{F}'(W_{gp}) = F_0'(W_{gp})$$

imply

$$S(W_{gp}) = \tilde{S}(W_{gp}) = F_0(W_{gp}) - \max_{W' \geq 0} \left\{ F_0'(W_{gp})(W_{gp} - W') + F_0(W') \right\} = 0.$$

Plugging $S(W_{gp}) = \tilde{S}(W_{gp}) = 0$ into the HJB equation and applying an Envelope Theorem, we obtain $F''(W_{gp}) = \tilde{F}''(W_{gp})$ and hence $a(W_{gp}) = \tilde{a}(W_{gp})$ and $F'''(W_{gp}) = \tilde{F}'''(W_{gp})$. Repeating the same argument after differentiating the first-order condition for a at W_{gp}, we have $F^{(4)}(W_{gp}) < \tilde{F}^{(4)}(W_{gp})$, since $dW'(W_{gp})/dW = d\tilde{W}'(W_{gp})/dW < 1$. Thus there exists $\varepsilon > 0$ such that $F'''(W) > \tilde{F}'''(W)$, $F''(W) < \tilde{F}''(W)$ and $F'(W) > \tilde{F}'(W)$ for all $W \in [W_{gp} - \varepsilon, W_{gp})$, which contradicts the fact that $\tilde{F}'(W)$ is cutting $F'(W)$ from above at $W = W_{gp}$. Thus, we must have $\tilde{W}_{gp} < \hat{W} < W_{gp}$.

QED

Proof of Lemma 4

Differentiating the HJB (8) wrt λ, we have

$$\begin{aligned} \frac{\partial F(W)}{\partial \lambda} &= \frac{-r}{(r+\lambda)^2}\left(a + F'(W)h(a) + \frac{1}{2}F''(W)r\sigma^2 Y^2\right) + \frac{r}{r+\lambda}\left(\frac{\partial F(W)}{\partial \lambda}\right)' h(a) + \\ &\quad \frac{r}{r+\lambda}\frac{1}{2}\left(\frac{\partial F(W)}{\partial \lambda}\right)'' r\sigma^2 Y^2 + \left(\frac{\partial F(W)}{\partial \lambda}\right)'(W - u(c)) \end{aligned}$$

Denoting $\frac{\partial F(W)}{\partial \lambda}$ as $G(W)$, we have a second-order differential equation

$$\begin{aligned} G(W) &= \frac{-1}{r+\lambda}\frac{r}{r+\lambda}\left(a+F'(W)h(a)+\frac{1}{2}F''(W)r\sigma^2Y^2\right)+G'(W)\left(W-u(c)+\frac{r}{r+\lambda}h(a)\right)+ \\ &\quad \frac{r}{r+\lambda}\frac{1}{2}G''(W)r\sigma^2Y^2 \\ &= \frac{-1}{r+\lambda}S(W)+G'(W)\left(W-u(c)+\frac{r}{r+\lambda}h(a)\right)+\frac{r}{r+\lambda}\frac{1}{2}G''(W)r\sigma^2Y^2, \end{aligned}$$

or

$$(r+\lambda)G(W) = -S(W)+G'(W)\left((r+\lambda)(W-u(c))+rh(a)\right)+\frac{1}{2}G''(W)r^2\sigma^2Y^2, \qquad (13)$$

with boundary conditions $G(0) = G(W_{gp}) = 0$, where $S(W) = \frac{r}{r+\lambda}(a+F'(W)h(a)+\frac{1}{2}F''(W)r\sigma^2Y^2)$ is a known function, and where W follows

$$dW_t = ((r+\lambda)(W_t-u(c))+rh(a))dt + rY\sigma dZ_t.$$

As before, we denote drift of W_t by μ and its volatility by ν.

The derivation of the equality in (11) follows DeMarzo and Sannikov (2006). Let

$$H_t := -\int_0^t e^{-(r+\lambda)s}S(W_s)ds + e^{-(r+\lambda)t}G(W_t).$$

We have

$$dH_t = -e^{-(r+\lambda)t}S(W_t)dt-(r+\lambda)e^{-(r+\lambda)t}G(W_t)dt+e^{-(r+\lambda)t}dG(W_t),$$

and thus, using Ito's lemma,

$$\begin{aligned} e^{(r+\lambda)t}dH_t &= -S(W_t)dt-(r+\lambda)G(W_t)dt+\left(G'(W_t)\mu(W_t)+\frac{1}{2}\nu(W_t)^2G''(W_t)\right)dt+ \\ &\quad G'(W_t)\nu(W_t)dZ_t. \end{aligned}$$

The dt terms sum up to zero by (13), and $\mathbb{E}[H_t]$ is bounded, i.e., H_t is a martingale. Thus,

$$G(W_0) = H_0 = \mathbb{E}[H_\tau] = \mathbb{E}\left[-\int_0^\tau e^{-(r+\lambda)t}S(W_t)dt + e^{-(r+\lambda)\tau}G(W_\tau)\right],$$

which, with the boundary conditions $G(W_\tau) = \frac{\partial F(W)}{\partial \lambda}\Big|_{W=0} = \frac{\partial F(W)}{\partial \lambda}\Big|_{W=W_{gp}} = 0$, which gives us the equality in (11).

To verify the boundary conditions at exit time, consider first $\frac{\partial F(W_{gp})}{\partial \lambda}$, where W_{gp} also depends on λ. Differentiating the value-matching condition $F(W_{gp}) = F_0(W_{gp})$ totally with respect to λ, we have

$$\frac{\partial F(W_{gp})}{\partial \lambda} + F'(W_{gp})\frac{\partial W_{gp}}{\partial \lambda} = F_0'(W_{gp})\frac{\partial W_{gp}}{\partial \lambda},$$

so,

$$\frac{\partial F(W_{gp})}{\partial \lambda} = (-F'(W_{gp}) + F_0'(W_{gp}))\frac{\partial W_{gp}}{\partial \lambda} = 0,$$

where the second inequality uses the smooth-pasting condition. The other boundary condition is easy to verify because $F(0) = 0$ at all λ, so obviously $\frac{\partial F(0)}{\partial \lambda} = 0$.

The strict inequality in (11), i.e., $G(W_0) < 0$, follows from $S > 0$ everywhere in $(0, W_{gp})$.

QED

Proof of Proposition 5

It is equivalent to prove that there exists a unique $W^s \in [0, W_{gp}]$ such that $F'(W) \geq \tilde{F}'(W)$ if and only if $W \leq W^s$.

0. Suppose $\tilde{W}_{gp} \leq W_{nj}$. We want to show that there does not exist $W^s < \tilde{W}_{gp}$ that solves $F'(W^s) = \tilde{F}'(W^s)$. Since $F'(0) > \tilde{F}'(0)$, the fact that $\tilde{W}_{gp} < W_{nj}$ implies $\tilde{F}'(\tilde{W}_{gp}) = F_0'(\tilde{W}_{gp}) < F'(\tilde{W}_{gp})$ following Proposition 2. So, generically, either there exists an even number of W^s solving $F'(W^s) = \tilde{F}'(W^s)$, or the solution does not exist.

1. Suppose there are at least two solutions, denoted as W^1 and W^2, where $W^1 < W^2 < \tilde{W}_{gp}$, such that $F'(W^j) = \tilde{F}'(W^j)$ for $j = 1, 2$. The fact that the signs of second derivative alternate, i.e., $F''(W^1) < \tilde{F}''(W^1)$ and $F''(W^2) > \tilde{F}''(W^2)$ implies

$$\begin{aligned}
(1 + \frac{\lambda}{r})S(W^1) &< (1 + \frac{\tilde{\lambda}}{r})\tilde{S}(W^1), \\
(1 + \frac{\lambda}{r})S(W^2) &> (1 + \frac{\tilde{\lambda}}{r})\tilde{S}(W^2).
\end{aligned}$$

Notice that by construction we have

$$F'(W) < \tilde{F}'(W) \text{ for } W \in \left(W^1, W^2\right).$$

And there exist $W^{1.5}$ such that $W^1 < W^{1.5} < W^2$ and

$$\begin{aligned}
F''(W^{1.5}) &= \tilde{F}''(W^{1.5}) \qquad (14)\\
F''(W) &< \tilde{F}''(W) \text{ for } W \in [W^1, W^{1.5}), \\
F''(W) &> \tilde{F}''(W) \text{ for } W \in \left(W^{1.5}, W^2\right).
\end{aligned}$$

The goal is to contradict the equality.

2. For any $W \in \left(W^{1.5}, W^2\right)$, the fact that $F'(W) < \tilde{F}'(W)$ and $W < W_{gp} < W_{nj}$ implies $\tilde{W}'(W) < W'(W) < W$. The derivative of $S(W)$ is given by

$$\begin{aligned} S'(W) &= -F''(W)(W - W'(W)), \\ &< -\tilde{F}''(W)(\tilde{W} - \tilde{W}'(W)), \\ &= \tilde{S}'(W). \end{aligned}$$

Then the premise that $(1+\frac{\lambda}{r})S\left(W^2\right) > (1+\frac{\tilde{\lambda}}{r})\tilde{S}\left(W^2\right)$ implies

$$\begin{aligned} &\Rightarrow \quad (1+\frac{\lambda}{r})\left(S\left(W^{1.5}\right) + \int_{W^{1.5}}^{W^2} S'(W)\,dW\right) > (1+\frac{\tilde{\lambda}}{r})\tilde{S}\left(W^2\right), \\ &\Rightarrow \quad (1+\frac{\lambda}{r})S\left(W^{1.5}\right) + (1+\frac{\tilde{\lambda}}{r})\int_{W^{1.5}}^{W^2} \tilde{S}'(W)\,dW > (1+\frac{\tilde{\lambda}}{r})\tilde{S}\left(W^2\right), \\ &\Rightarrow \quad (1+\frac{\lambda}{r})S\left(W^{1.5}\right) > (1+\frac{\tilde{\lambda}}{r})\tilde{S}\left(W^{1.5}\right), \end{aligned}$$

which contradicts (14). In this case Proposition 5 holds by setting $W^s = \tilde{W}_{gp}$.

3. [Existence of W^s and odd number of crossing points.] Suppose $\tilde{W}_{gp} \in (W_{nj}, W_{gp})$. From Proposition 4, we have that $\tilde{F}'(0) < F'(0)$. We also have $\tilde{F}'(\tilde{W}_{gp}) = \tilde{F}_0(\tilde{W}_{gp}) > F'(\tilde{W}_{gp})$, where the equality is the smooth-pasting condition and the inequality follows from Proposition 2 because $\tilde{W}_{gp} \in (W_{nj}, W_{gp})$. Thus, $\tilde{F}'$ and F' cross on $(0, \tilde{W}_{gp})$, and the number of crossing points $W^s \in (0, \tilde{W}_{gp})$, i.e., solutions to $\tilde{F}'(W^s) = F'(W^s)$, is odd.

4. Suppose there are at least three solutions, denoted as W^1, W^2, and W^3, where $W^1 < W^2 < W^3$, such that $F'(W^j) = \tilde{F}'(W^j)$ for $j = 1, 2, 3$. The fact that the signs of second derivative alternate, i.e. $F''(W^1) < \tilde{F}''(W^1)$, $F''(W^2) > \tilde{F}''(W^2)$ and $F''(W^3) < \tilde{F}''(W^3)$ implies

$$\begin{aligned} (1+\frac{\lambda}{r})S\left(W^1\right) &< (1+\frac{\tilde{\lambda}}{r})\tilde{S}\left(W^1\right), \\ (1+\frac{\lambda}{r})S\left(W^2\right) &> (1+\frac{\tilde{\lambda}}{r})\tilde{S}\left(W^2\right), \\ (1+\frac{\lambda}{r})S\left(W^3\right) &< (1+\frac{\tilde{\lambda}}{r})\tilde{S}\left(W^3\right). \end{aligned}$$

Notice that by construction we have

$$\begin{aligned} F'(W) &< \tilde{F}'(W) \text{ for } W \in \left(W^1, W^2\right), \\ F'(W) &> \tilde{F}'(W) \text{ for } W \in \left(W^2, W^3\right). \end{aligned}$$

And there exist W^a, W^b and W^c such that $W^1 < W^a < W^2 < W^b < W^c < W^3$ and

$$\begin{aligned} F''(W^a) &= \tilde{F}''(W^a) && (15) \\ F''(W^b) &= \tilde{F}''(W^b) && (16) \\ F''(W) &> \tilde{F}''(W) \text{ for } W \in (W^a, W^b), \\ F''(W) &< \tilde{F}''(W) \text{ for } W \in (W^b, W^c). \end{aligned}$$

The goal is to contradict the two equalities.

5. Consider two cases: $W^2 < W_{nj}$ and $W^2 \geq W_{nj}$. In the case of $W^2 < W_{nj}$, for any $W \in \left(W^1, W^2\right)$, the fact that $F'(W) < \tilde{F}'(W)$ implies $\tilde{W}'(W) < W'(W)$; the fact that $W^a < W < W^2 < W_{nj}$ implies $W'(W) < W$ and $F''(W) > \tilde{F}''(W)$. The derivative of $S(W)$ is given by

$$\begin{aligned} 0 &< S'(W) = -F''(W)\left(W - W'(W)\right), \\ &< -\tilde{F}''(W)\left(W - W'(W)\right), \\ &< -\tilde{F}''(W)\left(W - \tilde{W}'(W)\right), \\ &= \tilde{S}'(W). \end{aligned}$$

Then the premise that $(1+\frac{\lambda}{r})S\left(W^2\right) > (1+\frac{\tilde{\lambda}}{r})\tilde{S}\left(W^2\right)$ implies

$$\begin{aligned} & (1+\frac{\lambda}{r})\left(S(W^a) + \int_{W^a}^{W^2} S'(W)\,dW\right) > (1+\frac{\tilde{\lambda}}{r})\tilde{S}\left(W^2\right), \\ \Rightarrow\ & (1+\frac{\lambda}{r})S(W^a) + (1+\frac{\tilde{\lambda}}{r})\int_{W^a}^{W^2} \tilde{S}'(W)\,dW > (1+\frac{\tilde{\lambda}}{r})\tilde{S}(W_2), \\ \Rightarrow\ & (1+\frac{\lambda}{r})S(W^a) > (1+\frac{\tilde{\lambda}}{r})\tilde{S}(W^a), \\ \Rightarrow\ & \frac{(1+\frac{\lambda}{r})S(W^a) - a - F'(W^a)h(a)}{\frac{1}{2}r\sigma^2(h'(a))^2} > \frac{(1+\frac{\tilde{\lambda}}{r})\tilde{S}(W^a) - a - \tilde{F}'(W^a)h(a)}{\frac{1}{2}r\sigma^2(h'(a))^2}, \end{aligned}$$

which contradicts (15).

6. In the second case $W^2 \geq W_{nj}$, for any $W \in \left(W^2, W^3\right)$, the fact that $F'(W) > \tilde{F}'(W)$ implies $\tilde{W}'(W) > W'(W) > W$. Then the premise that $F''\left(W^b\right) = \tilde{F}''\left(W^b\right)$ implies $S'\left(W^b\right) > \tilde{S}'\left(W^b\right)$, thus there exists $\varepsilon > 0$ such that $W^b + \varepsilon \leq W^c$ and $0 > S'(W) > \tilde{S}'(W)$ for all $W \in [W^b, W^b + \varepsilon)$. The fact that $F''\left(W^b\right) = \tilde{F}''\left(W^b\right)$ implies

$$\begin{aligned} & \frac{(1+\frac{\lambda}{r})S\left(W^b\right) - a - F'(W^b)h(a)}{\frac{1}{2}r\sigma^2(h'(a))^2} = \frac{(1+\frac{\tilde{\lambda}}{r})\tilde{S}\left(W^b\right) - a - \tilde{F}'(W^b)h(a)}{\frac{1}{2}r\sigma^2(h'(a))^2}, \\ \Rightarrow\ & \frac{(1+\frac{\lambda}{r})S\left(W^b+\varepsilon\right) - a - h(a)F'(W^b+\varepsilon)}{\frac{1}{2}r\sigma^2(h'(a))^2} > \frac{(1+\frac{\tilde{\lambda}}{r})\tilde{S}\left(W^b+\varepsilon\right) - a - h(a)\tilde{F}'(W^b+\varepsilon)}{\frac{1}{2}r\sigma^2(h'(a))^2}, \\ \Rightarrow\ & F''(W^b+\varepsilon) > \tilde{F}''(W^b+\varepsilon), \end{aligned}$$

which contradicts the premise that $F''(W) < \tilde{F}''(W)$ for $W \in \left(W^b, W^c\right)$.

QED

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