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Abstract

We construct a continuous-time, New-Monetarist economy with general preferences that displays an endogenous, non-degenerate distribution of money holdings. Properties of equilibria are obtained analytically and equilibria are solved in closed form in a variety of cases. We study policy as incentive-compatible transfers financed with money creation. Lump-sum transfers are welfare-enhancing when labor productivity is low, but regressive transfers achieve higher welfare when labor productivity is high. We introduce illiquid government bonds and draw implications for the existence of liquidity-trap equilibria and policy mix in terms of "helicopter drops" and open-market operations.

JEL Classification: E40, E50
Keywords: money, inflation, risk sharing, liquidity traps.

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1 Introduction

We analyze a continuous-time New Monetarist economy, based on the competitive version of the Lagos and Wright (2005) model developed by Rocheteau and Wright (2005), LRW thereafter.[1] As in LRW, agents use a medium of exchange to finance random consumption opportunities, and make endogenous labor supply decisions. In contrast to LRW, but similar to Lucas (1980) and Bewley (1980, 1983), preferences are general and allow for wealth effects, leading to a continuous distribution of money holdings. Our first contribution is methodological: we show that the model remains tractable despite the unharnessed ex-post heterogeneity in money holdings. We characterize the properties of equilibria, including policy functions, value functions, and distributions, and we solve the model in closed form in a variety of cases. Our second contribution is to study policy in the form of incentive-compatible transfer schemes financed with money creation. We show analytically that these schemes can be designed to raise welfare, and in some cases approach the first best, by trading off the need to promote self-insurance and the need to provide risk sharing, thereby proving the Wallace (2014) conjecture. Our third contribution consists in adding illiquid government bonds as a policy instrument. We draw implications for the existence of liquidity traps and welfare-enhancing policies combining "helicopter drops" of money and open-market operations.

While we study a version of our model with money and bonds at the end of the paper, we focus most of our attention to a pure currency economy as it is the most transparent benchmark in which to study the monetary policy trade-off between self-insurance and risk-sharing. In our model, ex-ante identical households, who enjoy consumption and leisure flows, have the possibility to trade continuously in competitive spot markets.[2] At some random times households receive idiosyncratic preference shocks that generate utility for lumps of consumption. These represent large shocks that cannot be paid for by a contemporaneous income flow, such as health shocks, large housing repairs, durable goods expenditures, and so on. Following Kocherlakota (1998), lack

[1] The market structure with sequential competitive markets is analogous to the one in Rocheteau and Wright (2005) but it could be readily reinterpreted as one where households meet sellers bilaterally and at random and terms of trades are determined by bargaining. See our companion paper, Rocheteau, Weill, and Wong (2015), for such a reinterpretation.

[2] The assumption of continuous time has several advantages. First, the distribution of real balances obeys a smooth, infinitesimal law of motion known as a Kolmogorov forward equation. As a result, the distribution admits a smooth density, without spikes, except maybe at one boundary. Second, continuous time provides a sharp representation of the mismatch between flow endowments and lumpy spending that generates a role for liquidity. Third, under continuous time ex-post heterogeneity is generic even under the commonly-used quasi-linear preferences. Last, our methodology can be used to write Lagos-Wright economies in continuous time, which can be useful to integrate them with other continuous-time models, such as models of unemployment, models of price distribution, or menu cost models.

of enforcement and anonymity prevent households from borrowing to finance these spending shocks, thereby creating a role for liquidity. Because shocks are independent across households, the model generates heterogeneous individual histories and hence, possibly, heterogeneous holdings of money.

We provide a detailed characterization of the household's consumption and saving problem under a minimal set of assumptions on preferences. We show that households have a target for their real balances, which depends on their rate of time preference, the inflation rate, and the frequency of consumption opportunities. They approach this target gradually over time by saving a decreasing fraction of their labor income flow. When they are hit by a preference shock for lumpy consumption, households deplete their money holdings in full, if their wealth is below a threshold, or partially otherwise. Given the households' optimal consumption-saving behavior, we can characterize the stationary distribution of real money holdings in the population, and we solve for the value of money, thereby establishing the existence of an equilibrium. Under zero money growth ("laissez-faire"), the steady-state monetary equilibrium is unique, and it approaches the first best when households are patient. We study in detail the special case where households have linear preferences over consumption and labor flows. This version showcases the tractability of the model as the equilibrium can be characterized in closed form, and it admits as a limit, when labor productivity grows large, the New-Monetarist model of LRW.

We study monetary policy in the form of stationary transfer schemes financed by money creation. Incentive compatibility restricts transfers to be non-negative (because of lack of enforcement) and non-decreasing (because agents can hide their money) in an agent's real balances. We emphasize labor supply effects through which money creation affects output and welfare. We isolate a single parameter, labor productivity, that determines the speed at which households insure themselves against preference shocks, and that parametrizes the policy trade-off between providing risk sharing and promoting self-insurance. We find that, if we restrict transfers to be lump-sum, e.g., as in Kehoe, Levine, and Woodford (1992), then the optimal inflation rate is positive when labor productivity is low. For high labor productivity, however, positive inflation is suboptimal if the government is restricted to lump sum transfers. In that case, we prove that positive inflation raises welfare when engineered by a more general transfer scheme, in accordance with the Wallace (2014) conjecture. This scheme prescribes flat transfers for low levels of wealth, so as to provide risk sharing, and transfers that increase linearly with real balances for high levels of wealth in order to neutralize the disincentive effects of the inflation tax. We show that such an inflationary transfer scheme provides insurance but also raises aggregate output and welfare. We go beyond this conjecture and characterize transfer schemes that generate allocations arbitrarily close to the first best. For high labor productivity the optimal transfer as a function of real balances corresponds to a step function. For low labor productivity, the optimal transfer is lump sum.

In the pure currency economy the policymaker has a single instrument, fiat money, to address the trade-off between risk sharing and self insurance, hence the need of nonlinear transfers. We extend our pure currency by adding another asset, illiquid nominal government bonds, that can bear interest and hence promote self insurance in a similar, but perhaps more realistic way, than non-linear transfers. In equilibrium the poorest households hold money only, which creates an endogenous segmentation of asset markets – only households with sufficient wealth participate in the bonds market. We show an equivalence between liquidity-trap equilibria where the nominal interest rate is zero and equilibria of the pure-currency economy where a fraction of households do not deplete their real balances following a preference shock. We use this equivalence to draw some implications for the existence of liquidity-trap equilibria and for policy. Liquidity-trap equilibria occur where labor productivity is low, idiosyncratic risk is high, and bonds are scarce. Households have a high precautionary demand for assets because the pace of wealth accumulation is low. If the bond supply is low, the bond yield is driven to zero, so that households are indifferent between holding money and bonds. A combination of higher anticipated inflation through higher money growth and a lower bond-money ratio through open-market operations is welfare improving.

We also provide examples where interest-bearing illiquid bonds are essential in that they can raise welfare relative to the pure currency economy with lump-sum transfers. The optimal supply of bonds is chosen such that the nominal interest rate is positive and it is combined with lump sum transfers financed with money creation. As labor productivity falls it is optimal to raise both inflation and the nominal interest rate.

Literature

Our model can be viewed as a continuous-time, competitive version of Lagos and Wright (2005) with general preferences. Despite the presence of uninsurable idiosyncratic risk, the Lagos-Wright model delivers equilibria with degenerate distributions of money holdings which can be solved in closed form and can easily be integrated with the standard representative-agent model used in macroeconomics.[3] Yet, this gain in tractability comes at a cost: in the absence of ex-post

[3] For a recent review of the literature, see Lagos, Rocheteau, and Wright (2016).

heterogeneity, monetary policy is exclusively about enhancing the rate of return of currency, thereby making the Friedman rule omnipotent. Formulating tractable search-theoretic monetary models without restrictions on money holdings, and with non-degenerate distributions, has been considered challenging due to the interaction between bargaining and ex-post heterogeneity. Examples of such models include Camera and Corbae (1999), Zhu (2005), Molico (2006), and Chiu and Molico (2010, 2011), all in discrete time. While our preference shocks for lumpy consumption are reminiscent to random matching shocks in search models, we avoid the intricacies due to bargaining by assuming competitive prices.[4] Green and Zhou (1998, 2002) and Zhou (1999) assume price posting, undirected search, and indivisible goods, which leads to a continuum of steady states. In contrast, the laissez-faire monetary equilibrium of our model is unique. Menzio, Shi, and Sun (2013) assume directed search and free-entry of firms and characterize the monetary steady state under general preferences and a constant money supply. They only briefly discuss how one would solve their model with money growth, and they do not analyze the resulting policy trade-off, which is the main focus of our analysis. See also Sun and Zhou (2016) with an exogenous upper-bound on real balances.

Our approach is also closely related to incomplete market models where households self-insure against idiosyncratic income risk by accumulating assets: fiat money in Bewley (1980, 1983) and Lucas (1980), physical capital in Aiyagari (1994), and private IOUs in Huggett (1993).[5] We contribute to this literature by analyzing a tractable continuous-time model with a type of idiosyncratic risk that is reminiscent to the one in random matching models and with non-trivial labor supply decisions.[6] While incomplete markets are most often solved by way of numerical methods, a few papers have developed analytically tractable frameworks. In particular Scheinkman and Weiss (1986); Algan, Challe, and Ragot (2011); and Lippi, Ragni, and Trachter (2015) study Bewley economies with quasi-linear preferences, with a special attention to logarithmic preferences for consumption. Our model differs in a number of ways allowing for a comprehensive study of the welfare and output effects of inflation and the policy trade-off of monetary policy.[7] Amongst Bewley models who work

[4] Rocheteau, Weill, and Wong (2015) study a discrete-time version of the model with search and bargaining and alternating market structures. The model remains tractable and can be used to study transitional dynamics following one-time money injections.

[5] See Ljungqvist and Sargent (2004, chapters 16-17) and Heathcote, Storesletten, and Violante (2009) for surveys.

[6] In contrast to the seminal paper by Lucas (1980), we do not assume a cash-in-advance constraint (i.e., his condition (1.4)) since agents can finance their flow consumption with their current labor income, and we rule out credit arrangements from first principles. The nature of the idiosyncratic risk is also different: it takes the form of random arrivals of opportunities to consume lumpy amounts of consumption, which is analogous to the idiosyncratic liquidity risk in continuous-time random matching models. Also in contrast to Lucas (1980), we allow for endogenous labor supply we study money growth under various schemes and we can compute classes of equilibria in closed form.

[7] We characterize our model under general concave preferences and consider quasi-linear preferences only as a special case. Even for this special case, our model differs in important ways from the Scheinkman-Weiss model: risk

with numerical methods, Imrohoroglu (1992) and Dressler (2011) have studied the welfare cost of inflation.

Our work also contributes to a recent literature developing continuous time methods to analyze general equilibrium models with incomplete markets. Recently, Achdou, Han, Lasry, Lions, and Moll (2015) have proposed numerical tools based on mean-field-games techniques to study a wide class of heterogeneous-agent models in continuous time, with Huggett (1993) as their baseline. Our idiosyncratic lumpy consumption opportunities are similar to the uncertain lumpy expenditures in the Baumol-Tobin model of Alvarez and Lippi (2013). Our model of Section 5 with money and bonds is closely related with the following differences: we assume no cost to liquidate assets, and we do not take the consumption path (both in terms of flows and jump sizes) as exogenous; neither do we assume that labor income is exogenous.

2 The environment

Time, $t \in \mathbb{R}_+$, is continuous and goes on forever. The economy is populated with a unit measure of infinitely-lived households who discount the future at rate $r > 0$. There is a single perishable consumption good produced according to a linear technology that transforms h units of labor into h units of output. Households have a finite endowment of labor per unit of time, $\bar{h} < \infty$. Alternatively, one can normalize labor endowment to one and interpret $\bar{h}$ as labor productivity.

Households value consumption, c, and leisure flows, ℓ, according to an increasing and concave instantaneous utility function, $u(c, \ell)$. We assume that both consumption and leisure are normal goods, that $u(c, \ell)$ is bounded above, i.e. $\sup_{c \geq 0} u(c, \bar{h}) \equiv \|u\| < \infty$, and bounded below so that we can normalize $u(0, 0) = 0$. In addition to consuming and producing in flows, households receive preference shocks that generate lumps of utility for the consumption of discrete quantities of the good. Lumpy consumption opportunities represent large shocks (e.g., replacement of durables, health events and expenditures due to changes in family composition) that require immediate spending.[8] These shocks occur at Poisson arrival times, $\{T_n\}_{n=1}^{\infty}$, with intensity α. The utility of consuming y units of goods at time T_n is given by an increasing, concave, and bounded utility function, $U(y)$, and we normalize $U(0) = 0$.[9] Taken together, the lifetime expected utility of a

is idiosyncratic and arises from lumpy consumption opportunities, instead of an aggregate risk on agents' ability to work, and we impose a bound, $\bar{h}$, on flow labor supply that plays a key role for our normative analysis.

[8] One could also interpret the preference shocks as random consumption opportunities in a decentralized goods market with search-and-matching frictions. For such an interpretation, see Rocheteau, Weill, and Wong (2015).

[9] If we think of the shock as the replacement of durables, then $U(y) = \vartheta(y)/(r + \delta)$ is the discounted sum of the utility flows, $\vartheta(y)$, provided by a durable good, where δ is the Poisson arrival rate at which a particular durable

household can be written as:

$$\mathbb{E}\left[\int_0^{+\infty} e^{-rt} u\left(c_t, \bar{h} - h_t\right) dt + \sum_{n=1}^{\infty} e^{-rT_n} U\left(y_{T_n}\right)\right], \tag{1}$$

given some adapted and left-continuous processes for c_t, h_t, and y_t. We impose the following additional regularity conditions on households' utility functions. First, $U(y)$ is strictly increasing, strictly concave, and twice continuously differentiable; it also satisfies the Inada condition $U'(0) = +\infty$. Second, $u(c, \ell)$ can have either one of the following two specifications:

1. **Smooth-Inada (SI) preferences:** $u(c, \ell)$ is strictly concave, and twice continuously differentiable, and it satisfies Inada conditions with respect to both arguments, i.e., $u_c(0, \ell) = \infty$ and $u_c(\infty, \ell) = 0$ for all $\ell > 0$, $u_\ell(c, 0) = \infty$ for all $c > 0$;

2. **Linear preferences:** $u(c, \ell) = \min\{c, \bar{c}\} + \ell$, for some $\bar{c} \geq 0$.

The first specification facilitates the analysis because it implies smooth policy functions for households and strictly positive consumption and labor flows. The second specification is useful to establish close contacts with the literature as it corresponds to the quasi-linear preferences commonly used in monetary theory since Lagos and Wright (2005) to eliminate wealth effects and obtain equilibria with degenerate distributions of money balances.[10] In our model, distributions are *non*-degenerate even under quasi-linear preferences, because the feasibility constraint on labor, $h \leq \bar{h}$, can be binding for some agents in equilibrium.

In order to make money essential we assume that households cannot commit and there is no monitoring technology (Kocherlakota, 1998). As a result households cannot borrow to finance lumpy consumption since otherwise they would default on their debt. The only asset in the economy is fiat money: a perfectly recognizable, durable and intrinsically worthless object. The supply of money, denoted M_t, grows at a constant rate, $\pi \geq 0$, through lump-sum transfers to households. (We consider alternative transfer schemes in Section 4.2.) Trades of money and goods take place in spot competitive markets. The price of money in terms of goods is denoted ϕ_t.

For the purpose of studying policy and welfare, our first-best benchmark is the full-insurance allocation. Under SI preference, it is the time invariant allocation (c^{FI}, h^{FI}, y^{FI}) solving

$$u_c\left(c^{FI}, \bar{h} - h^{FI}\right) = u_\ell\left(c^{FI}, \bar{h} - h^{FI}\right) = U'\left(y^{FI}\right). \tag{2}$$

expires, and y is the quality of the durable.

[10] Lagos and Wright (2005) assume quasi-linear preferences of the form $u(c) + \ell$. See also Scheinkman and Weiss (1986) for similar preferences. The fully linear specification comes from Lagos and Rocheteau (2005). One can achieve the same amount of tractability with the larger class of preferences studied in Wong (2016), including constant return to scale, constant elasticity of substitution, and CARA.

Households equalize the marginal utilities of flow consumption, of leisure, and of lumpy consumption. Under linear preferences, flow labor and consumption can be at corners, so that

$$h^{FI} = \alpha y^{FI} = \min\{\alpha y^{\star}, \bar{h}\}, \tag{3}$$

and $c^{FI} = 0$, where $y^{\star}$ is the quantity that equalizes the marginal utility of lumpy consumption and the marginal disutility of work, $U'(y^{\star}) = 1$. If labor endowments are sufficiently large, then the first-best allocation is such that households consume $y^{\star}$ whenever they receive a preference shock and they supply $\alpha y^{\star}$ of their labor endowment. If the endowments are small, $\bar{h} < \alpha y^{\star}$, then $y^{\star}$ is not feasible, so households supply their whole labor endowment, $\bar{h}$, and share the output equally among the α households with a desire to consume.

3 Stationary monetary equilibrium

In this section we study stationary monetary equilibria where aggregate real balances, $\phi_t M_t$, are constant. It implies the rate of return on money, $\dot{\phi}_t/\phi_t$, is constant and equal to the negative of the inflation rate, i.e., $\phi_t = \phi_0 e^{-\pi t}$. In order to determine the equilibrium time-zero value of money, ϕ_0, we proceed as follows. First, we obtain from the government budget constraint that the real value of the lump sum transfer received by households is $\phi_t \dot{M}_t = \pi \phi_t M_t = \pi \phi_0 M_0$. In Section 3.1 we take this transfer as given and solve the household's consumption-saving problem and determine its choice of real balances. In Section 3.2, we characterize the distribution of real money balances as a function of the lump-sum transfer, $\pi \phi_0 M_0$. In Section 3.3, we use market clearing to establish the existence of ϕ_0.

3.1 The household's problem

We analyze the household's problem given any constant inflation rate, $\pi \geq 0$, and given any real lump sum transfer $\Upsilon = \pi \phi_0 M_0$. Let $W(z)$ denote the maximum attainable lifetime utility of a household holding z units of real balances. In our supplementary appendix we establish that $W(z)$ is a solution to the Bellman equation:

$$W(z) = \sup \int_0^{\infty} e^{-(r+\alpha)t} \Big(u\left(c_t, \bar{h} - h_t\right) + \alpha \big[U\left(y_t\right) + W\left(z_t - y_t\right) \big] \Big) \, dt, \tag{4}$$

with respect to left-continuous plans for $\{c_t, h_t, y_t\}$, a piecewise continuously differentiable plan for z_t, and subject to:

$$z_0 = z \tag{5}$$

$$0 \leq y_t \leq z_t \tag{6}$$

$$\dot{z}_t = h_t - c_t - \pi z_t + \Upsilon. \tag{7}$$

The effective discount factor, $e^{-(r+\alpha)t}$, in the household's objective, (4), is equal to the time discount factor, e^{-rt}, multiplied by the probability that no preference shock occurs during the time interval $[0,t)$, i.e., $\Pr(T_1 \geq t) = e^{-\alpha t}$. This effective discount factor multiplies the household's expected period utility at time t, conditional on $T_1 \geq t$. The first term of the period utility is the utility flow of consumption and leisure, $u\left(c_t, \bar{h} - h_t\right)$. The second term is the expected utility associated with a preference shock at time t, an event occurring with Poisson intensity α. This expected utility is the sum of $U(y_t)$ from consuming a lump of y_t units of consumption good and the continuation utility $W(z_t - y_t)$ from keeping $z_t - y_t$ real balances.

Equation (5) is the initial condition for real balances, and (6) is a feasibility constraint stating that real balances must remain positive before and after a preference shock. In particular, the constraint that $y_t \leq z_t$ follows from the absence of enforcement and monitoring technologies that prevent households from issuing debt. Finally, (7) is the law of motion for real balances. The rate of change in real balances is equal to the household's output flow net of consumption, $h_t - c_t$, plus the negative flow return on currency, $-\pi z$, and a flow lump-sum transfer of real balances, Υ.

Theorem 1 *For given Υ, Equation (4) has a unique bounded solution, $W(z)$. It is strictly increasing, strictly concave and continuously differentiable over $[0,\infty)$. It is twice continuously differentiable over $(0,\infty)$, except perhaps under linear preference, when this property may fail for at most two points. Moreover,*

$$W'(0) \leq \frac{r+\alpha}{\bar{h}}\left(\frac{\|u\|}{r} + \alpha\frac{\|U\|}{r}\right), \quad \lim_{z\to 0} W''(z) = -\infty, \ \textit{and} \ \lim_{z\to\infty} W'(z) = 0.$$

Finally, W solves the Hamilton-Jacobi-Bellman (HJB) equation:

$$rW(z) = \max\left\{u(c, \bar{h} - h) + \alpha\left[U(y) + W(z-y) - W(z)\right] + W'(z)\left(h - c - \pi z + \Upsilon\right)\right\}, \tag{8}$$

with respect to (c,h,y) and subject to $c \geq 0$, $0 \leq h \leq \bar{h}$ and $0 \leq y \leq z$.

Establishing that the value function is well behaved is important because it will later allow us to apply standard theorems in order to establish the existence of a unique stationary distribution of real balances, and to show that the mean of the distribution, ϕM, is continuous in Υ, which facilitates the proof of existence of an equilibrium. A perhaps surprising result is that $W'(0) < \infty$ even though $U'(0) = \infty$. Intuitively, a household with depleted money balances, $z = 0$, has a finite marginal utility for real balances because it has some positive time to accumulate real balances before the next opportunity for lumpy consumption, $\mathbb{E}[T_1] = 1/\alpha > 0$.[11]

The HJB equation, (8), has a standard interpretation as an asset-pricing condition. If we think of $W(z)$ as the price of an asset, the opportunity cost of holding that asset is $rW(z)$. The asset yields a utility flow, $u(c, \ell)$, and a capital gain, $U(y) + W(z-y) - W(z)$, in the event of a preference shock with Poisson arrival rate α. Finally, the value of the asset changes over time due to the accumulation of real balances, represented by the last term on the right side of (8), $W'(z_t)\dot{z}_t$.

Optimal lumpy consumption. From (8) a household chooses its optimal lumpy consumption in order to solve:

$$V(z) = \max_{0 \le y \le z} \{U(y) + W(z-y)\}. \tag{9}$$

In words, a household chooses its level of consumption in order to maximize the sum of its current utility, $U(y)$, and its continuation utility with $z - y$ real balances, $W(z-y)$. Because $U'(0) = \infty$ but $W'(0) < \infty$, a household always finds it optimal to choose strictly positive lumpy consumption, $y(z) > 0$, for all $z > 0$. Hence, the first-order condition of (9) is

$$U'(y) \ge W'(z-y), \tag{10}$$

with an equality if $y < z$. The following proposition provides a detailed characterization of the solution to (10).

Proposition 1 ***(Optimal Lumpy Consumption)*** *The unique solution to (10), $y(z)$, admits the following properties:*

1. $y(z)$ is continuous and strictly positive for any $z > 0$.

[11] The main technical challenge in Theorem 1 is to establish that $W(z)$ admits continuous derivatives of sufficiently high order. One approach would have been to find an explicit solution of the HJB equation and apply a sufficiency argument to show that this solution is equal to the value function of the household. Unfortunately, there are no explicit solutions in general, so we go the other way. That is, we show that the value function of the household necessarily solves a generalized HJB equation, using arguments from the theory of viscosity solutions (see, e.g., Bardi and Capuzzo-Dolcetta, 1997). Based on this generalized HJB, we are able to establish the desired smoothness properties of the value function.

2. *Both* $y(z)$ *and* $z - y(z)$ *are increasing and satisfy* $\lim_{z\to\infty} y(z) = \lim_{z\to\infty} z - y(z) = \infty$.

3. $y(z) = z$ *if and only if* $z \leq \bar{z}_1$, *where* $\bar{z}_1 > 0$ *solves* $U'(\bar{z}_1) = W'(0)$.

Finally, $V(z)$*, is strictly increasing, strictly concave, and continuously differentiable over* $(0, \infty)$ *with* $V'(z) = U'[y(z)]$.

Proposition 1 shows that, as long as real balances are below some threshold $\bar{z}_1$, the household finds it optimal to deplete its real balances in full upon receiving a preference shock. This follows because the utility derived from spending a small amount of real balances, $U'(0) = \infty$, is larger than the benefit from holding onto it, $W'(0) < \infty$. This result—the fact that liquidity constraints bind over a nonempty interval of the support of the wealth distribution—is in contrast with the standard incomplete-market model in continuous time where liquidity constraints never bind in the interior of the state space (Achdou, Han, Lasry, Lions, and Moll, 2015), and it will play a key role for the tractability of our model.

By induction we can construct a sequence of thresholds for real balances, $\{\bar{z}_n\}_{n=1}^{+\infty}$, such that:

$$\begin{aligned} z &\in [0, \bar{z}_1) \Longrightarrow z - y(z) = 0 \\ z &\in [\bar{z}_n, \bar{z}_{n+1}) \Longrightarrow z - y(z) \in [\bar{z}_{n-1}, \bar{z}_n), \ \forall n \geq 1. \end{aligned}$$

If a household's real balances belong to the interval $[\bar{z}_n, \bar{z}_{n+1})$, the post-trade real balances of the household following a preference shock, $z - y(z)$, belong to the adjacent interval, $[\bar{z}_{n-1}, \bar{z}_n)$. Hence, the household is insured against n consecutive preference shocks, i.e., it would take n shocks to deplete the real balances of the household. The properties of lumpy consumption, $y(z)$, and post-trade real balances, $z - y(z)$, are illustrated in Figure 1.

Optimal saving. Next, we characterize a household's optimal saving behavior. We first define the saving correspondence:

$$s(z) \equiv \left\{ h - c - \pi z + \Upsilon \,:\, (h, c) \text{ solves } (8) \right\}.$$

Proposition 2 ***(Optimal Saving Correspondence)*** *The saving correspondence,* $s(z)$*, is upper hemi-continuous, convex-valued, decreasing, strictly positive near* $z = 0$*, and admits a unique* $z^\star \in (0, \infty)$ *such that* $0 \in s(z^\star)$.

1. ***SI preferences.*** *The saving correspondence is singled-valued, strictly decreasing, and continuously differentiable over* $(0, \infty)$.

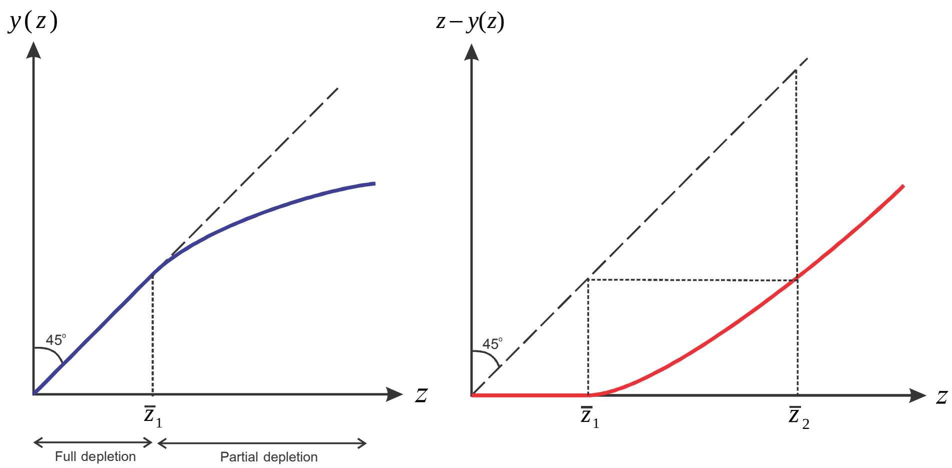

Figure 1: Left panel: Lumpy consumption. Right panel: Post-trade real balances.

2. ***Linear preferences.*** *The saving correspondence is equal to:*

$$s(z) = \left\{ \begin{array}{c} \bar{h} - \pi z + \Upsilon \\ [-\bar{c} - \pi z + \Upsilon, \bar{h} - \pi z + \Upsilon] \\ -\bar{c} - \pi z + \Upsilon \end{array} \right. \quad \textit{if} \quad W'(z) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right. 1. \tag{11}$$

The first part of Proposition 2 highlights three general properties of households' saving behavior. The first states that households save less when they hold larger real balances. The second property of $s(z)$ is that it is strictly positive near zero. The third property is that households have a target, $z^\star < \infty$, for their real balances.

The second part of Proposition 2 provides a tighter characterization of $s(z)$ under our two preference specifications. Under SI preferences, the HJB equation, (8), defines a strictly concave optimization problem leading to a smooth and strictly decreasing saving correspondence. Indeed, the first-order conditions for consumption and leisure are

$$u_c(c, \ell) = u_\ell(c, \ell) = W'(z). \tag{12}$$

Given that flow consumption and leisure are assumed to be normal goods, it follows that c and ℓ increase with z and so decrease with the marginal value of real balances. Under linear preferences (8) defines a linear optimization problem delivering a bang-bang solution for s. Households work maximally, $h = \bar{h}$, and consume nothing, $c = 0$, when real balances are low enough and $W'(z) > 1$. They stop working, $h = 0$, while consuming maximally, $c = \bar{c}$, when real balances are large enough and $W'(z) < 1$.

Next, we study the time path of a household's real balances conditional on not receiving a preference shock, namely, the solution to the initial value problem

$$\dot{z}_t = s(z_t) \text{ with } z_0 = 0. \tag{13}$$

Under linear preferences this problem is well defined, in the sense that $s(z)$ is single-valued, for all z except when $W'(z) = 1$ because there are multiple optimal values for s. In this case, we choose the s that is closest to zero. As a result, if $z^\star$ is such that $W'(z^\star) = 1$, this ensures that real balances remain constant and equal to their stationary point.[12] Given the unique solution to (13), we can define the time to reach z from $z_0 = 0$:

$$\mathcal{T}(z) \equiv \inf\{t \geq 0 \,:\, z_t \geq z |\, z_0 = 0\}. \tag{14}$$

Proposition 3 ***(Optimal Path of Real Balances)*** *The initial value problem (13) has a unique solution. This solution is strictly increasing for $t \in [0, \mathcal{T}(z^\star))$, where $z_{\mathcal{T}(z^\star)} = z^\star$, and it is constant and equal to $z^\star$ for all $t \geq \mathcal{T}(z^\star)$. Under SI preferences, $\mathcal{T}(z^\star) = \infty$. Under linear preferences, $\mathcal{T}(z^\star) < \infty$ if and only if $0 \in (-\bar{c} - \pi z^\star + \Upsilon, \bar{h} - \pi z^\star + \Upsilon)$.*

Proposition 3 shows that a household accumulates real money balances until it reaches its target $z^\star$. Under SI preferences real balances reach their target asymptotically at an exponential speed dictated by $|s'(z^\star)|$. Under linear preferences $s(z)$ may fail to be continuously differentiable at $z^\star$ and, as a result, the target may be reached in finite time. For instance, in the laissez-faire economy where $\pi = \Upsilon = 0$, $s(z)$ jumps downward at the target $z^\star$, i.e., $s(z) = \bar{h} > 0$ for all $z < z^\star$, while $s(z^\star) = 0$. Clearly, this implies that the target is reached in finite time $\mathcal{T}(z^\star) = z^\star/\bar{h} < +\infty$. In Figure 2 we illustrate the path for real balances and the spending behavior of a household subject to random preference shocks.

3.2 The stationary distribution of real balances

We now show that the household's policy functions, $y(z)$ and $s(z)$, induce a unique stationary distribution of real balances over the support $[0, z^\star]$. To this end, we define the minimal time that it takes for a household with z real balances at the time of a preference shock to accumulate strictly more than z' real balances following that shock:

$$\Delta(z, z') \equiv \max\left\{\mathcal{T}(z'_+) - \mathcal{T}\left[z - y(z)\right], 0\right\}, \tag{15}$$

[12] Under SI preferences a technical difficulty arises because $s(z)$ is not continuously differentiable at $z = 0$, and hence the standard existence and uniqueness theorems for ODEs do not apply. One can nevertheless construct the unique solution of (13) by starting the ODE at some $z > 0$ and "run it backward" until it reaches zero.

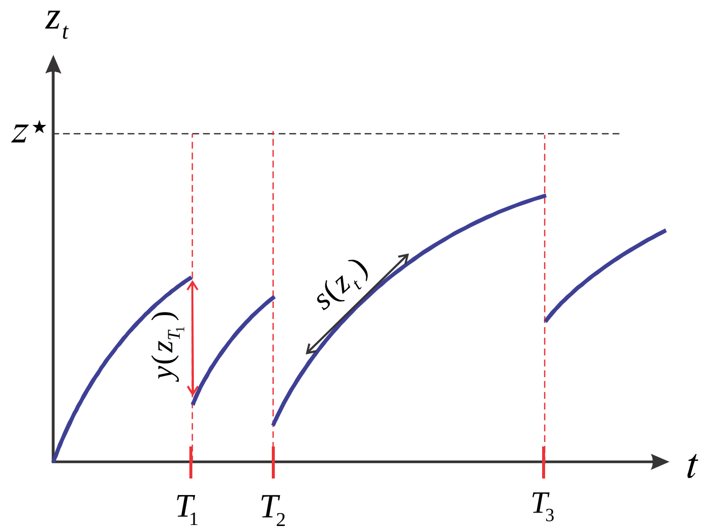

Figure 2: Optimal path of real balances

for $z, z' \in [0, z^\star]$. Notice that $\Delta(z, z^\star) = \infty$ since the household never accumulates more than the target. Let $F(z)$ denote the cumulative distribution function of a candidate stationary equilibrium. For given Υ it must solve the fixed-point equation:

$$1 - F(z') = \int_0^\infty \alpha e^{-\alpha u} \int_0^\infty \mathbb{I}_{\{u \geq \Delta(z,z')\}} dF(z)\, du = \int_0^\infty e^{-\alpha\Delta(z,z')}\, dF(z), \tag{16}$$

where the second equality is obtained by changing the order of integration. The right side of (16) calculates the measure of households with real balances strictly greater than z'. First, it partitions the population into cohorts indexed by the date of their last preference shock. There is a density measure, $\alpha e^{-\alpha u}$, of households who had their last preference shocks u periods ago. Second, in each cohort there is a fraction $dF(z)$ of households who held z real balances immediately before the shock. Those households consumed $y(z)$, which left them with $z - y(z)$ real balances. If $u \geq \Delta(z, z')$, then sufficient time has elapsed since the preference shock for their current holdings to be strictly greater than z'. A key observation is that the fixed-point problem in (16) is equivalent to the problem of finding a stationary distribution for the discrete-time Markov process with transition probability function:

$$Q(z, [0, z']) \equiv 1 - e^{-\alpha\Delta(z,z')}. \tag{17}$$

The function Q is the transition probability of the discrete-time Markov process that samples the real balances of a given household at the arrival times $\{T_n\}_{n=1}^\infty$ of its preference shocks. This observation allows us to apply standard results for the existence and uniqueness of stationary distributions of discrete-time Markov processes.[13] We obtain:

Proposition 4 ***(Stationary Distribution of Real Balances)*** *For given Υ the fixed point problem, (16), admits a unique solution, $F(z)$. This solution is continuous in the lump-sum transfer parameter, Υ, in the sense of weak-convergence.*

In addition to obtaining existence and uniqueness of a stationary distribution, Proposition 4 shows that F is continuous in Υ because all policy functions are appropriately continuous in that parameter. This continuity property is helpful to establish the existence of a steady-state equilibrium as it ensures that the market-clearing condition is continuous in the price of money, ϕ.

3.3 The real value of money

Equating the aggregate supply of real balances, $\phi_0 M_0$, with the aggregate demand of real balances as measured by the mean of the distribution F, we obtain the market-clearing condition

$$\phi_0 M_0 = \int_0^\infty z \, dF(z \,|\, \pi\phi_0 M_0), \tag{18}$$

where the right side makes it explicit that the stationary distribution depends on ϕ_0 via the the lump-sum transfer, $\Upsilon = \pi\phi_0 M_0$, since the household's path for real balances, z_t, depends on Υ. From (18) money is neutral at a steady state in the sense that aggregate real balances are determined independently of M_0.[14] As is standard, however, a change in the money growth rate will have real effects by affecting the rate of return on household savings. We now define an equilibrium:

Definition 1 *A stationary monetary equilibrium is composed of: a value function, $W(z \,|\pi\phi_0 M_0\,)$, and associated policy functions, that solve the household's optimization problem (4); a distribution of real balances, $F(z \,|\, \pi\phi_0 M_0)$, that solves the condition for stationarity (16); a price, $\phi_0 > 0$, that solves the market-clearing condition (18).*

In the definition, our notations emphasize that the value function and the distribution of real balance depend on ϕ via the real lump sum transfer, $\pi\phi M$.

In order to establish the existence of an equilibrium we study (18) at its boundaries. As ϕ_0 approaches zero, the left side of (18) goes to zero, but the right side remains strictly positive because,

[13] In Section 3.4, we use this characterization to provide a closed-form solution of the distribution in the case of equilibria with full depletion of real balances ($z^\star < \bar{z}_1$). In the general case, numerical calculations are needed and we use an equivalent characterization in terms of Kolmogorov forward equations, as described in our supplementary numerical appendix.

[14] In Rocheteau, Weill, and Wong (2015) we show in a discrete-time version of our model that one-time money injections are not neutral in the short run.

from Proposition 2, households accumulate strictly positive real balances even when they receive no lump-sum transfer, $\Upsilon = 0$. As ϕ_0 tends to infinity, the left side of (18) becomes larger than the right side because the lump sum transfer becomes so large that households only consume and stop working. Finally, Proposition 4 established that the stationary distribution, F, is continuous in $\Upsilon = \pi\phi_0 M_0$. Hence, we can apply the intermediate value theorem and obtain:

Proposition 5 ***(Existence and Uniqueness)*** *For all $\pi \geq 0$ there exists a stationary monetary equilibrium. Moreover, the laissez-faire equilibrium, $\pi = \Upsilon = 0$, is unique.*

From Proposition 5 a monetary equilibrium exists for all inflation rates. Indeed, we show in Proposition 2 that, as a result of the Inada condition on $U(y)$, $s(z)$ is always strictly positive near $z = 0$. In the laissez-faire where $\pi = \Upsilon = 0$ the equilibrium has a simple recursive structure allowing uniqueness to be proved. From Theorem 1 the value and policy functions are uniquely determined independently of F. From Proposition 4, F is uniquely determined given the policy functions.

3.4 Equilibria with full depletion of real balances

In this section we study the class of equilibria with *full depletion*, in which households find it optimal to spend all their money holdings whenever a preference shock occurs, i.e., $y(z) = z$ for all $z \in [0, z^\star]$. In this case our model is very tractable and it lends itself to a tight characterization of decision rules and distributions. We also show that full depletion occurs under appropriate parameter restrictions. These results will be used in the following sections.

The optimal path for real balances under full depletion. The ODE for the optimal path of real balances, (7), can be rewritten as:

$$\dot{z}_t = h(\lambda_t) - c(\lambda_t) - \pi z_t + \Upsilon, \tag{19}$$

where $\lambda_t \equiv W'(z_t)$ is the marginal value of real balances, while $h(\lambda_t)$ and $c(\lambda_t)$ are the solutions to

$$\max_{c \geq 0, h \leq \bar{h}} \left\{ u(c, \bar{h} - h) + \lambda \left(h - c - \pi z + \Upsilon \right) \right\}. \tag{20}$$

To solve for λ_t we apply the envelope condition to differentiate the HJB (8) with respect to z along the optimal path of money holdings. This leads to the ODE:

$$r\lambda_t = \alpha \left[U'(z_t) - \lambda_t \right] - \pi\lambda_t + \dot{\lambda}_t, \tag{21}$$

where we use $V'(z_t) = U'[y(z_t)] = U'(z_t)$ from Proposition 1. The pair, (z_t, λ_t), solves a system of two ODEs, (19) and (21).[15] We can show that the stationary point of this system is a saddle point and the optimal solution to the household's problem is the associated saddle path.

The stationary distribution of real balances under full depletion. Under full depletion, $y(z) = z$, the time it takes for a household to accumulate z' real balances following a preference shock is $\Delta(z, z') = \mathcal{T}(z'_+)$. Hence, from (17), the transition probability function,

$$Q(z, [0, z']) = 1 - e^{-\alpha \mathcal{T}(z'_+)}, \tag{22}$$

does not depend on z. In words, the probability that a household holds less than z' is independent of its real balances just before its last lumpy consumption opportunity, z. This result is intuitive since households "re-start from zero" after a lumpy consumption opportunity. It follows that the stationary probabilities coincide with the transition probabilities, i.e.,

$$F(z') = Q\left(z, [0, z']\right). \tag{23}$$

Finally, the equilibrium equation for the price level, (18), simplifies as well:

$$\phi_0 M_0 = \int_0^\infty z dF(z \,|\, \pi\phi_0 M_0) = \int_0^\infty \left[1 - F(z \,|\, \pi\phi_0 M_0)\right] dz = \int_0^{z^\star} e^{-\alpha \mathcal{T}(z \,|\, \pi\phi_0 M_0)} dz, \tag{24}$$

where our notation highlights that the time to accumulate real balances, $\mathcal{T}$, is a function of the real lump sum transfer, $\Upsilon = \pi\phi_0 M_0$.

Verifying full depletion. From the first-order condition, (10), households find it optimal to deplete their money holdings in full when a lumpy consumption opportunity occurs; $y(z) = z$ for all $z \in [0, z^\star]$, if and only if

$$U'(z^\star) \geq W'(0) = \lambda_0. \tag{25}$$

In order to verify this condition one must solve for the equilibrium price, ϕ_0, and the associated real transfer, $\Upsilon = \pi\phi_0 M_0$. We turn to this task in the following proposition.

Proposition 6 (Sufficient Conditions for Full Depletion) *Under either SI or linear preferences, there exists a threshold for the inflation rate, π_F, such that, for all $\pi \geq \pi_F$, all stationary monetary equilibria feature full depletion.*

[15] A similar system of ODEs holds under partial depletion, where $U'(z_t)$ is replaced by $U'[y(z_t)]$. Hence, in order to solve for this system, one also needs to solve for the unknown function $y(z)$. In Appendix B, we provide a numerical solution to this problem.

Under linear preferences, given any $\pi \geq 0$, there exists a threshold for labor productivity, $\bar{h}_F^{\pi}$, such that, for all $\bar{h} \geq \bar{h}_F^{\pi}$, there exists a unique stationary monetary equilibrium, and this equilibrium must feature full depletion.

Proposition 6 identifies two conditions on exogenous parameters ensuring full depletion. If inflation is large enough, then money holdings become "hot potatoes": they depreciate quickly so that households always find it optimal to spend all their money when given the opportunity. Under linear preferences, if labor productivity is large enough, then households spend all of their money holdings when a preference shock hits because they anticipate that they can rebuild their money inventories quickly.

3.5 The LRW economy

We now show that the model under linear preferences, $u(c, \ell) = \min\{c, \bar{c}\} + \ell$, can easily be solved with pencil and paper, and closed-form solutions, including the non-degenerate distribution of real balances, can be obtained for a broad set of parameter values.[16] When labor productivity grows very large, $\bar{h} \to \infty$, equilibrium converges to the LRW equilibrium with linear value functions and a degenerate distribution of money holdings.

We focus on laissez-faire equilibria ($\pi = 0$) with similar spending patterns as in LRW, i.e., households deplete their real balances in full when they receive a preference shock. From (20) households choose $\dot{z} = h \leq \bar{h}$ to maximize $\dot{z}(\lambda - 1)$ where λ solves the envelope condition (21). The solution is such that $z_t = \bar{h}t$ for all $t \leq \mathcal{T}(z^{\star}) = z^{\star}/\bar{h}$, where t is the length of time since the last preference shock, and $z^{\star}$ is the stationary solution to (21). The marginal value of money at the target is $\lambda = 1$ because a household who keeps its real balances constant must be indifferent between not working and working at a disutility cost of one in order to accumulate one unit of real balances worth λ. From (21):

$$U'(z^{\star}) = 1 + \frac{r}{\alpha}. \tag{26}$$

The marginal utility of lumpy consumption is equal to the marginal disutility of labor augmented by a wedge, r/α, due to discounting. If households are more impatient, or if preference shocks are less frequent, households reduce their targeted real balances.

[16] While we focus on linear preferences, we could achieve the same amount of tractability with the larger class of preferences studied in Wong (2016), including, for example, constant return to scale, constant elasticity of substitution, and CARA.

From (22)-(23) the steady-state distribution of real balances is a truncated exponential distribution,

$$F(z) = 1 - e^{-\frac{\alpha z}{\bar{h}}} \, \mathbb{I}_{\{z<z^\star\}} \text{ for all } z \in \mathbb{R}_+. \tag{27}$$

Note that it has a mass point at the targeted real balances, $1 - F(z^\star_-) = e^{-\alpha z^\star/\bar{h}}$, which is increasing with $\bar{h}$. From market clearing, (24), aggregate real balances are:

$$\phi_0 M_0 = \frac{\bar{h}}{\alpha}\left(1 - e^{-\frac{\alpha z^\star}{\bar{h}}}\right). \tag{28}$$

Aggregate real balances are smaller than the target, $\phi M < z^\star$, and they are increasing with the household's labor endowment.

We now check the condition for full depletion of money balances, (25). Integrating (21) over $[t, \mathcal{T}(z^\star)]$ and using the change of variable $z = \bar{h}t$, we obtain a closed-form expression for λ as a function of $z \in [0, z^\star]$,

$$\lambda(z) = 1 + \alpha \int_z^{z^\star} e^{-\left(\frac{r+\alpha}{\bar{h}}\right)(x-z)} \left[\frac{U'(x) - U'(z^\star)}{\bar{h}}\right] dx. \tag{29}$$

The marginal value of real balances is equal to the marginal disutility of labor, one, plus a discounted sum of the differences between the marginal utility of lumpy consumption on the path going from z to $z^\star$, $U'(z_t)$, and at the target, $U'(z^\star)$. It is easy to check that $\lambda'(z) < 0$, i.e., the value function is strictly concave, and as z approaches $z^\star$ the marginal value of real balances approaches one. From (29) the condition for full depletion, (25), can be expressed as

$$\frac{r}{\alpha} \geq \alpha \int_0^{\mathcal{T}(z^\star)} e^{-(r+\alpha)\tau} \left\{U'\left[z(\tau)\right] - U'(z^\star)\right\} d\tau. \tag{30}$$

The right side of (30) is monotone decreasing in $\bar{h}$ (since $\mathcal{T}(z^\star) = z^\star/\bar{h}$) and it approaches 0 as $\bar{h}$ tends to $+\infty$. So, we confirm Proposition 6 by showing that the equilibrium features full depletion if and only if $\bar{h}$ is above some threshold. Alternatively, (30) holds if households are sufficiently impatient because the cost of holding money outweighs the insurance benefits from hoarding real balances.

The following proposition establishes that as $\bar{h}$ tends to infinity the equilibrium approaches an equilibrium with degenerate distribution and linear value function, analogous to the one in LRW.

Proposition 7 ***(Convergence to LRW)*** *As $\bar{h} \to \infty$ the measure of households holding $z^\star$ tends to one, the value of money approaches $z^\star/M_0$, and $W(z)$ converges to $z - z^\star + \alpha\left[U(z^\star) - z^\star\right]/r$.*

4 Policy

A central objective of the literature on pure currency economies is to characterize welfare-enhancing policies taking into account the frictions in the environment (e.g., lack of monitoring and lack of enforcement) that make money essential. The policymaker seeks to resolve the fundamental trade-off between risk sharing and self insurance using as a policy tool a transfer scheme financed with money creation. Transfers must be consistent with the inability of the government to monitor agents and to enforce trades, i.e., they are not contingent on private histories and they are non-negative. Finally, transfers must be non-decreasing since households can hide their assets when claiming the transfer. This problem is challenging because it requires to solve households' optimization problem under possibly non-linear transfers and to determine how different transfer schemes affect the distribution of real balance and the value of money.

The literature has followed different paths to address these challenges. Part of the literature abstracts from risk sharing by studying models with degenerate distributions so that policy has a single objective, to promote self insurance. If lump-sum taxes are available, it suffices to pay interest on currency financed with taxes. In the absence of lump-sum taxation (because of lack of enforcement) this requires non-linear transfers that reward the accumulation of large real balances (e.g., Hu, Kennan, and Wallace, 2009; Andolfatto, 2010). Another branch of the literature studies pure currency economies with heterogenous agents so that policy faces a trade-off, i.e., promoting risk sharing may reduce incentives to self-insure, but then it is restricted to lump sum transfer schemes (e.g., Kehoe, Levine, and Woodford, 1992; Molico, 2006). Results take the form of examples where inflation is beneficial, but there are also many cases where inflation by means of lump-sum transfer is not optimal. Finally, Wallace (2014) conjectured that, once general transfers are allowed, some inflation is always optimal. That is, one can always find transfers financed by money creation that dominate laissez faire. In this section we address the optimal design of monetary policy in the context of the economy described in Section 3.5 in three steps.

In the first step, we follow Kehoe, Levine, and Woodford (1992, Section 6) and study money growth through lump-sum transfers.[17] We go beyond their analysis because our economy features

[17] Kehoe, Levine, and Woodford (1992) consider a discrete-time version of the Scheinkman and Weiss (1986) economy with two types of agents, buyers and sellers, that alternate through time, which leads to aggregate uncertainty. They focus on two-state Markov equilibria where sellers hold all the currency at the end of each period and transfers of money are lump sum. Section 6, which is the closest to what we do, specializes on logarithmic preferences. In contrast, we have idiosyncratic shocks, no heterogeneity except the one coming from money holdings, and a rich distribution of money holdings with full continuous support.

a distribution of money holdings with a continuous support (instead of two mass points) and endogenous labor supply. Dealing with such heterogeneity is analytically challenging because one needs to determine how money transfers and the inflation tax affect the labor supplies of all agents in the distribution and aggregate real balances. In this context, our key contribution is to show that optimal inflation is decreasing in a fundamental parameter of the economy, labor productivity, $\bar{h}$, and it is zero if labor productivity is large enough.

In the second step, we depart from lump-sum transfers in instances where such transfers generate no welfare gain. We prove the Wallace conjecture by designing an incentive-compatible transfer scheme that combines a lump-sum component for risk sharing and a regressive component to promote self insurance. This scheme generates welfare gains for the whole class of equilibria studied in Section 3.5.

Wallace (2014) acknowledged that his "conjecture is weak in that it says only that some intervention is beneficial". He asked: "Can we hope for stronger conclusions— perhaps, a characterization of when an improvement comes from a small progressive scheme and when it comes from a small regressive scheme? I think not". The third step addresses this question. Namely, we characterize policies that not only generate welfare gains but implement allocations close to the first best. Our answer to Wallace's question is that these near-optimal schemes are progressive for low labor productivity, and regressive for high labor productivity.

4.1 Money growth through lump-sum transfers

We investigate the effects of anticipated inflation implemented with lump-sum transfers on output and welfare. Analyzing the case of money growth is challenging analytically because the transfer, $\Upsilon = \pi\phi_0 M_0$, that affects individual problems and the distribution of real balances is endogenous and depends on the mean of the distribution. We will see that despite this difficulty a large class of equilibria can be obtained in closed form.

In order to study the trade-off between risk sharing and self insurance, it is instructive to focus on equilibria with full depletion, $y(z^\star) = z^\star$. In the presence of money growth, $\pi > 0$, the target for real balances can take two expressions depending on whether the feasibility constraint, $h(z^\star) \leq \bar{h}$, is slack or binding:

$$z^\star \equiv \min\{z_s, z_b\}, \tag{31}$$

where

$$z_s \equiv \left(U'\right)^{-1}\left(1+\frac{r+\pi}{\alpha}\right), \quad \text{and} \quad z_b \equiv \frac{\bar{h}}{\pi}+\phi_0 M_0. \tag{32}$$

The quantity z_s can be interpreted as the *ideal target* that households aim for: it equalizes the marginal utility of lumpy consumption, $U'(z)$, and the cost of holding real balances, $1+(r+\pi)/\alpha$. It is feasible to reach only if $\bar{h}+\Upsilon=\bar{h}+\pi\phi M \geq \pi z_s$. The quantity z_b is the highest level of real balances feasible to accumulate, given households' finite labor endowment, $\bar{h}$, the inflation tax on real balances, πz, and the lump-sum transfer, Υ. Thus z_b is a *constrained target*. From (31) the *effective target*, $z^\star$, is the minimum between these two quantities.

From (19) the trajectory for individual real balances is $z_t = z_b(1-e^{-\pi t})$. Given that the time since the last preference shock is exponentially distributed, the distribution of real balances is

$$F(z) = 1-\left(\frac{z_b - z}{z_b}\right)^{\frac{\alpha}{\pi}} \quad \text{for all } z < z^\star, \tag{33}$$

and $F(z)=1$ for all $z \geq z^\star$. If $z_b \leq z_s$ then households reach $z^\star=\min\{z_b, z_s\}$ only asymptotically, and the distribution of real balances has no mass point. In contrast, if $z_b > z_s$ then households reach $z^\star$ in finite time and the distribution has a mass point at $z=z^\star$. Substituting the closed-form expressions for $\mathcal{T}(z\,|\,\pi\phi M) = -\log\left(1-z/z_b\right)/\pi$ and z_b into the market-clearing condition (24), we find after a few lines of algebra that aggregate real balances solve:

$$\frac{\phi_0 M_0}{\bar{h}/\pi+\phi_0 M_0} = \frac{\pi}{\alpha+\pi}\left\{1-\left(1-\min\left\{1, \frac{z_s}{\bar{h}/\pi+\phi_0 M_0}\right\}\right)^{\frac{\alpha+\pi}{\pi}}\right\}. \tag{34}$$

The left side is strictly increasing in ϕ and the right side is decreasing in ϕ. Hence, (34) has a unique solution, and there is a unique candidate equilibrium with full depletion. Finally, the condition for full depletion of money balances is given by (30) where r is replaced with $r+\pi$ and $\mathcal{T}(z\,|\,\pi\phi_0 M_0) = -\log\left(1-z/z_b\right)/\pi$.

We now define aggregate output and households' ex-ante welfare by:

$$\begin{aligned}\mathcal{H}(\pi,\bar{h}) &\equiv \int h(z;\pi,\bar{h})dF(z;\pi,\bar{h}) \\ \mathcal{W}(\pi,\bar{h}) &\equiv \int \left[-h(z;\pi,\bar{h})+\alpha U(z)\right]dF(z;\pi,\bar{h}).\end{aligned}$$

The pointwise limits for those quantities when labor productivity goes to infinity are denoted by $\mathcal{H}^\infty(\pi) \equiv \lim_{\bar{h}\to\infty}\mathcal{H}(\pi,\bar{h})$ and $\mathcal{W}^\infty(\pi) \equiv \lim_{\bar{h}\to\infty}\mathcal{W}(\pi,\bar{h})$. The following proposition shows that the effects of money growth on $\mathcal{H}$ and $\mathcal{W}$ are qualitatively different depending on the size of $\bar{h}$.

Proposition 8 ***(Output and welfare effects of inflation)*** *In the quasi-linear economy:*

1. ***Large labor productivity***. *Both* $\mathcal{H}^{\infty}(\pi)$ *and* $\mathcal{W}^{\infty}(\pi)$ *are decreasing with* π*.*

2. ***Low labor productivity***. *If* $U(z)/[zU'(z)]$ *is bounded above near zero, then there exists some minimum inflation rate,* $\underline{\pi}$*, and a continuous function* $\bar{H} : [0,\infty) \to \mathbb{R}_+$ *with limits* $\lim_{\pi\to 0} \bar{H}(\pi) = \lim_{\pi\to\infty} \bar{H}(\pi) = 0$*, such that, for all* $\pi \geq \underline{\pi}$ *and* $\bar{h} \in \left[0, \bar{H}(\pi)\right]$*, there exists an equilibrium with binding labor,* $h(z^\star) = \bar{h}$*, and full depletion. In this equilibrium* $\mathcal{H}(\pi, \bar{h})$ *attains its first-best level,* $\bar{h}$*, and* $\mathcal{W}(\pi, \bar{h})$ *increases with* π*.*

3. ***Large inflation.*** *As* $\pi \to \infty$*,* $\mathcal{H}(\pi, \bar{h}) \to 0$ *and* $\mathcal{W}(\pi, \bar{h}) \to 0$*.*

The size of the labor productivity, $\bar{h}$, determines the speed at which households can reach their targeted real balances, and the extent of ex-post heterogeneity across households that prevails in equilibrium. As a result, $\bar{h}$ proves to be a key parameter to determine the extent to which lump-sum transfers of money provide risk-sharing and deter self-insurance and, ultimately, how they affect households' ex-ante welfare.

With large labor productivity, $\bar{h} \to \infty$, there is no role for risk-sharing as all households reach their target almost instantly. However, money growth implemented with lump-sum transfers reduces the rate of return of money, which adversely affects the incentives to self insure, as measured by $z^\star$. Hence, aggregate output, which is approximately $\alpha z^\star$, and social welfare, approximately, $\alpha\left[U(z^\star) - z^\star\right]$, are decreasing with the inflation rate. These are the standard comparative statics in models with degenerate distributions (e.g., Lagos and Wright, 2005).

With low labor productivity, risk-sharing considerations dominate because even though π reduces $z^\star$, it takes a long time for households to reach $z^\star$. Indeed, in the laissez-faire equilibrium the time that it takes, in the absence of any shock, to reach the target, $\mathcal{T}(z^\star) = z^\star/\bar{h}$, can be arbitrarily large when $\bar{h}$ is small. Consider the regime where the equilibrium features both full depletion, $y(z^\star) = z^\star$, and binding labor, $h(z^\star) = \bar{h}$. From Part 2 of Proposition 8 this regime occurs when the inflation rate is not too low and the labor endowment not too high. Because households cannot reach their ideal target, z_s, they all supply $\bar{h}$ irrespective of their wealth, and thus aggregate output is constant and equal to $\bar{h}$. This output level is also the full-insurance one, $h^{FI} = \bar{h}$. Indeed, the condition for the binding labor constraint is $\bar{h}/\pi + \bar{h}/\alpha \leq z_s < y^\star$, which implies $\bar{h} < \alpha y^\star$, and from (3) $h^{FI} = \bar{h}$. In addition, aggregate real balances are equal to the first-best level of consumption, $\phi_0 M_0 = \bar{h}/\alpha$. Hence, risk-sharing is the only consideration for policy as the only source of inefficiency arises from the non-degenerate distribution of real balances.

Wealthy households who hold more real balances than the socially desirable level of consumption, $z > \bar{h}/\alpha$, pay a tax equal to $\pi(z - \bar{h}/\alpha)$ while poor households who hold fewer real balances than the socially-desirable level of consumption, $z < \bar{h}/\alpha$, receive a subsidy equal to $\pi(\bar{h}/\alpha - z)$. Hence, moderate inflation moves individual consumption levels toward the first best, thereby smoothing consumption across households and raising their ex-ante welfare.

A calibrated example In the following we complement Proposition 8 and use a calibrated example to study the relationship between the optimal inflation rate and $\bar{h}$.[18] We normalize a unit of time to a year and we set $r = 4\%$. The inflation rate is $\pi = 2\%$. We adopt a CRRA specification for the utility of lumpy consumption, $U(y) = y^{1-a}/(1-a)$. Provided that $\bar{h} \geq \alpha$ the first-best level of lumpy consumption is 1. The remaining parameters, a, α, and $\bar{h}$, are calibrated to the distribution of the balances of transaction accounts in the 2013 SCF.[19] These calibration targets are matched with $a = .31$, $\alpha = 3.21$ and $\bar{h} = 6.26$.

In Figure 3 we distinguish four regimes: full versus partial depletion of real balances ($y(z^\star) = z^\star$ versus $y(z^\star) < z^\star$) and slack versus binding labor constraint ($h(z^\star) < \bar{h}$ versus $h(z^\star) = \bar{h}$). For sufficiently high π and sufficiently low $\bar{h}$, the equilibrium features full depletion and binding labor; this corresponds to the area marked III in the figure. The lower bond for inflation and the upper boundary of this area correspond respectively to $\underline{\pi}$ and $\log\left[\bar{H}(\pi)\right]$ in Proposition 8. As π is reduced below $\underline{\pi}$ the equilibrium features partial depletion of real balances (areas I and II). Finally, provided that $\bar{h}$ is sufficiently large, the equilibrium features both full depletion and slack labor (area IV). The equilibria we characterize in closed form correspond to III and IV. Note that for all $\bar{h} \leq \alpha = 3.21$ (i.e., $\log \bar{h} \leq 1.14$) the first-best level of output is $h^{FI} = \bar{h}$, achieved in areas II and III.

Proposition 8 establishes that for high $\bar{h}$ inflation is detrimental to society's welfare whereas for low $\bar{h}$ positive inflation implemented with lump-sum transfers raises welfare relative to the laissez-faire. In Figure 3 we illustrate these results by plotting with a black, thick curve the welfare-maximizing inflation rate as a function of the labor endowment. As $\bar{h}$ increases the optimal inflation rate decreases, and for a sufficiently high value of $\bar{h}$ the laissez-faire equilibrium dominates

[18] Arguably, pure currency economies are not directly comparable to actual economies with multiple assets. The purpose of our calibration exercise is to provide a numerical example illustrating some properties of our model with parameter values that are plausible. See Appendix B for details on how to solve numerically the system of delay differential equations.

[19] We adopt the following three targets: the ratio of the balances of the 80th-percentile household to the average balances, $F^{-1}(.8)/\phi M$, the ratio of the average balances to the average income, $\phi M/H$, and the semi-elasticity of money demand, $\eta \equiv \partial \log \phi M/(100 \times \partial \pi)$. In the 2013 SCF, $F^{-1}(.8)/\phi M = 1.23$ and $\phi M/H = .39$. Aruoba, Waller and Wright (2011) estimate that $\eta = -.06$. Transaction accounts in SCF include checking, savings, money market, and call accounts, but they do not include currency.

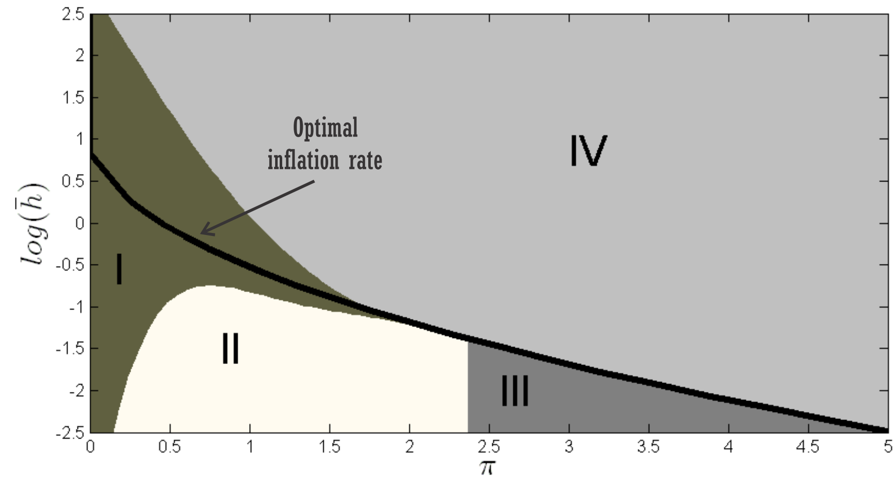

Figure 3: Region I: Slack labor & Partial depletion; Region II: Binding labor & Partial depletion; Region III: Binding labor & Full depletion; Region IV: Slack labor & Full depletion.

any equilibrium with positive inflation. Moreover, when equilibria with full depletion and binding labor exist (area III) then the optimal inflation rate is the highest one that is consistent with such an equilibrium. A higher inflation rate would relax the labor constraint (area IV) and would reduce output below its efficient level, $\bar{h}$. For values of $\bar{h}$ that are large enough such that region III does not exist, then the optimal inflation rate corresponds to an equilibrium with slack labor and partial depletion (area I). In area IV with slack labor and full depletion a reduction of the inflation rate is always welfare improving.

4.2 Beyond lump-sum transfers

We have shown in Proposition 8 and Figure 3 that when $\bar{h}$ is sufficiently large, inflation implemented through lump-sum transfers is welfare-worsening as the social cost of lowering $z^\star$ outweighs the risk-sharing benefits associated with lump-sum transfers. In contrast, Wallace (2014) conjectures that money creation is almost always optimal in pure-currency economies, as long as one can depart from lump-sum transfers. In accordance with this conjecture, we establish in the following that inflation is optimal once one allows for more general, incentive-compatible, transfer schemes. This conjecture is hard to verify for economies with a rich heterogeneity as ours, as one needs to determine how transfers affect individual labor supplies throughout the endogenous distribution and their overall welfare effect.

Suppose new money, $\dot{M} = \pi M$, is injected through the following transfer scheme:

$$\tau(z) = \begin{cases} \tau_0 & z \leq z_\pi^\star \\ \tau_z z - \tau_1 \quad \text{if} & z \in (z_\pi^\star, z_0^\star] \\ \pi z & z > z_0^\star \end{cases}, \tag{35}$$

where $z_\pi^\star$ solves $U'(z_\pi^\star) = 1 + (r + \pi)/\alpha$. The real transfer, $\tau(z)$, is non-negative because in pure currency economies with no enforcement taxation is not feasible. The transfer is non-decreasing so that households have no incentive to hide some of their money balances. Hence, $\tau_0 \geq 0$ and $\tau_z \geq 0$. Moreover, we assume that $\tau(z)$ is continuous, $\tau_z = (\pi z_0^\star - \tau_0) / (z_0^\star - z_\pi^\star)$ and $\tau_1 = (\pi z_\pi^\star - \tau_0) z_0^\star / (z_0^\star - z_\pi^\star)$. From the government budget constraint, the sum of the transfers to households net of the inflation tax must be zero, $\int [\tau(z) - \pi z] \, dF_\tau(z) = 0$, where the distribution F_τ is now indexed by the transfer scheme. Hence, $\tau_z \geq \pi$ and $\tau_1 \leq 0$. So the first tier is a lump-sum transfer, the second is a linear regressive transfer, and the third tier is neutral.

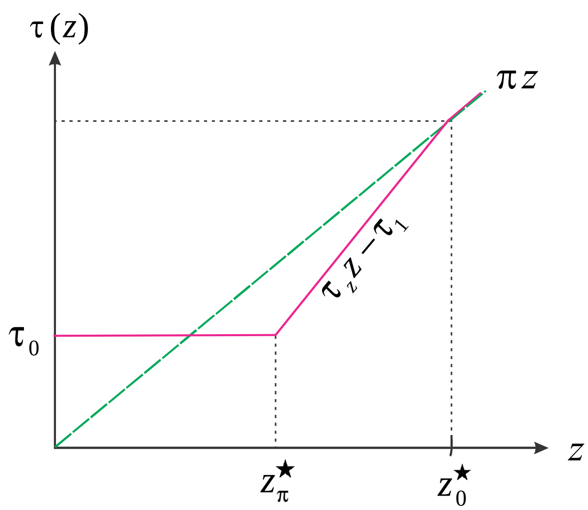

Figure 4: A socially beneficial inflationary scheme

Our proposed scheme, illustrated in Figure 4, takes into account the trade-off between self-insurance and risk sharing in economies with non-degenerate distributions. It has a lump-sum component, τ_0, that improves risk sharing by transferring wealth from the richest households to the poorest ones. The threshold for real balances below which households receive τ_0 is $z_\pi^\star$, the target for real balances in an economy with lump-sum transfers only. It has a regressive component, $\tau_z z - \tau_1$, that specifies a transfer increasing with real balances for all households holding z between $z_\pi^\star$ and $z_0^\star$. The purpose of this component is to mitigate the disincentive effect of inflation on households' willingness to accumulate real balances. As a result, households accumulate the same amount they would in the laissez-faire equilibrium, $z_0^\star$.

In the following proposition we denote by $\bar{h}_F^0$ the threshold for labor endowments above which there is full depletion in the laissez-faire equilibrium ($\pi = 0$); in other words (30) holds.

Proposition 9 ***(The Wallace conjecture)*** *Suppose that $\bar{h} \geq \bar{h}_F^0$, the equilibrium features full depletion. There exists an incentive-feasible, inflationary transfer scheme, $\tau(z)$ given by (35) with $\pi > 0$, such that:*

(i) Society's welfare is higher under $\tau(z)$ relative to the laissez-faire.

(ii) Aggregate real balances and output are higher under $\tau(z)$ relative to the laissez-faire.

(iii) The target for real balances is $z_0^\star$, the same as under laissez-faire.

In order to prove that the transfer scheme is socially beneficial we show that it not only redistributes wealth, but also raises aggregate real balances. In order to make the second claim we establish that it takes longer to accumulate $z_0^\star$ under the transfer scheme, τ, than under laissez faire. Relative to laissez faire, households accumulate real balances at a faster pace when they are poor, because $\tau(z) - \pi z > 0$, and at a slower pace when they are rich, because $\tau(z) - \pi z < 0$. Even though the sum of the net transfers across households is zero, only a fraction of the households become sufficiently rich to be net contributors to the scheme before they are hit by a new preference shock. As a result, the burden on the rich households outweighs the subsidies they received while being poor, and hence they reach their desired real balances later relative to the laissez faire. It follows that there is a larger fraction of households who are producing making aggregate real balances larger under the inflationary scheme. In summary, the transfer scheme, τ, raises society's welfare by redistributing a higher stock of real balances from rich to poor households without giving incentives to households to lower their targeted real balances.

4.3 Near-efficient policies

We now characterize policies that implement allocations close to the first best, and we study how such policies vary with labor productivity. We start with the case where labor productivity is low and we assume that the utility for lumpy consumption is linear with a satiation point, $U(y) = A\min\{y, \bar{y}\}$. Similar preferences have been used in Kehoe, Levine, and Woodford (1992, Section 5) and Green and Zhou (2005, Section 6).

Proposition 10 ***(Implementation of the First Best).*** *Assume $U(y) = A\min\{y, \bar{y}\}$. If $\bar{h} < \alpha\bar{y}\left[\alpha(A-1) - r\right] / (\alpha A - r)$, then there is a monetary equilibrium under a lump-sum transfer* *scheme,* $\tau(z) = \pi \bar{h}/\alpha$ *for all* z *with* $\pi \in \left[\alpha\bar{h}/(\alpha\bar{y} - \bar{h}), \alpha(A-1) - r\right]$*, that exactly implements a first best.*

If $\bar{h} < \alpha\bar{y}$, then a first best allocation is one where there is full employment, $h = \bar{h}$. We implement this outcome with positive money growth through lump sum transfers. We construct an equilibrium in which inflation is sufficiently high so that households have to work full time just to maintain their targeted real balances at a level less than $\bar{y}$. Because the marginal utility of lumpy consumption is constant and equal to A for all $z \in [0, \bar{y}]$, there is no welfare loss associated with households' ex-post heterogeneity. Provided that the rate of time preference is not too large, there is a range of positive inflation rates that implement a first-best allocation. The inflation rate cannot be too low since otherwise households might find it optimal not to work when they reach the target for real balances. For instance, if $\pi = 0$ households reach the target in a finite amount of time, and hence a fraction of them do not supply any labor. The inflation rate cannot too high or households will not find it optimal to accumulating real balances.

For high labor productivity a first-best allocation is such that $y = y^\star$ where $U'(y^\star) = 1$. We implement allocations that are close to the first best with a transfer scheme that is a step function,

$$\tau(z \mid \pi) = \begin{cases} 0 & \text{if} \quad z < z^\star(\pi) \\ \pi z & \text{if} \quad z \geq z^\star(\pi) \end{cases}, \tag{36}$$

where $z^\star(\pi) = y^\star - \Delta/\bar{h}$, for $\Delta = 2\left[\pi y^\star + \alpha U(y^\star)\right]/|U''(y^\star)|$.[20] If households hold real balances above $z^\star$, then they receive a proportional transfer that exactly compensates for the inflation tax, i.e., their asset position is protected against inflation. However, if they hold less than $z^\star$, then they no longer receive the proportional transfer. When $z^\star$ is close to $y^\star$, this scheme rewards households who hold real balances close to the first best and to punish those who hold too little real balances. For this scheme to be nearly efficient, there must be few households along the equilibrium path who hold less than $z^\star$, which requires $\bar{h}$ to be large. We formalize this notion in the following proposition by measuring the welfare loss relative to the first best in consumption-equivalent units, i.e., we compute the fraction δ of the first-best consumption that an household would be willing to pay in order to move from the equilibrium to the first best.

It should be noted that the transfer scheme (36) generates a surplus corresponding to the inflation tax levied on households with $z < z^\star$: after making lump sum transfers, the government

[20] This transfer scheme is a generalization of the one studied in Bajaj, Hu, Rocheteau and Silva (2017) in the context of an economy with a degenerate distribution of money holdings.

has money left in hand that it uses to purchase consumption goods. We assume that this surplus is thrown away, which creates a welfare loss. Our efficiency result below holds in spite of this loss.

Proposition 11 ***(Near-Efficient Schemes with High Labor Productivity.)*** *Assume that* $-ry^\star + \alpha\,[U(y^\star) - y^\star] > 0$*. Then, there is some* $\underline{\pi}$ *such that, given any* $\pi > \underline{\pi}$*, as long as* $\bar{h}$ *is large enough, there exists a monetary equilibrium that features full depletion of real balances with scheme* $\tau(z \mid \pi)$*. Moreover, the welfare loss in terms of first-best consumption is:*

$$\delta = \frac{1}{\bar{h}} \times \frac{1}{y^\star} \int_0^{y^\star} \{[U(y^\star) - y^\star] - [U(z) - z]\}\, dz + o\left(\frac{1}{\bar{h}}\right). \tag{37}$$

The proof of the Proposition is challenging because of the discontinuity in the transfer scheme. This discontinuity implies that the value function has a concave kink at $z^\star$. As a result, the standard optimality verification argument, which requires smoothness, must be extended to handle our particular case. The condition $-ry^\star + \alpha\,[U(y^\star) - y^\star] > 0$ is necessary for households to have incentives to hold real balances close to the first best, $y^\star$. It states that the opportunity cost of holding real balances, $ry^\star$, is less than the expected surplus from holding such real balances in the event of a preference shock for lumpy consumption, $\alpha\,[U(y^\star) - y^\star]$. It is the typical condition for the implementation of the first best in monetary models with degenerate distribution (see, e.g., Hu, Kennan, and Wallace, 2009).

The Proposition shows that the transfer scheme $\tau(z \mid \pi)$ leads to nearly efficient allocation as long as $\bar{h}$ is large enough. Moreover, it provides an intuitive estimate of the welfare loss. Namely, to a first-order approximation in $1/\bar{h}$, the welfare loss is equal to the average surplus that is lost by households with real balances strictly below the target, $z^\star$. Notice that the estimate of δ does not account for the welfare loss incurred by households at the target. This is because the target $z^\star$ is close to the first-best output, $y^\star$, and the first-best output already maximizes utilitarian welfare. Hence, the welfare loss incurred by households at $z^\star$ is of second order.

Finally, we note that the transfer scheme in Proposition 11 can be replicated without transfers but with indivisible bonds. Bonds are of the pure-discount variety that pay $z^\star$ in terms of goods to their bearer at maturity with Poisson arrival rate μ. They are perfectly recognizable and can be traded at no cost. Their real price is $q = \mu z^\star/(\varrho + \mu)$, where where ϱ is the real rate of return on bonds. Under the conditions stated in Proposition 11 the supply of bonds, B, can be chosen to implement an equilibrium with $q = z^\star$ and $\varrho = 0$, i.e., bonds are traded at their face value. Precisely, B must be equal to the measure of households at their targeted wealth, $B = 1 - F(z^\star_-)$.

In equilibrium, households accumulate real balance until they reach their targeted wealth, at which point they switch to holding bonds. When they receive a preference shock, they fully deplete their wealth, spending real balances if $z < z^{\star}$, and bonds if $z = z^{\star}$.

In the following section we elaborate on the idea that government bonds can serve as a policy instrument to provide incentives to self insure, just like regressive transfers do.

5 Beyond pure currency: Money and illiquid bonds

We now consider a second instrument, beside fiat money, to tackle the policy trade-off between risk sharing and self insurance. We introduce illiquid nominal government bonds that can bear interest. With this extension, monetary policy can be implemented through both "helicopter drops" and open market operations by purchasing or selling government bonds in exchange for money. We will show the equivalence between equilibria of the economy with money and bonds that feature a zero nominal interest rate – liquid traps – and equilibria of the pure currency economy that feature partial depletion. We will use this equivalence and the analytical results from Section 4 to obtain insights for the existence of liquidity trap equilibria and for policy. We will also provide examples where interest-bearing bonds raise welfare relative to the pure currency economy with lump-sum transfers, and we will compute the optimal policy for different productivity levels.

In addition to fiat money, suppose there is a supply, B_t, of short-term, pure-discount, nominal bonds that pay one unit of money at the time of maturity that occurs at Poisson arrival rate $\mu > 0$. Bonds that expire are replaced with newly-issued bonds. The supply of bonds is growing at the same rate π as the money supply to keep the ratio, B_t/M_t, constant and equal to B_0/M_0. The price of money and bonds (in terms of goods) are denoted by ϕ_t and q_t. We focus on stationary equilibria in which $q_t B_t$ and $\phi_t M_t$ are constant, i.e., $\dot{\phi}_t/\phi_t = \dot{q}_t/q_t = -\pi$.

In a stationary equilibrium, the expected real rate of return on bonds, denoted by ϱ, is constant and solves

$$\varrho q_t = \mu\left(\phi_t - q_t\right) + \dot{q}_t \Rightarrow \varrho = \mu\left(\frac{\phi_0}{q_0} - 1\right) - \pi. \tag{38}$$

The rate of return of bonds has two components. First, with intensity μ, this bond matures into one unit of money, generating a capital gain of $\phi_0/q_0 - 1$. Second, the value of the bond, which is a claim on a unit of money, depreciates at the rate of inflation, π. If bonds can serve as means of payment, then money and bonds are perfect substitutes, $\phi_0 = q_0$, and so they generate the same rate of return, $\varrho = -\pi$. In what follows we assume that bonds are not as liquid as money: in the event of a preference shock only fiat money can be used to finance lumpy consumption, e.g., because it is the only asset that can be authenticated instantly. However, households are free to trade bonds in between lumpy consumption opportunities.[21] We will see later that the illiquidity of bonds can also be justified on normative grounds.

In Supplementary Appendix I, we extend the analysis of the households' problem of Section 3 to the present environment. As in Theorem 1, we study the maximum attainable lifetime utility of a household with ω units of wealth, $W(\omega)$. We show that it is strictly increasing and strictly concave, with $W'(0) < +\infty$ and $W'(+\infty) = 0$, that it is continuously differentiable over $[0, \infty)$, and that it solves the Hamilton-Jacobi-Bellman equation:

$$rW(\omega) = \max_{c,h,y,z} \left\{ u(c, \bar{h} - h) + \alpha \left[U(y) + W(\omega - y) - W(\omega) \right] + W'(\omega)\dot{\omega} \right\}, \tag{39}$$

subject to $c \geq 0$, $0 \leq h \leq \bar{h}$, $0 \leq y \leq z \leq \omega$, and $\dot{\omega} = h - c + \varrho(\omega - z) - \pi z + \Upsilon$ where, by the budget constraint of the government, $\Upsilon = \mu B_0 (q_0 - \phi_0) + \pi(\phi_0 M_0 + q_0 B_0)$.[22] In any equilibrium, the nominal interest rate on government bonds is bounded below by zero, $\varrho + \pi \geq 0$. If the inequality is strict, then $y = z$, households do not hold more money than what they intend to spend in case of a preference shock.

The first-order condition with respect to y is

$$\alpha \left[U'(y) - W'(\omega - y) \right] \geq (\varrho + \pi) \, W'(\omega), \tag{40}$$

with an equality if $y < \omega$. The left side is the same as in the pure-currency economy: it is the expected net utility of consuming a marginal unit of good at the time of a preference shock. The right side is the expected opportunity cost of holding real balances until the next preference shock. Following the same reasoning as in Proposition 1, $y(\omega)$ is strictly positive and increasing with wealth. Because $W'(0) < +\infty$, the poorest households hold only money.

From the HJB equation, (39), the marginal value of wealth, $\lambda_t = W'(\omega_t)$, solves:

$$(r + \pi) \, \lambda_t = \alpha \left\{ U' \left[y(\omega_t) \right] - \lambda_t \right\} + \dot{\lambda}_t. \tag{41}$$

Interestingly, (41) is identical to its version in the pure currency economy. Intuitively, the choice of real balances is interior for all levels of wealth and, as a result, the marginal value of wealth

[21] It takes an infinitesimal amount of time to authenticate bonds, but that delay is large enough to miss an opportunity to consume. The idea that assets are not acceptable because they lack recognizability has been formalized in Lester, Postlewaite, and Wright (2012), Li, Rocheteau, and Weill (2012), and Hu (2013). Alternative explanations for the coexistence of money and interest-bearing bonds include Zhu and Wallace (2007) and Lagos (2013).

[22] In the household's budget constraint we assumed that the portfolio of bonds is fully diversified so that the return, $\varrho(\omega - z)$, is deterministic.

coincides with the marginal value of real balances. If $y(\omega) \leq \omega$ does not bind, then the Envelope Theorem applied to (39) gives

$$(r - \varrho)\,\lambda(\omega) = \alpha\,[\lambda(\omega - y) - \lambda(\omega)] + \dot{\lambda}. \tag{42}$$

The left side is the opportunity cost of wealth measured by the difference between the rate of time preference and the rate of return on bonds. The first term on the right side is the change in the marginal value of wealth following an opportunity for lumpy consumption. The targeted wealth, $\omega^\star$, corresponds to the stationary solution to (42), $\dot{\lambda}_t = 0$. Together with (40) it implies

$$\left(1 + \frac{r - \varrho}{\alpha}\right) W'(\omega^\star) \geq W'\left[\omega^\star - y(\omega^\star)\right]. \tag{43}$$

The strict concavity of $W(\omega)$ implies $r > \varrho$. Even though bonds are illiquid, in the sense that they cannot be used to finance lumpy consumption, they do provide insurance services by allowing households to replenish their holdings of liquid assets after a preference shock. By market clearing, the richest households must hold some bonds, so that (43) holds at equality.

From the policy functions one can construct $\Delta(\omega, \omega')$, the minimal time that it takes for a household with wealth ω at the time of a preference shock to accumulate strictly more than ω'. A stationary distribution of wealth, $F(\omega)$, is a solution to

$$1 - F(\omega') = \int_0^\infty \alpha e^{-\alpha u} \int_0^\infty \mathbb{I}_{\{u \geq \Delta(\omega,\omega')\}} dF(\omega)\, du = \int_0^\infty e^{-\alpha\Delta(\omega,\omega')}\, dF(\omega). \tag{44}$$

By market clearing, bonds have to be held, which implies that the richest households do not deplete their wealth in full when a preference shock occurs. It is a key difference between equilibria of the economy with illiquid bonds and equilibria of the pure-currency economy: the former must feature partial depletion of wealth. Using that from (38) $q_0 = \mu\phi_0/(\varrho + \pi + \mu)$, the market-clearing conditions for real balances and bonds are:

$$\frac{\mu\phi_0 B_0}{\varrho + \pi + \mu} = \int_0^\infty [\omega - z(\omega)]\, dF(\omega) \tag{45}$$

$$\phi_0 M_0 = \int_0^\infty z(\omega) dF(\omega), \tag{46}$$

where the right sides of (45) and (46) depend on the lump-sum transfer, Υ, and the real return on bonds, ϱ. A stationary monetary equilibrium is composed of a value function, $W(\omega)$, a distribution of wealth, $F(\omega)$, a price of money, $\phi_0 > 0$, and a real interest rate, ϱ, solving (39), (44), (45) and (46). The following proposition follows directly from market clearing (bonds have to be held) and the fact that $W'(0) < +\infty$ (the poorest households want to spend all their wealth).

Proposition 12 ***(Properties of Equilibrium)*** *Any equilibrium features:*

(i) Endogenous segmentation: There is a threshold for wealth, $\underline{\omega} \in (0, \omega^{\star})$, below which households hold all their wealth in the form of money, i.e., bonds are held by households with wealth above $\underline{\omega}$.

(ii) Liquidity premium on bonds: The real interest rate on illiquid bonds is less then the discount rate.

Our model generates a form of segmentation that is similar to the one in Grossman and Weiss (1983) and Alvarez, Atkeson, and Kehoe (2002). Only a fraction of households hold and trade bonds. In contrast to Alvarez, Atkeson, and Kehoe (2002), this segmentation does not require a fixed cost in participating in the bond market – there is an opportunity cost, of course, since by participating in the bond market a households holds a less liquid asset. The second result that bonds command a liquidity premium equal to $r - \varrho > 0$, is in contrast with LRW where the rate of return of illiquid bonds is r.[23] Indeed, in LRW, illiquid bonds have no insurance value because the household can reach its targeted real balances instantly. It is through this liquidity premium that changes in the supply of bonds can affect the real interest rates, ϱ.

We now focus on a subset of equilibria called a *liquidity traps* where the nominal interest rate on bonds reaches its lower bound, $\varrho + \pi = 0$. In such equilibria money and bonds are perfect substitutes as savings vehicles.

Proposition 13 ***(Liquidity-Trap Equilibria: An Equivalence Result.)*** *Consider a pure-currency economy with money growth rate, π, and initial money supply, $M_0 = 1$. Denote, $\{y^1(z), h^1(z), c^1(z), \phi^1, F^1(z)\}$, a steady-state equilibrium with partial depletion, and let*

$$\beta \equiv \frac{\int_0^\infty \left[z - y^1(z)\right] dF^1(z)}{\int_0^\infty y^1(z) dF^1(z)} > 0. \tag{47}$$

Then, for all $B_0 \leq \beta$, there is an equilibrium of the economy with initial money supply, $M_0 = 1$, and initial bonds supply, B_0, $\{y^2(\omega), h^2(\omega), c^2(\omega), z^2(\omega), \phi^2, \varrho^2, F^2(\omega)\}$, with $\varrho^2 = -\pi$, $F^2(\omega) = F^1(\omega)$, $y^2(\omega) = y^1(\omega)$, $h^2(\omega) = h^1(\omega)$, $c^2(\omega) = c^1(\omega)$, $z^2(\omega) \in \left[y^1(\omega), \omega\right]$, and $\phi_0^2 = \phi_0^1 / (1 + B_0)$. Conversely, any liquidity-trap equilibrium of two-asset economy corresponds to an equilibrium of the pure-currency economy that features partial depletion.

[23] The idea that government bonds can pay a liquidity premium even if they are not used as medium of exchange provided that they allow agents to reallocate liquidity in the presense of idiosyncratic preference shocks can be found in Kocherlakota (2003). See also Berentsen, Camera, and Waller (2007), Li and Li (2013), Lagos and Zhang (2015), and Geromichalos and Herrenbrueck (2016).

Any equilibrium of the pure-currency economy with partial depletion is equivalent in terms of allocations, aggregate wealth, and welfare to a liquidity-trap equilibrium of the economy with money and bonds. In order to get some intuition for this equivalence result, note that in pure-currency economies households accumulate money balances for two motives: $y(z)$ for a transaction motive and $z - y(z)$ for a precautionary motive. In equilibria that feature partial depletion the second motive is active, i.e. $z - y(z) > 0$ for some z in the support of the real balance distribution. Suppose we introduce a small supply of illiquid nominal bonds in such an economy. If the nominal interest rate on nominal bonds is strictly positive, then households want to fulfill their precautionary motive with bonds only. However, if the supply of bond is small enough, i.e. $B_0/M_0 \leq \beta$, the bond market would not clear. Hence, in equilibrium, the nominal interest rate must fall to zero.

We now use Proposition 13 and results from Section 3.5 to establish conditions for the existence of liquidity-trap equilibria.

Corollary 2 ***(Existence of Liquidity-Trap Equilibria.)*** *Consider the economy with linear preferences and constant money supply. There exists a liquidity-trap equilibrium if and only if $\bar{h} < h_F^0$, where h_F^0 solves*

$$\frac{r}{\alpha} = \alpha \int_0^{z^\star/h_F^0} e^{-(r+\alpha)\tau} \left\{ U'\left[z(\tau)\right] - U'(z^\star) \right\} d\tau,$$

and $B_0/M_0 \leq \beta$.

Liquidity-trap equilibria exist when labor productivity is low and bonds are scarce. Indeed, when $\bar{h}$ is low, households have a high precautionary demand for assets because the pace of wealth accumulation is low. If the bond supply is low, the bond yield is driven to zero, so that households are indifferent between holding money and bonds. One can show after some algebra that h_F^0 is increasing in α, which means liquidity traps occur when the idiosyncratic risk, measured by α, is high. In contrast, liquidity-trap equilibria do not exist for any bond-money ratio in times of high productivity and low idiosyncratic uncertainty, in which case $\varrho \in (-\pi, r)$.[24]

[24] From a theoretical standpoint, this result is consistent with the absence of liquidity-trap equilibria (away from the Friedman rule) in the LRW model with illiquid bonds. Williamson (2012), Andolfatto and Williamson (2015), and Rocheteau, Wright, and Xiao (2015) obtain liquidity-trap equilibria in models with degenerate distributions where bonds are partially acceptable as means of payment and markets are segmented or liquidity is reallocated through intermediaries. Related to what we do, Guerrieri and Lorenzoni (2015) study liquidity traps in an heterogenous-agents, incomplete-market model and characterize transitional dynamics following a one-time aggregate shock. A treatment of liquidity traps in New-Keynesian models has been provided by Eggertsson and Woodford (2003). Those models differ in fundamental ways from ours: they assume complete financial markets, they introduce money in the utility function, and in the absence of nominal rigidities a liquidity trap corresponds to the Friedman rule.

We can now use results from Section 4 to obtain policy recommendations. The condition under which a liquidity trap exists, namely, a low $\bar{h}$, is the same condition for which risk-sharing considerations matter the most and inflation is beneficial. So even though open-market operations, interpreted as changes in the ratio B_0/M_0, are ineffective in liquidity traps, anticipated inflation through a higher growth rate of governments' liabilities, can raise welfare. According to the example in Figure 3, for very low values of $\bar{h}$, welfare can be improved by raising inflation to a level that induces full depletion of real balances; meanwhile, to prevent the nominal interest rate from increasing, B_0/M_0 has to be driven to zero eventually. So while an open-market operation alone is ineffective, a combination of open-market operations and "helicopter drops" is useful.

For intermediate values of $\bar{h}$, inflation is still beneficial but the bond-money ratio can stay positive. In order to illustrate this point we compute the combination of π and i that generates the highest welfare for the parameter values in Section 4.1. The optimal policy, $\pi = 1\%$ and $i = 4.75\%$, implies $\phi_0 M_0/q_0 B_0 = 5.24$, $\Upsilon = 0.0019$, which is about 0.04% of aggregate output. In the absence of bonds, the optimal policy under lump-sum transfers is $\pi = 0$. So the presence of interest-bearing illiquid bonds is socially beneficial and it is accompanied by some positive inflation rate. If labor productivity is reduced by half, the optimal policy is $\pi = 2\%$ and $i = 5.75\%$. It is optimal to increase the rate of growth of nominal assets, which allows to finance larger transfers and interest payments on nominal bonds. The money-to-bond ratio is reduced to $\phi_0 M_0/q_0 B_0 = 2.91$, but Υ increases to 0.0043, which is about 0.15% of aggregate output.

6 Other applications

In the following we illustrate additional insights and other tractable cases of our pure currency economy. We first provide an example with quadratic preferences, allowing us to characterize in closed form the transitional dynamics following a one-time money injection. Second, we assume general preferences over c and h but linear and stochastic preferences over lumpy consumption, in order to discuss the effects of inflation on households' spending behavior.

6.1 Money in the short run

Suppose now that preferences are quadratic: $U(y) = Ay - y^2/2$ and $u(c, \bar{h} - h) = \varepsilon c - c^2/2 - h^2/2$.[25] From (12), and assuming interiority, the optimal choices of consumption and labor in a steady

[25] Notice that these preferences do not satisfy the Inada conditions imposed earlier. But previous results are not needed as we are able to solve the equilibrium in closed form.

state are are $c_t = \varepsilon - \lambda_t$ and $h_t = \lambda_t$. Under full depletion of real balances the stationary solution to the system of ODEs, (19)-(21), is $\lambda^\star = \varepsilon/2$ and $z^\star = A - (1 + r/\alpha)\,\varepsilon/2$. We assume that $A > (1 + r/\alpha)\,\varepsilon/2$ to guarantee $z^\star > 0$. Along the saddle path trajectory

$$\lambda(z) = \frac{\nu}{2}\,(z - z^\star) + \lambda^\star, \tag{48}$$

where $\nu = \left(r + \alpha - \sqrt{(r+\alpha)^2 + 8\alpha}\right)/2 < 0$. It follows that the household's policy functions are:

$$\begin{aligned} c(z) &= \frac{\varepsilon - \nu\,(z - z^\star)}{2} && (49)\\ h(z) &= \frac{\varepsilon + \nu\,(z - z^\star)}{2}. && (50) \end{aligned}$$

As households get richer their marginal value of wealth decreases, their consumption flow increases, and their supply of labor decreases. The condition for full depletion is $A - z^\star > -\,\nu z^\star/2 + \lambda^\star$ and $c(z)$ is interior for all z if $c(0) \geq 0$. It can be shown that the set of parameter values for which these restrictions hold is non-empty.

The saddle path of (19)-(21) is such that $z_t = z^\star\left(1 - e^{\nu t}\right)$ where t is the length of the time interval since the last preference shock. Given that t is exponentially distributed the distribution of real balances is:

$$F(z) = 1 - \left(\frac{z^\star - z}{z^\star}\right)^{-\frac{\alpha}{\nu}} \quad \text{for all } z \leq z^\star. \tag{51}$$

In contrast to the model of Section 3.5 the distribution of real balances has no mass point at $z^\star$ as households reach their target asymptotically. Market clearing gives

$$\phi M = \int_0^{z^\star} [1 - F(z)]\,dz = \frac{\nu}{\nu - \alpha} z^\star. \tag{52}$$

As before aggregate real balances depend on all preference parameters (r, ε, A) but not on M: money is neutral in the long run.

We now turn to the transitional dynamics following a one-time increase in the money supply, from M to γM, where $\gamma > 1$. In general, one should take into account that the rate of return of money, $\dot{\phi}/\phi$, might vary along the transitional path. Here, however, we guess and verify the existence of an equilibrium where the value of money adjusts instantly to its new steady-state value, ϕ/γ. Along the equilibrium path aggregate real balances, $Z = \phi M$, are constant. To check that our proposed equilibrium is indeed an equilibrium we show that the goods market clears at any point in time. From (49) and (50) it is easy to check that aggregate consumption is $C \equiv \int c(z)dF_t(z) + \alpha \int z dF_t(z) = [\varepsilon - \nu\,(Z - z^\star)]\,/2 + \alpha Z$ while aggregate output is $H =$

$\int h(z)dF_t(z) = [\varepsilon + \nu (Z - z^\star)]/2$. From (52) it follows that $C + \alpha Z = H$, i.e., the goods market clears. The predictions of the model for aggregate quantities are consistent with the quantity theory: the price level moves in proportion to the money supply and real quantities are unaffected. So, from an aggregate viewpoint, money is neutral in the short run.[26]

However, money affects the distribution of real balances and consumption levels across households, which is relevant for welfare under strictly concave preferences. We compute society's welfare at the time of the money injection as $\int W(z)dF_0(z)$ where

$$F_0(z) = F\left[\gamma z - (\gamma - 1)Z\right]. \tag{53}$$

According to (53) the measure of households who hold less than z real balances immediately after the money injection is equal to the measure of households who were holding less than $\gamma z - (\gamma - 1)Z$ just before the shock: they received a lump-sum transfer of size $(\gamma - 1)Z$ and their real wealth is scaled down by a factor γ^{-1} due to the increase in the price level. The value function, $W(z)$, being strictly concave ($\lambda(z)$ is a decreasing function of z), the reduction in the spread of the distribution leads to an increase in welfare.

6.2 Inflation and velocity

Suppose now that $U(y) = Ay$ where A is an i.i.d. draw from some distribution $\Psi(A)$.[27] We will use this version of the model to capture the common wisdom according to which households spend their real balances faster on less valuable commodities as inflation increases, thereby generating a misallocation of resources.[28]

We conjecture that $W(z)$ is linear with slope λ. The HJB equation, (8), becomes:

$$rW(z) = \max_{c,h} \left\{ u(c, \bar{h} - h) + \alpha \int V(z) + \lambda(h - c - \pi z + \Upsilon) \right\} \tag{54}$$

where $V(z) \equiv \int V(z, A)d\Psi(A)$ with

$$V(z, A) \equiv \max_{0 \le y \le z} \left\{ Ay + W(z - y) \right\} = \max_{0 \le y \le z} (A - \lambda)\, y + W(z). \tag{55}$$

[26] This result is certainly not general, but it is a useful benchmark suggesting that the effects of a one-time money injection on aggregate real quantities will crucially depend on preferences that determine the relationship between labor supply decisions and wealth. In Rocheteau, Weill, and Wong (2015) we study transitional dynamic following one-time money injections in a discrete-time version of our model with search and bargaining and quasi-linear preferences. We show that the money injection affects the rate of return of money, aggregate real balances, and output levels.

[27] For a significant extension of our model with linear utility for lumpy consumption, see Herrenbrueck (2014). The model is extended to account for quantitative easing and the liquidity channel of monetary policy.

[28] This wisdom has proved difficult to formalize in models with degenerate distributions. See Lagos and Rocheteau (2005); Ennis (2009); Liu, Wang, and Wright (2011), and Nosal (2011) for several attempts to generate the 'hot potato' effect in this class of models.

From (55) the household spends all his real balances whenever $A > \lambda$. Differentiating (54) and using that $V'(z) - \lambda = \int_{\lambda}^{\bar{A}} (A - \lambda) d\Psi(A)$, λ solves:

$$(r + \pi)\,\lambda = \alpha \left[\int_{\lambda}^{\bar{A}} (A - \lambda)\, d\Psi(A) \right] = \alpha \int_{\lambda}^{\bar{A}} [1 - \Psi(A)]\, dA. \tag{56}$$

Equation (56) has the interpretation of an optimal stopping rule. According to the left side of (56), by spending its real balances the household saves the opportunity cost of holding money, as measured by $r + \pi$. According to the middle term in (56), if the household does not spend its real balances, then it must wait for the next preference shock with $A \geq \lambda$. Such a shock occurs with Poisson arrival rate $\alpha\,[1 - \Psi(\lambda)]$, in which case the expected surplus from spending one unit of real balances is $\mathbb{E}\,[A - \lambda |\, A \geq \lambda] = \int_{\lambda}^{\bar{A}} (A - \lambda)\, d\Psi(A)/\,[1 - \Psi(\lambda)]$. Finally, the right side of (56) is obtained by integration by parts. It is straightforward to check that there is a unique, $\lambda^{\star}$, solution to (56), and this solution is independent of the household's real balances as initially guessed. As inflation increases $\lambda^{\star}$ decreases and, in accordance with the "hot potato" effect, households spend their money holdings on goods for which they have a lower marginal utility of consumption. Given $\lambda^{\star}$ (12) describes the flow of consumption, $c^{\star}$, and hours, $h^{\star}$.

The real balances of a household who depleted its money holdings t periods ago are $z_t = (h^{\star} - c^{\star} + \pi\phi M)\,(1 - e^{-\pi t})/\pi$. The probability that a household does not receive a preference shock with $A \geq \lambda^{\star}$ over a time interval of length t is $e^{-\alpha[1-\Psi(\lambda^{\star})]t}$. Consequently,

$$F(z) = 1 - \left[\frac{h^{\star} - c^{\star} + \pi(\phi M - z)}{h^{\star} - c^{\star} + \pi\phi M} \right]^{\frac{\alpha[1-\Psi(\lambda)]}{\pi}} \quad \text{for all } z \leq \frac{h^{\star} - c^{\star} + \pi\phi M}{\pi}. \tag{57}$$

By market clearing, (18),

$$\phi M = \frac{h^{\star} - c^{\star}}{\alpha\,[1 - \Psi(\lambda^{\star})]}. \tag{58}$$

Aggregate real balances fall with inflation: because households save less, $h^{\star} - c^{\star}$ is lower, and because they spend their real balances more rapidly, $\alpha\,[1 - \Psi(\lambda^{\star})]$ increases. The velocity of money, denoted $\mathcal{V}$, is defined as nominal aggregate output divided by the stock of money. From (58),

$$\mathcal{V} \equiv \frac{h^{\star}}{\phi M} = \frac{\alpha\,[1 - \Psi(\lambda^{\star})]}{1 - \frac{c^{\star}}{h^{\star}}}. \tag{59}$$

The velocity of money increases with inflation for two reasons: households spend their real balances more often following preference shocks, $1 - \Psi(\lambda^{\star})$ increases, and the saving rate, $(h^{\star} - c^{\star})/h^{\star}$, decreases. A monetary equilibrium exists if $h^{\star} - c^{\star} > 0$, which holds if the inflation rate is not too large and the preference shocks are sufficiently frequent.

Finally, if preferences over flow consumption and leisure are also linear, then all households supply $\bar{h}$ provided that $\pi < \alpha \int_1^{\bar{A}} [1 - \Psi(A)] \, dA - r$. So inflation has no effect on aggregate output. Welfare at a steady-state monetary equilibrium is

$$\mathcal{W} = \alpha \int \int AzdF(z)d\Psi(A) - \bar{h} = \bar{h} \left[\frac{\int_{\lambda^\star}^{\bar{A}} Ad\Psi(A)}{1 - \Psi(\lambda^\star)} - 1 \right].$$

It is increasing with $\lambda^\star$ and hence decreasing with π. As inflation increases output is consumed by households with lower marginal utilities, which reduces social welfare.

7 Conclusion

We constructed a continuous-time, pure-currency economy in which households are subject to idiosyncratic preference shocks for lumpy consumption. We offered a complete characterization of steady-state equilibria for general preferences. We provided closed-form solutions for a class of equilibria where households fully deplete their money holdings periodically and for special classes of preferences. We studied both analytically and numerically a version of our economy with quasi-linear preferences resembling the New-Monetarist framework of Lagos and Wright (2005) and Rocheteau and Wright (2005). The equilibrium of this economy features a non-degenerate distribution of real balances and a trade-off for policy between self-insurance and risk sharing parameterized by labor productivity. We derived a number of analytical results on this policy trade-off.

We studied incentive-compatible transfers financed with money creation and designed such transfers to raise welfare by optimally trading off risk-sharing and self-insurance. We showed that the shape of the optimal transfers depends on labor productivity. We extended our analysis to monetary policy implemented with both money growth and illiquid nominal bonds. We used our results for pure currency economies to establish that liquidity traps occur when labor productivity is low and idiosyncratic risk is high. Money growth through "helicopter drops" accompanied by open-market operations to reduce the bond-money ratio are welfare enhancing.

Our model can easily be extended in several directions. It can incorporate search and bargaining in order to feature a non degenerate distribution of prices, as shown in Rocheteau, Weill, and Wong (2015). We adopted a discrete-time version of this model to study transitional dynamics following monetary shocks. The model remains highly tractable and delivers new insights for the short-run effects of money. One can incorporate idiosyncratic employment risk, e.g., by adding a frictional labor market, and private assets, such as capital and claims on firms' profits. The model will have implications for how the distribution of liquidity affects firms' entry, employment, and interest rates.

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Appendix A: Proofs of Propositions

PROOF OF THEOREM 1. The Theorem summarizes a series of results from the Supplementary Appendix, proved in a more general case in which there are two assets, money and an illiquid bond. Lemma I.3 shows that the Bellman equation has a unique bounded solution, and that this solution is concave, continuous, and increasing. Lemma I.5 shows that the value function is strictly increasing. Proposition I.7 shows that the value function is a viscosity solution of the HJB equation. Proposition I.13 uses this and other results to show that the value function is continuously differentiable. Proposition I.15 shows that the derivative of the value function is strictly decreasing. This implies that the value function is strictly concave, that it is twice continuously differentiable almost everywhere, and that it is a classical solution of the HJB equation, i.e., that it satisfies (8).

Proposition I.17 shows that the value function is twice continuously differentiable in a neighborhood of any $z > 0$, except perhaps if saving rate is zero, and under linear preferences if $W'(z) = 1$. Since there is a unique level of real balances such that the saving rate is zero (see Proposition 2), and since $W'(z)$ is strictly decreasing, this means that the value function is twice continuously differentiable except for two levels of real balances. Under SI preferences, Proposition III.4 shows that the value function is twice continuously differentiable even when the saving rate is zero. Under linear preferences, Lemma V.6 shows that, in equilibrium, the value function is twice continuously differentiable over the support of the distribution of real balances.

To derive the bound on $W'(0)$, consider some small $\varepsilon > 0$. By working full time, $h_t = \bar{h}$ and consuming nothing, the household can reach ε at time T_ε solving $z_{T_\varepsilon} = \varepsilon$, where $\dot{z}_t = \bar{h} + \Upsilon - \pi z_t$. Solving this ODE explicitly gives:

$$T_\varepsilon = -\frac{1}{\pi}\log\left(1 - \frac{\pi}{\bar{h}+\Upsilon}\varepsilon\right) = \frac{\varepsilon}{\bar{h}+\Upsilon} + o(\varepsilon).$$

Since utility flows are bounded below by zero, we must have that $W(0) \geq e^{-(r+\alpha)T_\varepsilon}W(\varepsilon)$, which implies in turn that:

$$0 \leq W(\varepsilon) - W(0) \leq \left(1 - e^{-(r+\alpha)T_\varepsilon}\right)W(\varepsilon).$$

Dividing both side by ε and taking the limit $\varepsilon \to 0$, we obtain that:

$$W'(0) \leq \frac{r+\alpha}{\bar{h}+\Upsilon}W(0).$$

By taking the sup norm on both sides of (4) we obtain that $r\|W\| \leq \|u\| + \alpha\|U\|$, and the result follows. Finally, the result that $\lim_{z\to\infty} W'(z) = 0$ follows from the fact that $W(z)$ is concave and bounded, while the result that $\lim_{z\to 0} W''(z) = \infty$ is proved in Corollary I.20 of the Supplementary Appendix. ■

PROOF OF PROPOSITION 1. The results follow directly because $U(y)$ and $W(z)$ are both strictly concave and continuously differentiable, because $U'(0) = \infty$ while $W'(0) < \infty$, and because $U'(\infty) = W'(\infty) = 0$. ■

PROOF OF PROPOSITION 2. Note that the saving rate correspondence can be written as $s(z) = h - c - \pi z + \Upsilon$, where $(h, c) \in X(\lambda)$ and $X(\lambda) = \arg\max \left\{u(c, \bar{h} - h) + \lambda\left(h - c - \pi z + \Upsilon\right)\right\}$, with respect to $c \geq 0$, $0 \leq h \leq \bar{h}$. With linear preferences, $X(\lambda) = (\bar{c}, 0)$ if $\lambda < 1$, $X(\lambda) = [0, \bar{c}] \times [0, \bar{h}]$ if $\lambda = 1$ and $X(\lambda) = (0, \bar{h})$ if $\lambda > 1$. With SI preferences, one can easily check that $X(\lambda)$ is singled-valued and continuous, that the optimal consumption choice, $c(\lambda)$, is strictly decreasing, and that the optimal labor choice, $h(\lambda)$, is increasing (see Lemma I.11 in the Supplementary Appendix for details). Combined with the fact, established in Theorem 1, that $W'(z)$ is strictly decreasing and continuous, all the statements of the Lemma follow except for $s(z) > 0$.

To establish that $s(z) > 0$ near zero, recall that the value function is twice differentiable almost everywhere. Consider any $z > 0$ such that $W''(z)$ exists. Then, we can apply the envelope condition to the right side of the HJB equation (see Theorem 1 in Milgrom and Segal, 2002). We obtain that:

$$(r + \alpha + \pi)W'(z) = \alpha V'(z) + W''(z)s(z). \tag{60}$$

From Proposition 1 we have that $V'(z) = U'\left[y(z)\right]$. Since $y(z) \leq z$ and $\lim_{z\to 0} U'(z) = \infty$, it follows that $\lim_{z\to 0} V'(z) = \infty$. From $W'(0) < \infty$ and (60), $\lim_{z\to 0} W''(z)s(z) = -\infty$. Since $W''(z) \leq 0$, it then follows that $s(z) > 0$ for some z close enough to zero. Since $s(z)$ is decreasing, it follows that $s(z) > 0$ for all z close enough to zero. ■

PROOF OF PROPOSITION 3. The proof is based on results from two sections of the Supplementary Appendix: Section III.3, which studies the initial value problem in the case of SI preferences, and Section IV.2, which explicitly solves for the solution to this problem in the case of linear preferences. ■

PROOF OF PROPOSITION 4. See Section V.1 of the Supplementary Appendix for the detailed application of Theorem 12.12 and 12.13 in Stokey, Lucas, and Prescott (1989). ■

Lemma 1 *At the target level of real balances, $z^\star$:*

$$W'(z^\star) = \frac{\alpha}{r + \alpha + \pi} V'(z^\star).$$

PROOF OF PROPOSITION 5. First, we note that the stationary distribution cannot be concentrated at $z = 0$, since $Q(z, \{0\}) = 0$ for all z. Hence, when $\phi = 0$, the left-hand side of (18) is zero and so is less than the right-hand side, which is strictly positive. When $\phi \to \infty$, we have from the upper bound of Theorem 1 that $W'(z) \to 0$ for all $z \in [0, \infty)$. This implies that labor supply is zero and consumption is strictly positive for all $z \in [0, z^\star]$, hence the saving rate is $s(z) < -\pi z + \Upsilon$. Plugging $s(z^\star) = 0$, it follows that $z \leq z^\star < \Upsilon/\pi$ for all real balances z in the support of the stationary distribution, $[0, z^\star]$, implying that the right-hand side of (18) is less than the left-hand side. Finally, note that (18) is continuous in ϕ because, by Proposition 4, the stationary distribution is continuous in ϕ in the sense of weak convergence. The result then follows by an application of the intermediate value theorem. ■

PROOF OF LEMMA 1. Note that, at $z^\star$, there exists some optimal consumption and labor choices, $(c^\star, h^\star)$ such that $h^\star - c^\star - \pi z^\star + \Upsilon = 0$. Hence, from the HJB:

$$\begin{aligned}(r+\alpha)W(z) &\geq u(c^\star, \bar{h} - h^\star) + \alpha V(z) + W'(z)\left(h^\star - c^\star - \pi z + \Upsilon\right) \\ (r+\alpha)W(z^\star) &= u(c^\star, \bar{h} - h^\star) + \alpha V(z^\star),\end{aligned}$$

where the inequality in the first equation follows because we evaluate the right side of the HJB at a point that may not achieve the maximum. Taking the difference between these two equations, and recalling that $h(z^\star) - c(z^\star) - \pi z^\star + \Upsilon = 0$, we obtain that:

$$(r+\alpha)\left[W(z) - W(z^\star)\right] \geq \alpha\left[V(z) - V(z^\star)\right] - \pi W'(z)\left(z - z^\star\right)$$

The result follows by dividing both sides by $z - z^\star$, for $z > z^\star$ and then for $z < z^\star$, and taking the limit as $z \to z^\star$, keeping in mind that $V(z)$ is differentiable at $z^\star$ and that $W(z)$ is continuously differentiable. ■

Lemma 2 *Under either SI or linear preferences, for $z \in [0, z^\star]$:*

$$W'(z) = \frac{\alpha}{r+\alpha+\pi}\int_z^{z^\star} V'(x)dG(x \,|\, z), \;\; \textit{where } G(x \,|\, z) \equiv 1 - e^{-(r+\alpha+\pi)[\mathcal{T}(x_+) - \mathcal{T}(z)]}$$

PROOF OF LEMMA 2. First, recall from Theorem 1 that the value function is twice continuously differentiable over $(0, \infty)$, except perhaps under linear preferences, when this property may not hold for at most two points. Hence, we can take derivatives on the right side of the HJB equation along the path of real balances z_t, except perhaps at two points. Applying the envelope condition, we obtain that:

$$(r+\alpha+\pi)W'(z_t) = \alpha V'(z_t) + W''(z_t)\,\dot{z}_t,$$

if $z_t < z^\star$, except perhaps at two points. At $z = z^\star$, Lemma 1 shows that

$$(r+\alpha+\pi)W'(z^\star) = \alpha V'(z^\star).$$

In all cases we can integrate this formula forward over the time interval $[t, \mathcal{T}(z^\star)]$ and we obtain that:

$$W'(z_t) = \int_t^{\mathcal{T}(z^\star)} \alpha V'(z_s) e^{-(r+\alpha+\pi)(s-t)}\, ds + e^{-(r+\alpha+\pi)[\mathcal{T}(z^\star)-t]} \frac{\alpha V'(z^\star)}{r+\alpha+\pi}. \tag{61}$$

Consider the integral on the right side of (61). The inverse of z_t when restricted to the time interval $[0, \mathcal{T}(z^\star)]$ is the strictly increasing function $\mathcal{T}(x)$, the time to reach the real balances x starting from time zero. Let $M(x) \equiv 1 - e^{-(r+\alpha+\pi)[\mathcal{T}(x)-t]}$ and note that $M \circ z(s) = 1 - e^{-(r+\alpha+\pi)(s-t)}$. With these notations, the first integral can be written:

$$\begin{aligned} \int_t^{\mathcal{T}(z^\star)} \alpha V'(z_s)\, e^{-(r+\alpha+\pi)(s-t)}\, ds &= \frac{\alpha}{r+\alpha+\pi} \int_t^{\mathcal{T}(z^\star)} V' \circ z_s \, d\left[M \circ z_s\right] \\ = \quad \frac{\alpha}{r+\alpha+\pi} \int_{x \in [z, z^\star]} V'(x)\, dM(x) &= \frac{\alpha}{r+\alpha+\pi} \int_{x \in [z, z^\star)} V'(x)\, dG(x \,|\, z). \end{aligned}$$

where the second equality follows by an application of the change of variable formula for Lebesgue-Stieltjes integral (see Carter and van Brunt, 2000, Theorem 6.2.1), and the second line follows because $G(x \,|\, z) = M(x)$ for all $x \in [z, z^\star)$. The result follows by noting that the second integral can be written: $\frac{\alpha V'(z^\star)}{r+\alpha+\pi} \times \left[G(z^\star \,|\, z) - G(z^\star_- \,|\, z)\right]$. ■

PROOF OF PROPOSITION 6. To establish the first point of the Proposition, we note that, at the target $z^\star$, $h^\star - c^\star - \pi z^\star + \Upsilon = 0$, where $(c^\star, h^\star)$ are optimal consumption and labor choices when $z = z^\star$. Since, in equilibrium, $\Upsilon = \pi \int_0^{z^\star} z dF(z) < \pi z^\star$, we obtain that $h^\star - c^\star > 0$. This implies that the marginal value of real balances satisfies $W'(z^\star) \geq \underline{\lambda} > 0$, where the constant $\underline{\lambda}$ is independent of the rate of inflation, π. With linear preferences, $\underline{\lambda} = 1$. With SI preferences, $\underline{\lambda}$ solves $h(\underline{\lambda}) - c(\underline{\lambda}) = 0$. Next, we use Lemma 1:

$$(r+\alpha+\pi)W'(z^\star) = U'\left[y(z^\star)\right].$$

Since $W'(z^\star) \geq \underline{\lambda}$, this implies that $\lim_{\pi\to\infty} y(z^\star) = 0$. Finally, since we have established in Theorem 1 that $W'(0) \leq (r+\alpha)/\overline{h} \times (\|u\| + \alpha\|U\|)/r$, we obtain that $W'(0) < U'\left[y(z^\star)\right]$ for π large enough. Therefore, the solution of the optimal lumpy consumption problem is $y(z^\star) = z^\star$, i.e., there is full depletion. We conclude that $\lim_{\pi\to\infty} z^\star = \lim_{\pi\to\infty} y(z^\star) = 0$.

The second part of the Proposition, which deals with linear preference, requires some notations and results from Section 3.5. The proof can be found at the beginning of the proof of Proposition 8, in the paragraph "(i) Large labor endowments". ■

PROOF OF PROPOSITION 7. From (27) $\lim_{\bar{h}\to\infty} F(z) = 0$ for all $z < z^\star$ and $F(z) = 1$ for all $z \geq z^\star$. From (28) $\lim_{\bar{h}\to\infty} \phi M = z^\star$. Finally, from (29) we compute the value function in closed form:

$$W(z) = z + W(z^\star) - z^\star - \frac{\alpha}{r+\alpha}\int_z^{z^\star}\left[1 - e^{-\frac{(r+\alpha)(u-z)}{\bar{h}}}\right]\left[U'(u) - U'(z^\star)\right] du \quad \forall z < z^\star \tag{62}$$

$$W(z^\star) = \frac{\alpha}{r}\left\{U(z^\star) - z^\star - \frac{\alpha}{r+\alpha}\int_0^{z^\star}\left[1 - e^{-\frac{(r+\alpha)u}{\bar{h}}}\right]\left[U'(u) - U'(z^\star)\right] du\right\}. \tag{63}$$

From (62) $\lim_{\bar{h}\to\infty} W(z) = z + W(z^\star) - z^\star$ and from (63) $\lim_{\bar{h}\to\infty} W(z^\star) = \alpha\left[U(z^\star) - z^\star\right]/r$. ■

PROOF OF PROPOSITION 8. Part (i): Large labor endowment. Fix some $\pi \geq 0$. We first note that $y(z^\star) \leq z^\star \leq z_s$, hence equilibrium aggregate demand is bounded by αz_s independently of $\bar{h}$. Equilibrium aggregate supply can be written:

$$F(z_-^\star)\bar{h} + \left[1 - F(z_-^\star)\right] h^\star.$$

To remain bounded as $\bar{h} \to \infty$, it must be the case that $\lim_{\bar{h}\to\infty} F(z_-^\star) = 0$. This also implies that, for $\bar{h}$ large enough, there is an atom at $z^\star$, so that $W'(z^\star) = 1$ and $z^\star = z_s$. Because F converges to a Dirac distribution concentrated at $z^\star = z_s$, we have that $\lim_{\bar{h}\to\infty} \phi M = z_s$.

Next we argue that, as $\bar{h}$ is large enough, $y(z^\star) = y(z_s) = z_s$, i.e., all equilibria must feature full depletion. For this we use the expression for $W'(z)$ derived in Lemma 2:

$$\begin{aligned} W'(0) &= \frac{\alpha}{r+\alpha+\pi}\int_0^{z^\star} U'\left[y(z)\right] dG(z\,|\,0) \leq \frac{\alpha}{r+\alpha+\pi}\int_0^{z^\star} \max\{U'(z), W'(0)\}\, dG(z\,|\,0) \\ &\leq \frac{\alpha}{r+\alpha+\pi}\left[G(z_s^-\,|\,0)\int_{z\in[0,z_s)} \max\left\{U'(z), W'(0)\right\}\frac{dG(z\,|\,0)}{G(z_s^-\,|\,0)} + \left[1 - G(z_s^-\,|\,0)\right]\max\{U'(z_s), W'(0)\}\right], \end{aligned}$$

as long as $\bar{h}$ is large enough. To obtain the inequality of the first line, we have used that $U'\left[y(z)\right] = U'(z)$ if there is full depletion, while $U'\left[y(z)\right] = W'\left[z - y(z)\right] \leq W'(0)$ if there is partial depletion. To obtain the second line, we have used that $z^\star = z_s$ as long as $\bar{h}$ is large enough. Substituting the expression for $\mathcal{T}(z\,|\,\pi\phi M)$ into the definition of $G(z\,|\,0)$, we obtain that:

$$G(z\,|\,0) = \begin{cases} 1 - \left(1 - \frac{z}{z_b}\right)^{1+\frac{r+\alpha}{\pi}} & \text{if} \quad z < z_s \\ 1 & \text{if} \quad z = z_s. \end{cases}$$

Given that z_b goes to infinity as $\bar{h}$ goes to infinity, one sees that $G(z\,|\,0)$ converges weakly to a Dirac distribution concentrated at z_s. We also have:

$$\frac{G'(z\,|\,0)}{G(z_s^-\,|\,0)} = \left(1 + \frac{r+\alpha}{\pi}\right)\frac{\frac{1}{z_b}\left(1 - \frac{z}{z_b}\right)^{\frac{1+\alpha}{\pi}}}{1 - \left(1 - \frac{z_s}{z_b}\right)^{1+\frac{1+\alpha}{\pi}}} \leq \frac{\left(1 + \frac{r+\alpha}{\pi}\right)\frac{1}{z_b}}{1 - \left(1 - \frac{z_s}{z_b}\right)^{1+\frac{1+\alpha}{\pi}}} \to \frac{1}{z_s},$$

as $\bar{h}$ goes to infinity (since it implies $z_b \to \infty$). Thus, the conditional probability distribution, $G(z \,|\, 0)/G(z_s^- \,|\, 0)$, has a density that can be bounded uniformly in $\bar{h}$. Finally, our bound for $W'(0)$ in Theorem 1 can be written, in the case of linear preferences, as

$$W'(0) \leq \frac{r+\alpha}{\bar{h}} \left(\frac{\bar{h}+\bar{c}}{r} + \alpha \frac{\|U\|}{r} \right) \to 1 + \frac{\alpha}{r},$$

as $\bar{h} \to \infty$. Taken together, these observations imply that:

$$\int_{z \in [0, z_s)} \max \left\{ U'(z), W'(0) \right\} \frac{dG(z \,|\, 0)}{G(z_s^- \,|\, 0)} \leq \frac{1}{z_s} \int_0^{z_s} \max \left\{ U'(z), 1 + \frac{\alpha}{r} \right\} \, dz + \varepsilon$$

for some $\varepsilon > 0$ as long as $\bar{h}$ is large enough (note that the integral on the right side is well defined since $U(z) = \int_0^z U'(z)\, dx$). Together with the fact that $G(z_s^- \,|\, 0) \to 0$ as $\bar{h} \to \infty$, we obtain that:

$$G(z_s^- \,|\, 0) \int_{z \in [0, z_s)} \max \left\{ U'(z), W'(0) \right\} \frac{dG(z \,|\, 0)}{G(z_s^- \,|\, 0)} \to 0 \text{ and } 1 - G(z_s^- \,|\, 0) \to 1$$

as $\bar{h} \to \infty$. Hence, for any $\varepsilon > 0$, $W'(0) \leq \frac{\alpha}{r+\alpha+\pi} \max\{U'(z_s), W'(0)\} + \varepsilon$ as long as $\bar{h}$ is large enough. Picking $\varepsilon < \frac{r+\alpha}{r+\alpha+\pi} U'(z_s)$, we obtain that $W'(0) < \max\{W'(0), U'(z_s)\}$, which implies that $W'(0) < U'(z_s)$, for $\bar{h}$ large enough, i.e., there is full depletion.

Because $H = \alpha \phi M$ under full depletion, and because the distribution of real balances converges towards a Dirac distribution concentrated at z_s, we obtain that $\lim_{\bar{h} \to \infty} H = H^{\infty}(\pi) = \alpha z_s$, which is decreasing in π. Aggregate welfare can be written as $\alpha \int U(z)\, dF(z) - H$, the average utility enjoyed from lumpy consumption net of the average disutility of supplying labor. As $\bar{h} \to \infty$, F converges weakly to a Dirac distribution concentrated at z_s, and H converges to αz_s. It follows that welfare converges to $\mathcal{W}^{\infty} = \alpha \left[U(z_s) - z_s \right]$, which is decreasing with π.

Part (ii): Small labor endowment. We have shown that there exists a unique candidate equilibrium with full depletion. In this candidate equilibrium, the condition for binding labor is that $z_s \geq z_b$ or, using the definition of z_s:

$$U'(z_b) \leq 1 + \frac{r+\pi}{\alpha}.$$

Recall that $z_b = \frac{\bar{h}}{\pi} + \frac{\bar{h}}{\alpha}$ is an increasing function of $\bar{h}$. Since marginal utility is decreasing, the condition for binding labor can be written:

$$\bar{h} \in \left[0, \bar{H}(\pi)\right] \text{ where } \bar{H}(\pi) = \frac{\alpha \pi}{\alpha + \pi} \left(U'\right)^{-1} \left(1 + \frac{r+\pi}{\alpha}\right).$$

One immediately sees that $\lim_{\pi \to 0} \bar{H}(\pi) = \lim_{\pi \to \infty} \bar{H}(\pi) = 0$.

Next, we turn to the sufficient condition for full depletion. Using Lemma 2 we have, in the candidate equilibrium with full depletion:

$$W'(0) = \frac{\alpha}{r+\alpha+\pi}\int_0^{z_b} U'(z)dG(z) \text{ where } G(z) = 1-\left(1-\frac{z}{z_b}\right)^{1+\frac{r+\alpha}{\pi}}.$$

Substituting the expression for $G(z)$ in the integral, we obtain:

$$W'(0) = \frac{\alpha}{\pi}\frac{1}{z_b}\int_0^{z_b} U'(z)\left(1-\frac{z}{z_b}\right)^{\frac{r+\alpha}{\pi}} dz \leq \frac{\alpha}{\pi}\frac{U(z_b)}{z_b},$$

where the inequality follows by using $(1-z/z_b)^{\frac{r+\alpha}{\pi}} \leq 1$, integrating, and keeping in mind that $U(0)=0$. Full depletion obtains if $W'(0) \leq U'(z_b)$. Using the above upper bound for $W'(0)$, we obtain that a sufficient condition for full depletion is:

$$\frac{\pi}{\alpha} \geq \frac{U(z_b)}{z_b U'(z_b)}.$$

Note that $z_b \leq (U')^{-1}(1)$, that the function $z \mapsto [U(z)-U(0)]/[zU'(z)]$ is continuous over $(0,(U')^{-1}(1)]$ and, by our maintained assumption in the Lemma, bounded near zero. Hence, it is bounded over the closed interval $[0,(U')^{-1}(1)]$. Therefore, the condition for full depletion is satisfied if:

$$\pi \geq \underline{\pi} \equiv \alpha \times \sup_{z\in[0,(U')^{-1}(1)]} \frac{U(z)}{zU'(z)}.$$

Output effect of inflation. In the regime with binding labor, $h(z)=\bar{h}$ for all $z \in \text{supp}(F)$. Hence, for all $\bar{h} \in \left[0,\hat{h}\right]$ and for all $\pi \in [\underline{\pi},\bar{\pi}]$, $H=\bar{h}$.

Welfare effect of inflation. From (34) in the regime with binding labor, $\phi M = \bar{h}/\alpha$. Hence, an increase in the money growth rate through lump-sum transfers is a mean-preserving reduction in the distribution of real balances. In this regime social welfare is measured by

$$\mathcal{W} = \int\left[-h(z)+\alpha U(z)\right]dF(z) = -\bar{h}+\alpha\int U(z)dF(z).$$

Given the strict concavity of $U(y)$ money growth leads to an increase in welfare.

Part (iii): Large inflation. From (31), as $\pi\to\infty$, $z^\star\to 0$, $\phi M\to 0$, $H\to 0$, and $\mathcal{W}\to 0$. ■

PROOF OF PROPOSITION 9.

The proof is structured as follows. Given a policy, (π,τ), we conjecture that households behave as follows: $y(z)=z$ for all $z\in[0,z_0^\star]$; $h(z)=\bar{h}$ for all $z<z_0^\star$, and $h(z_0^\star)=0$. We also assume that parameters are such that $\bar{h}+\tau(z)-\pi z>0$ for all $z\in[0,z_0^\star)$. Given this conjecture we will show that: (i) Aggregate real balances under τ are larger than under laissez faire ($\tau_0 = \tau_1 = \tau_z = 0$). (ii) Welfare under τ is larger than under laissez-faire. The second part of the proof will consist in checking that: (iii) For π small enough, there is a transfer scheme, τ, of the form described in (35), that balances the government budget; (iv) Households' conjectured behavior is optimal.

Guessing that the equilibrium features full depletion, and keeping in mind that $\tau(z_0^\star) = \pi z_0^\star$ by construction, the government budget constraint under the transfer scheme, τ, is:

$$\int [\tau(z) - \pi z] \, dF_\tau(z) = \int_0^{\mathcal{T}(z_0^\star;\tau)} \{\tau [z(t)] - \pi z(t)\} \, \alpha e^{-\alpha t} dt = 0, \tag{64}$$

where $\mathcal{T}(z_0^\star;\tau)$ is the time to accumulate $z_0^\star$ under the transfer scheme τ and $z(t)$ is the solution to

$$\begin{aligned} \dot{z} &= \bar{h} + \tau(z) - \pi z \quad \text{for all } z < z_0^\star. \\ &= 0 \text{ if } z = z_0^\star. \end{aligned} \tag{65}$$

We denote $Z_\tau \equiv \int [1 - F_\tau(z)] \, dz$ the aggregate real balances under the transfer scheme, τ, and $Z_0 \equiv \int [1 - F_0(z)] \, dz$ the aggregate real balances under laissez faire. Moreover, denote $\mathcal{T}_\tau \equiv \mathcal{T}(z_0^\star;\tau)$ and $\mathcal{T}_0 = \mathcal{T}(z_0^\star, 0)$ under laissez-faire.

RESULT #1: $\mathcal{T}_\tau > \mathcal{T}_0$ and $Z_\tau > Z_0$.

PROOF: By construction the transfer scheme in (35) is such that there is a level of real balances, $z_{\hat{t}}$, with $\hat{t} \in (0, \mathcal{T}_\tau)$, below which the net transfer to the household is positive, since $\tau_0 > 0$, and above which the net transfer is negative, since from (64) the sum of those transfers must be 0:

$$\begin{aligned} \tau(z_t) - \pi z_t &> 0 \text{ for all } t \in \left(0, \hat{t}\right) \\ \tau(z_t) - \pi z_t &< 0 \text{ for all } t \in \left(\hat{t}, \mathcal{T}_\tau\right). \end{aligned}$$

Dividing the government budget constraint by $\alpha e^{-\alpha \hat{t}}$, (64) becomes:

$$\int_0^{\hat{t}} [\tau(z_t) - \pi z_t] \frac{\alpha e^{-\alpha t}}{\alpha e^{-\alpha \hat{t}}} dt + \int_{\hat{t}}^{\mathcal{T}_\tau} [\tau(z_t) - \pi z_t] \frac{\alpha e^{-\alpha t}}{\alpha e^{-\alpha \hat{t}}} dt = 0. \tag{66}$$

Given $\hat{t}$, $\alpha e^{-\alpha t}/\alpha e^{-\alpha \hat{t}}$ is decreasing in t, $\alpha e^{-\alpha t}/\alpha e^{-\alpha \hat{t}} > 1$ for all $t < \hat{t}$ and $\alpha e^{-\alpha t}/\alpha e^{-\alpha \hat{t}} < 1$ for all $t > \hat{t}$. It follows that

$$\int_0^{\hat{t}} [\tau(z_t) - \pi z_t] \frac{\alpha e^{-\alpha t}}{\alpha e^{-\alpha \hat{t}}} dt > \int_0^{\hat{t}} [\tau(z_t) - \pi z_t] \, dt \tag{67}$$

$$\int_{\hat{t}}^{\mathcal{T}_\tau} [\tau(z_t) - \pi z_t] \frac{\alpha e^{-\alpha t}}{\alpha e^{-\alpha \hat{t}}} dt < \int_{\hat{t}}^{\mathcal{T}_\tau} [\tau(z_t) - \pi z_t] \, dt. \tag{68}$$

From (66) and the two inequalities, (67)-(68),

$$\int_0^{\mathcal{T}_\tau} [\tau(z_t) - \pi z_t] \frac{\alpha e^{-\alpha t}}{\alpha e^{-\alpha \hat{t}}} dt = 0 > \int_0^{\mathcal{T}_\tau} [\tau(z_t) - \pi z_t] \, dt. \tag{69}$$

From (65) and (69),

$$\int_0^{\mathcal{T}_\tau} \left[\tau\left(z_t\right) - \pi z_t\right] dt = \int_0^{\mathcal{T}_\tau} \left[\dot{z}_t - \bar{h}\right] dt = z_0^\star - \bar{h}\mathcal{T}_\tau < 0,$$

where we used that $z_0 = 0$ and $z_{\mathcal{T}_\tau} = z_0^\star$. So $\mathcal{T}_\tau > \mathcal{T}_0 = z_0^\star/\bar{h}$. As a result the measure of households holding their targeted real balances is

$$1 - F_\tau(z_0^\star) = e^{-\alpha \mathcal{T}_\tau} < e^{-\alpha z_0^\star/\bar{h}} = 1 - F_0(z_0^\star).$$

The law of motion for aggregate real balances is $\dot{Z}_\tau = F_\tau(z_0^\star)\bar{h} - \alpha Z_\tau$. At a steady state, $Z_\tau = F_\tau(z_0^\star)\bar{h}/\alpha$, which is larger than $Z_0 = F_0(z_0^\star)\bar{h}/\alpha$ under laissez faire.■

Social welfare is measured by the sum of utilities across households:

$$\mathcal{W}_\tau = \int \left[-h(z) + \alpha U(z)\right] dF_\tau(z) = \alpha \int \left[-z + U(z)\right] dF_\tau(z), \tag{70}$$

where the second equality is obtained by market clearing, $\int h(z) dF_\tau(z) = \alpha \int y(z) dF_\tau(z) = \alpha \int z dF_\tau(z)$. Using that $U(z) - z = \int_0^z \left[U'(x) - 1\right] dx + U(0)$ (70) can be rewritten as

$$\mathcal{W}_\tau = \alpha \int \int \left[U'(x) - 1\right] \mathbb{I}_{\{0 \leq x \leq z\}} dx dF_\tau(z) + \alpha U(0). \tag{71}$$

Changing the order of integration,

$$\begin{aligned} \int \int \left[U'(x) - 1\right] \mathbb{I}_{\{0 \leq x \leq z\}} dx dF_\tau(z) &= \int \int \mathbb{I}_{\{0 \leq x \leq z\}} dF_\tau(z) \left[U'(x) - 1\right] dx \\ &= \int \left[1 - F_\tau(z)\right] \left[U'(x) - 1\right] dx. \end{aligned} \tag{72}$$

Plugging (72) into (71):

$$\mathcal{W}_\tau = \alpha \int \left[1 - F_\tau(x)\right] \left[U'(x) - 1\right] dx + \alpha U(0). \tag{73}$$

RESULT #2: *Social welfare under τ is higher than welfare at the laissez-faire.*

PROOF: The welfare gain under τ relative to laissez faire is:

$$\begin{aligned} \mathcal{W}_\tau - \mathcal{W}_0 &= \alpha \int \left[1 - F_\tau(x)\right] \left[U'(x) - 1\right] dx - \alpha \int \left[1 - F_0(x)\right] \left[U'(x) - 1\right] dx \\ &= \alpha \int \left[F_0(x) - F_\tau(x)\right] \left[U'(x) - 1\right] dx. \end{aligned} \tag{74}$$

Given our conjecture that the equilibrium features full depletion we have:

$$F_\tau\left(z_{\tau,t}\right) = F_0\left(z_{0,t}\right) = 1 - e^{-\alpha t},$$

where $z_{\tau,t}$ and $z_{0,t}$ denote the real balances of a household who received its last preference shock t periods ago under the transfer scheme τ and under laissez-faire, respectively. Integrating the law of motion of real balances, (65):

$$\begin{aligned} z_{0,t} &= \bar{h}t \text{ for all } t < z_0^\star/\bar{h} \\ z_{\tau,t} &= \bar{h}t + \int_0^t \left[\tau\left(z_{\tau,x}\right) - \pi z_{\tau,x}\right] dx \text{ for all } t \le \mathcal{T}_\tau. \end{aligned}$$

By definition of the transfer scheme,

$$\begin{aligned} \tau\left(z_t\right) - \pi z_t &> 0 \text{ for all } t \in \left(0, \hat{t}\right) \\ \tau\left(z_t\right) - \pi z_t &< 0 \text{ for all } t \in \left(\hat{t}, \mathcal{T}_\tau\right), \end{aligned}$$

and, from (64), $\int_0^{\mathcal{T}_\tau} \left[\tau\left(z_{\tau,x}\right) - \pi z_{\tau,x}\right] dx < 0$. It follows that there is a $\tilde{t} \in \left(\hat{t}, \mathcal{T}_0\right)$ such that $z_{0,\tilde{t}} = z_{\tau,\tilde{t}} = z_s$. For all $t \in \left(0, \tilde{t}\right)$, $z_{0,t} < z_{\tau,t}$. For all $t \in \left(\tilde{t}, \mathcal{T}_\tau\right)$ and $z_{0,t} > z_{\tau,t}$. Equivalently, $F_\tau(z) < F_0(z)$ for all $z < z_s$ and $F_\tau(z) > F_0(z)$ for all $z > z_s$. From (74):

$$\mathcal{W}_\tau - \mathcal{W}_0 = \alpha \left\{ \int_0^{z_s} \left[F_0(x) - F_\tau(x)\right] \left[U'(x) - 1\right] dx + \int_{\tilde{z}}^{z_0^\star} \left[F_0(x) - F_\tau(x)\right] \left[U'(x) - 1\right] dx \right\}. \quad (75)$$

By the definition of z_s and the fact that $U'(z)$ is decreasing:

$$\begin{aligned} \left[F_0(x) - F_\tau(x)\right] \left[U'(x) - 1\right] &> \left[F_0(x) - F_\tau(x)\right] \left[U'(\tilde{x}) - 1\right] \quad \text{for all } x \in (0, z_s) \\ \left[F_0(x) - F_\tau(x)\right] \left[U'(x) - 1\right] &> \left[F_0(x) - F_\tau(x)\right] \left[U'(\tilde{x}) - 1\right] \quad \text{for all } x \in (z_s, z_0^\star). \end{aligned}$$

Plugging these two inequalities into (75):

$$\mathcal{W}_\tau - \mathcal{W}_0 \ge \alpha \left[U'(z_s) - 1\right] \int_0^{z_0^\star} \left[F_0(x) - F_\tau(x)\right] dx. \quad (76)$$

We proved that

$$\int z dF_\tau(z) = \int \left[1 - F_\tau(z)\right] dz \ge \int z dF_0(z) = \int \left[1 - F_0(z)\right] dz.$$

Hence, $\int_0^{z_0^\star} \left[F_0(x) - F_\tau(x)\right] dx > 0$. Moreover, $U'(z_s) \ge U'(z_0^\star) = 1 + r/\alpha$. Hence, $U'(\hat{z}) - 1 > 0$. It follows from (76) that $\mathcal{W}_\tau > \mathcal{W}_0$.■

The transfer scheme, τ, is fully characterized by π and τ_0 since $\tau_z = \left(\pi z_0^\star - \tau_0\right) / \left(z_0^\star - z_\pi^\star\right)$ and $\tau_1 = \left(\pi z_\pi^\star - \tau_0\right) z_0^\star / \left(z_0^\star - z_\pi^\star\right)$. We now establish that for a given inflation rate, π, there exists a lump-sum component, τ_0, that balances the government budget.

RESULT #3: *For π sufficiently small, there is a $\tau_0 \in (0, \pi z_\pi^\star)$ such that $\int \left[\tau(z) - \pi z\right] dF_\tau(z) = 0$ holds and $\dot{z}_t > 0$ for all $t \in [0, \mathcal{T}(z_0^\star))$.*

PROOF: The government budget constraint, (64), can be re-expressed as

$$\Gamma(\tau_0) \equiv \int_0^{\mathcal{T}(z_0^\star,\tau_0)} \{\tau[z(t)] - \pi z(t)\}\, \alpha e^{-t} dt = 0,$$

By direct integration of the ODE for real balances, (65), one obtains that both $z(t)$ and $\mathcal{T}(z_0^\star, \tau_0)$ are continuous functions of τ_0. Since $\tau(z)$ is, by construction, continuous in z, we obtain that $\Gamma(\tau_0)$ is also continuous in τ_0. If $\tau_0 = 0$, then from (35):

$$\tau(z) - \pi z = \begin{cases} -\pi z & \text{if} \quad z \leq z_\pi^\star \\ \left(\frac{\pi z_\pi^\star}{z_0^\star - z_\pi^\star}\right)(z - z_0^\star) & \quad\; z \in (z_\pi^\star, z_0^\star] \end{cases}.$$

Hence, $\tau(z_t) - \pi z_t < 0$ for all $t \in (0, \mathcal{T}(z_0^\star))$ and $\Gamma(0) < 0$. If $\tau_0 = \pi z_\pi^\star$, then from (35):

$$\tau(z) = \begin{cases} \pi z_\pi^\star & \text{if} \quad z \leq z_\pi^\star \\ \pi z & \quad\; z \in (z_\pi^\star, z_0^\star] \end{cases}$$

Consequently, $\tau[z(t)] - \pi z(t) > 0$ for all $t < \mathcal{T}(z_\pi^\star)$ and $\tau(z_t) - \pi z_t = 0$ for all $t \geq T_\pi^\star$. Hence, $\Gamma(\pi z_\pi^\star) > 0$. By the Intermediate Value Theorem there is $\tau_0 \in (0, \pi z_\pi^\star)$ such that $\Gamma(\tau_0) = 0$. Finally, for the transfer scheme to be feasible, it must be that $\dot{z} > 0$ for all $z < z_0^\star$. This requires $\bar{h} + \tau_0 - \pi z_\pi^\star > 0$, since net transfers achieve their minimum at $z = z_\pi^\star$. This condition will be satisfied for π sufficiently small. ■

Finally, we need to check that household's conjectured behavior is optimal: households find it optimal to supply $\bar{h}$ units of labor until they reach $z_0^\star$ and to deplete their money holdings in full when a preference shock occurs. The ODE for the marginal value of money is:

$$(r + \pi)\lambda_t = \alpha\left[U'(z_t) - \lambda_t\right] + \lambda_t \tau'(z_t) + \dot{\lambda}_t. \tag{77}$$

RESULT #4: *For* π and τ_0 *sufficiently small the solution to (77) is such that:* $\lambda_t > 1$ *for all* $t < T_\tau$*,* $\lambda_{T_\tau} = 1$*, and* $\lambda_t \leq U'(z_0^\star)$ for all $t \in [0, T_\tau]$.

PROOF: Integrating (77), we obtain that the marginal value of money solves:

$$\lambda_t = 1 + \int_t^{\mathcal{T}(z_\pi^\star)} e^{-(r+\pi+\alpha)(s-t)} \alpha \left\{U'(z_s) - U'(z_\pi^\star)\right\} ds + e^{-(r+\pi+\alpha)[\mathcal{T}(z_\pi^\star)-t]}\left[\lambda_{\mathcal{T}(z_\pi^\star)} - 1\right], \tag{78}$$

for all $t \leq \mathcal{T}_\pi^\star$, and

$$\lambda_t = 1 + \int_t^{\mathcal{T}_\tau} \alpha e^{-(r+\pi+\alpha-\tau_z)(s-t)} \left[U'(z_s) - U'(z_0^\star) + \frac{\tau_z - \pi}{\alpha}\right] ds, \tag{79}$$

for all $t \geq \mathcal{T}(z_\pi^\star)$, where we used that $\lambda_{\mathcal{T}_\tau} = 1$ and $\mathcal{T}(z_\pi^\star) = -\frac{1}{\pi}\ln\left[1 - \pi z_\pi^\star/(\bar{h} + \tau_0)\right]$. For all $t \in (\mathcal{T}_\pi^\star, \mathcal{T}_\tau)$, $z_t < z_0^\star$ and hence $U'(z_t) > U'(z_0^\star)$. Given that $\tau_z > \pi$ it follows from (79) that

$\lambda_t > 1$ for all $t \in (\mathcal{T}^\star_\pi, \mathcal{T}_\tau)$. Similarly, for all $t < \mathcal{T}^\star_\pi$, $z_t < z^\star_\pi$ and hence $U'(z_t) > U'(z^\star_\pi)$. Given that $\lambda_{\mathcal{T}(z^\star_\pi)} > 1$, it follows from (78) that $\lambda_t > 1$ for all $t \leq \mathcal{T}(z^\star_\pi)$.

For the second part of the Lemma we note that, when $\pi = \tau_0 = 0$, we have that $\lambda_t < U'(z^\star_0)$ for all $t \in [0, \mathcal{T}(z^\star_0)]$. Since λ is continuous with respect to (t, π, τ_0) and since $\mathcal{T}(z^\star_0)$ is finite at $\pi = \tau = 0$, we obtain by uniform continuity that $\lambda_t < U'(z^\star_0)$ for (π, τ_0) sufficiently small.

Note that, by Result #3, the τ_0 balancing the government budget constraint is less than $\pi z^\star_\pi$, hence it goes to zero as π goes to zero. Hence, when π is sufficiently small, the solution to the ODE (77) satisfies all the properties of Result #4. This allows us to construct a candidate value function for all $z \in [0, z^\star_0]$. Namely, we let $\lambda(z) \equiv \lambda_{\mathcal{T}(z)}$, where λ_t is the solution to the ODE (77):

$$W(z) \equiv W(0) + \int_0^z \lambda(x)\, dx \text{ where } rW(0) \equiv \lambda(0)\left(\bar{h} + \tau_0\right).$$

Next, we construct a candidate value function for $z \geq z^\star_0$:

RESULT #5: *For* π and τ_0 *sufficiently small, there exists a continuously differentiable and bounded function,* $W(z)$, *and two absolutely continuous functions,* $V(z)$ and $\lambda(z)$, *such that*: *For* $z \leq z^\star_0$, $W(z)$, $V(z)$ *and* $\lambda(z)$ *are the functions constructed following Result #4*; For $z \geq z^\star_0$:

$$W(z) = W(z_0) + \int_{z_0}^z \lambda(x)\, dx \tag{80}$$

$$V(z) = \max_{y \in [0,z]} U(y) + W(z-y) \tag{81}$$

$$(r+\alpha)\lambda(z) = V'(z) - \bar{c}\lambda'(z) \text{ almost everywhere} \tag{82}$$

$$\lambda(z) \in [0,1]. \tag{83}$$

■

Proof. We construct a solution to the problem (80)-(83) as follows. Suppose that we have constructed a solution over some interval $[z^\star_0, Z]$, where $Z \geq z^\star_0$. We first observe that:

$$U'(z^\star_0) = 1 + \frac{r}{\alpha} \geq \sup_{x \in [0, z^\star_0]} \lambda(x) = \sup_{x \in [0,Z]} \lambda(x), \tag{84}$$

where the first equality and the first inequality follow from our construction of $W(z)$ and $\lambda(z)$ over $[0, z^\star_0]$, and the last equality follows because $\lambda(z) \leq 1$ for $z \in [z^\star_0, Z]$. We now show how to extend this solution over the interval $[Z, Z + z^\star_0]$. First, we let:

$$\tilde{V}(z) \equiv \max_{y \in [z-Z, z]} U(y) + W(z-y), \tag{85}$$

which is well defined for all $z \in [Z, Z + z^\star_0]$, given that we have constructed $W(z)$ for all $z \leq Z$ and since $z - y \leq Z$ by the choice of our constraint set. Note that, in principle, the function $\tilde{V}(z)$ differs from $V(z)$ because it imposes the constraint that $y \geq z - Z$. Our goal is to show that, nevertheless, $\tilde{V}(z) = V(z)$. Precisely, if one extends $\lambda(z)$ over $[Z, Z + z_0^\star]$ using (82), and define $W(z)$ using (80), then the household never finds it optimal to choose $y < z_0^\star$, implying that the additional constraint we imposed to define $\tilde{V}(z)$ is not binding.

We first establish that $\tilde{V}(z)$ is absolutely continuous and $\tilde{V}'(z) \leq U'(z_0^\star)$. Consider first $z \in [Z, Z + z_0^\star/2]$. Given (84), it follows that the solution to (85) must be greater than $z_0^\star$. By implication since $z - Z \leq z_0^\star/2$, the solution y to (85) must be greater than $z - Z + z_0^\star/2$. Given this observation and after making the change of variable $x = z - y$, we obtain that

$$\tilde{V}(z) \equiv \max_{x \in [0, Z - z_0^\star/2]} U(z - x) + W(x).$$

The objective is continuously differentiable with respect to z, and its partial derivative is $U'(z-x) \leq U'(z_0^\star/2)$ given that $z \leq Z + z_0^\star/2$ and $x \leq Z - z_0^\star/2$. Proceeding to the interval $z \in [Z + z_0^\star/2, Z + z_0^\star]$, we make the change of variable $x = z - y$ in (85) and obtain that $\tilde{V}(z) = \max_{x \in [0Z]} U(z-x) + W(x)$. Again, the objective is continuously differentiable with a partial derivative with respect to z equal to $U'(z - x) \leq U'(z_0^\star/2)$, since $z \geq Z + z_0^\star/2$ and $x \leq Z$. Hence, in both cases, given that the objective has a bounded partial derivative with respect to z, we can apply Theorem 2 in Milgrom and Segal (2002): $\tilde{V}(z)$ is absolutely continuous and the envelope condition holds, i.e., $\tilde{V}'(z) = U'[y(z)]$ whenever this derivative exists. By condition (84), it follows that $y(z) \geq z_0^\star$, hence $\tilde{V}'(z) \leq U'(z_0^\star)$, as claimed.

Next, we construct a solution over $[Z, Z + z_0^\star]$. Given that the function $\tilde{V}(z)$ constructed above is absolutely continuous, we can integrate the ODE (82) with $\tilde{V}'(z)$ and we obtain a candidate solution:

$$\tilde{\lambda}(z) = \lambda(Z) e^{-\frac{r+\alpha}{\bar{c}}(z-Z)} + \frac{\alpha}{\bar{c}} \int_Z^z \tilde{V}'(x) e^{-\frac{r+\alpha}{\bar{c}}(z-x)} \, dx$$

Given that $\lambda(Z) \leq 1$ and $\tilde{V}'(x) \leq U'(z_0^\star) = 1 + r/\alpha$, one sees after direct integration that $\tilde{\lambda}(z) \leq 1 \leq U'(z_0^\star)$ for all $z \in [Z, Z + z_0^\star]$. Now let

$$\tilde{W}(z) = W(Z) + \int_Z^z \tilde{\lambda}(x) \, dx.$$

We now show that, if we extend $W(z)$ by $\tilde{W}(z)$, $\lambda(z)$ by $\tilde{\lambda}(z)$, and $V(z)$ by $\tilde{V}(z)$ over the interval $[Z, Z + z_0^\star]$, we obtain a solution of the problem (80)-(82) over $[Z, Z + z_0^\star]$: indeed, we have just shown that $\tilde{\lambda}(z) = \tilde{W}'(z) \leq U'(z_0^\star)$ for all $z \in [Z, Z + z_0^\star]$, implying that the constraint $y \geq z - Z$ we imposed in the definition of $\tilde{V}(z)$ is not binding. That is:

$$V(z) = \max_{y \in [0,z]} U(y) + W(z - y) = \max_{y \in [z-Z, Z]} U(y) + W(z - y) = \tilde{V}(z).$$

Hence, we have extended the solution from $[z_0^\star, Z]$ to $[Z, Z + z_0^\star]$. Notice that the argument does not depend on Z: we can start with $Z = z_0^\star$, and repeat this extension until we obtain a solution defined over $[z_0^\star, \infty)$.

Finally, we show that $\underline{W(z) \text{ is bounded}}$. By construction we have:

$$\begin{aligned}
\lambda(z) &= \lambda(z_0^\star) e^{-\frac{r+\alpha}{\bar{c}}(z - z_0^\star)} + \frac{\alpha}{\bar{c}} \int_{z_0^\star}^{z} V'(x) e^{-\frac{r+\alpha}{\bar{c}}(x - z_0^\star)}\, dx \\
W(z) &= W(z_0^\star) + \int_{z_0^\star}^{z} \lambda(y)\, dy.
\end{aligned}$$

Plugging the first equation into the second, keeping in mind that $\lambda(z_0^\star) = 1$, and changing the order of integration we obtain:

$$\begin{aligned}
W(z) &= W(z_0^\star) + \frac{\bar{c}}{r+\alpha}\left(1 - e^{-\frac{r+\alpha}{\bar{c}}(z - z_0^\star)}\right) + \frac{\alpha}{r+\alpha}\int_{z_0^\star}^{z} V'(x)\left[1 - e^{-\frac{r+\alpha}{\bar{c}}(z - x))}\right] dy \\
&\leq W(z_0^\star) + \frac{\bar{c}}{r+\alpha} + \frac{\alpha}{r+\alpha}\left[V(z) - V(z_0^\star)\right] \\
&\leq W(z_0^\star) + \frac{\bar{c}}{r+\alpha} + \frac{\alpha}{r+\alpha}\left[W(z) + \|U\| - W(z_0^\star)\right],
\end{aligned}$$

where the first inequality follows because $1 - e^{-\frac{r+\alpha}{\bar{c}}(z-x)} \leq 1$ for all $x \in [z_0^\star, z]$, and the second inequality because $W(z) \leq V(z) \leq W(z) + \|U\|$. Rearranging and simplifying we obtain that

$$W(z) \leq W(z_0) + \frac{\bar{c} + \alpha\|U\|}{r},$$

establishing the claim. ■

RESULT #6: *For π sufficiently small and τ_0 chosen, as in RESULT #3, to balance the government budget constraint, the households conjectured behavior is optimal.*

Proof. Consider the candidate value function constructed in Result #4 and #5. By construction, $W(z)$ is continuously differentiable and it solves the HJB equation:

$$(r+\alpha)W(z) = \max_{c \geq 0, 0 \leq h \leq \bar{h}, 0 \leq y \leq z} \left\{\min\{c, \bar{c}\} + \bar{h} - h + \alpha\left[U(y) + W(z-y)\right] + W'(z)\left[h - c + \tau(z) - \pi z\right]\right\}.$$

Then, the optimality verification argument of Section VII in the supplementary appendix establishes that $W(z)$ is equal to the maximum attainable utility of a households, and that the associated decision rules are optimal. ■

PROOF OF PROPOSITION 10. We construct an equilibrium featuring full depletion and such that $z^\star < \bar{y}$. The ODE for the marginal value of real balances, (21), becomes

$$(r+\pi)\lambda_t = \alpha(A - \lambda_t) + \dot{\lambda}_t, \quad \forall t \in [0, \mathcal{T}(z^\star)], \tag{86}$$

with $\dot{\lambda}_{\mathcal{T}(z^\star)} = 0$. With Poisson arrival rate, α, the household spends all its real balances, which generates a marginal surplus equal to $A - \lambda$. The solution is $\lambda_t = \mathbb{E}\left[e^{-(r+\pi)T_1} A\right] = \alpha A/(r+\pi+\alpha)$, for all $t \in \mathbb{R}_+$. It is straightforward to check that $A > \lambda_0 = \alpha A/(r+\pi+\alpha)$, which guarantees that full depletion is optimal. From $U'(\bar{y}^-) = A \geq 1 + (r+\pi)/\alpha$, so the unconstrained target is such that $z_s \geq \bar{y}$. The condition $\pi \geq \alpha\bar{h}/(\alpha\bar{y} - \bar{h})$ implies $z_b = \bar{h}(1/\pi + 1/\alpha) \leq \bar{y}$. So $z^\star = z_b = \bar{h}(1/\pi + 1/\alpha)$ and the equilibrium features full employment, $h = \bar{h}$. Finally, $\bar{h} < \alpha\bar{y}\left[\alpha\left(A-1\right) - r\right]/\left(\alpha A - r\right)$ implies $\bar{h} < \alpha\bar{y}$, which implies that the first best is such that $h = \bar{h}$. ■

PROOF OF PROPOSITION 11. We start with a formal definition of δ. With linear preferences, equilibrium welfare is:

$$W = \int \left\{\min\{\bar{c}, c(z)\} + \bar{h} - h(z) + \alpha U[y(z)]\right\} \, dF(z).$$

First-best welfare is, for large enough $\bar{h}$

$$W^\star = \bar{h} + \alpha\left[U(y^\star) - y^\star\right].$$

The welfare loss relative to the first-best allocation is then defined as the $\delta \in [0,1]$ solving:

$$\bar{h} + \alpha\left\{U[y^\star(1-\delta)] - y^\star\right\} = \int \left\{\min\{\bar{c}, c(z)\} + \bar{h} - h(z) + \alpha U[y(z)]\right\} \, dF(z).$$

We also provide an explicit formula for the threshold inflation $\underline{\pi}$ appearing in the Proposition. We start by fixing some $\varepsilon > 0$ small enough such that

$$\alpha\left[U(y^\star) - U(\varepsilon)\right] - (r+\alpha)y^\star > 0 \tag{87}$$

The existence of such ε is guaranteed by the maintained assumption that $\alpha U(y^\star) - (r+\alpha)y^\star > 0$. Given such ε, we choose

$$\underline{\pi} \equiv \frac{(r+\alpha)y^\star}{\varepsilon}. \tag{88}$$

With these preliminaries in mind, we turn to the proof of the Proposition.

RESULT #1: *Sufficient optimality conditions for the household's problem.*

We guess that, under the conditions stated in the Proposition, the household optimal policy is

- For $z < z^\star$: flow consume $c = 0$, flow work $h = \bar{h}$, and lumpy consume $y = z$.
- For $z = z^\star$: flow consume $c = 0$, flow work $h = 0$, and lumpy consume $y = z$.

- For $z > z^\star$: flow consume $c = \bar{c}$, flow work $h = 0$ and lumpy consume some $y(z) > z^\star$.

The corresponding equations for the value function is:

$$\begin{aligned}
\text{for } z < z^\star, \qquad (r+\alpha)W(z) &= \alpha\left[U(z) + W(0)\right] + W'(z)\left(\bar{h} - \pi z\right) && (89)\\
\text{for } z = z^\star, \qquad (r+\alpha)W(z^\star) &= \bar{h} + \alpha\left[U(z^\star) + W(0)\right] && (90)\\
\text{for } z > z^\star, \qquad (r+\alpha)W(z) &= \bar{h} + \bar{c} + \alpha \max_{y\in[0,z]}\left[U(y) + W(z-y)\right] - W'(z)\bar{c}. && (91)
\end{aligned}$$

Below we solve for $W(z)$ explicitly for $z \le z^\star$, and we provide an implicit construction of $W(z)$ for $z > z^\star$. A non-standard feature of $W(z)$, which is a consequence of the discontinuity of the transfer scheme, is that it is not differentiable at $z^\star$, with a concave kink. This means in particular that the standard optimality verification argument, which assumes continuous differentiability, does not apply directly here. In Supplementary Appendix VI we extend that argument to our case and we prove sufficiency. That is, the stated optimal consumption-saving policy is optimal if the function $W(z)$ is bounded, continuously differentiable for $z \neq z^\star$, and satisfies two Hamilton-Jacobi-Bellman equations. First, for $z \neq z^\star$:

$$(r+\alpha)W(z) = \max\left\{u(c, \bar{h} - h) + \alpha\left[U(y) + W(z-y)\right] + W'(z)\left[h - c - \pi z + \tau(z)\right]\right\},$$

with respect to $c \ge 0$, $h \in [0, \bar{h}]$ and $y \in [0, z]$. Second, for $z = z^\star$,

$$(r+\alpha)W(z) = \max\left\{u(c, \bar{h} - h) + \alpha\left[U(y) + W(z-y)\right]\right\},$$

with respect to $c \ge 0$, $h \in [0, \bar{h}]$, $y \in [0, z]$ and subject to $h - c - \pi z + \tau(z) = 0$.

Given that $u(c, \bar{h} - c) = \min\{c, \bar{c}\} + \bar{h} - h$ and given the transfer scheme $\tau(z \,|\, \pi)$, we obtain that the stated consumption-saving policy is optimal if there exists a bounded function $W(z)$, continuously differentiable for $z \neq z^\star$, satisfying (89)-(91) as well as:

$$\begin{aligned}
W'(z) &< U'(z^\star) \text{ for } z < z^\star && (92)\\
W'(z) &> 1 \text{ for } z < z^\star && (93)\\
W'(z) &< 1 \text{ for } z > z^\star. && (94)
\end{aligned}$$

Condition (94) is sufficient for full depletion, that is $z = \arg\max_{y\in[0,z]} U(y) + W(z-y)$.

RESULT #2: *A closed-form expression for $W(z)$ for $z \le z^\star$.*

We use (89)-(90) to construct a guess for the function $W(z)$. First, taking limit in (89) as $z \uparrow z^\star$ and comparing with (90), we obtain that, for $W(z)$ to be continuous, it must be that

$W'(z^\star-) = \bar{h}/(\bar{h} - \pi z^\star)$. Second, taking derivative in (89), we obtain that:

$$(r+\alpha+\pi)\lambda(z) = \alpha U'(z) + \lambda'(z)\left[\bar{h} - \pi z\right],$$

where we used our notation $\lambda(z) \equiv W'(z)$. After integration, using the terminal condition derived above, $\lambda(z^\star-) = \bar{h}/(\bar{h} - \pi z^\star$, we obtain:

$$\lambda(z) = \frac{\bar{h}}{\bar{h} - \pi z}\left[\frac{\alpha}{\bar{h}}\int_z^{z^\star} U'(y)\left(\frac{\bar{h} - \pi y}{\bar{h} - \pi z}\right)^{\frac{r+\alpha}{\pi}} dy + \left(\frac{\bar{h} - \pi z^\star}{\bar{h} - \pi z}\right)^{\frac{r+\alpha}{\pi}}\right]. \tag{95}$$

Hence:

$$W(z) = W(0) + \int_0^z \lambda(y)\,dy,$$

where $W(0)$ is obtained by equating the limit at $z \uparrow z^\star$ with (90)

$$rW(0) = \bar{h} + \alpha U(z^\star) - (r+\alpha)\int_0^{z^\star} \lambda(z)\,dz.$$

Conversely, one easily shows that the function $W(z)$ thus constructed satisfies (89) and (90). One can also check that this function is convex near $z^\star$ with a slope strictly larger than one in a left-neighborhood of $z^\star$.

RESULT #3: *Verification of the full time work condition, (93), for all $\bar{h}$ large enough.*

We first derive a uniform lower bound over the interval $z \in [\varepsilon, 1]$ and shows that it is greater than one for large enough $\bar{h}$. Clearly, in (95), the multiplicative term and the second term in the square bracket are both minimized at $z = \varepsilon$ over $z \in [\varepsilon, 1]$. The integral term is clearly positive. Therefore,

$$\begin{aligned}\lambda(z) \geq \frac{\bar{h}}{\bar{h} - \pi\varepsilon}\left(\frac{\bar{h} - \pi y^\star}{\bar{h} - \pi\varepsilon}\right)^{\frac{r+\alpha}{\pi}} &= 1 + \frac{1}{\bar{h}}\left[\pi\varepsilon - (r+\alpha)\left(y^\star - \varepsilon\right)\right] + o\left(\frac{1}{\bar{h}}\right) \\ &> 1 + \frac{1}{\bar{h}}\left[\pi\varepsilon - (r+\alpha)y^\star\right] + o\left(\frac{1}{\bar{h}}\right).\end{aligned}$$

therefore, condition (88) ensures that, as long as $\pi > \underline{\pi}$, this expression is strictly greater than one for all $\bar{h}$ large enough. The condition $\pi > \underline{\pi}$ ensures that $\lambda(z^\star-)$ is sufficiently greater than one – this is the sense in which the incentives to escape inflation and reach the target must be sufficiently large.

Next, we derive a uniform lower bound for $\lambda(z)$ over the interval $[0, \varepsilon]$ and we show that this lower bound is strictly greater than one. The multiplicative term in (95) is greater than one. The derivative of the integral between the square brackets is:

$$\begin{aligned}\frac{d}{dz}\int_z^{z^\star} U'(y)\left(\frac{\bar{h} - \pi y}{\bar{h} - \pi z}\right)^{\frac{r+\alpha}{\pi}} dy &= -U'(z) + \frac{r+\alpha}{\bar{h} - \pi z}\int_z^{z^\star} U'(y)\left(\frac{\bar{h} - \pi y}{\bar{h} - \pi z}\right)^{\frac{r+\alpha}{\pi}} dy \\ &\leq -U'(\varepsilon) + \frac{r+\alpha}{\bar{h} - \pi\varepsilon}U(y^\star) < 0\end{aligned}$$

as long as $\bar{h}$ is large enough. The second term in the square bracket of (95) is increasing in z and decreasing in $z^\star$. Therefore, as long as $\bar{h}$ is large enough, we obtain that:

$$\lambda(z) \quad \geq \quad \frac{\alpha}{\bar{h}} \int_{\varepsilon}^{y^\star - \frac{\Delta}{\bar{h}}} U'(y) \left(\frac{\bar{h} - \pi y}{\bar{h} - \pi \varepsilon} \right)^{\frac{r+\alpha}{\pi}} dy + \left(\frac{\bar{h} - \pi y^\star}{\bar{h}} \right)^{\frac{r+\alpha}{\pi}}.$$

By an application of the Dominated Convergence Theorem, one easily sees that the integral on the right-hand side converges towards $U(y^\star) - U(\varepsilon)$ as $\bar{h} \to \infty$. The second-term on the right-side can be written $1 - (r+\alpha)y^\star/\bar{h} + o(1/\bar{h})$. Taken together, we obtain

$$\lambda(z) \geq 1 + \frac{1}{\bar{h}} \Big\{ \alpha \left[U(y^\star) - U(\varepsilon) \right] - (r+\alpha) y^\star \Big\} + o\left(\frac{1}{\bar{h}} \right).$$

Therefore, condition (87) ensures that the right-side will be strictly greater than one for all $\bar{h}$ large enough.

RESULT #4: *Verification of the full depletion condition, (92).*

We obtain a uniform upper bound for $\lambda(z)$ as follows. For $z \leq z^\star \leq y^\star$ we have that $\bar{h}/(\bar{h} - \pi z)$ is less than $\bar{h}/(\bar{h} - \pi y^\star)$. In the square bracket both $(\bar{h} - \pi y)/(\bar{h} - \pi z)$ and $(\bar{h} - \pi z^\star)/(\bar{h} - \pi z)$ are less than one. Therefore:

$$\lambda(z) \leq \frac{\bar{h}}{\bar{h} - \pi y^\star} \left[\frac{\alpha}{\bar{h}} U(y^\star) + 1 \right] = 1 + \frac{1}{\bar{h}} \left[\pi y^\star + \alpha U(y^\star) \right] + o\left(\frac{1}{\bar{h}} \right).$$

On the other hand, with $z^\star = y^\star - \Delta/\bar{h}$, $U'(z^\star) = U'(y^\star) - U''(y^\star)\Delta/\bar{h} + o(1/\bar{h}) = 1 + |U''(y^\star)|\Delta/\bar{h} + o(1/\bar{h})$, since $U''(z) < 0$ and $U'(y^\star) = 1$ by assumption. Our choice of Δ ensures that $\Delta > \left[\pi y^\star + \alpha U(y^\star) \right] / |U''(y^\star)|$, and so (94) holds for all $\bar{h}$ is large enough.

RESULT #5: *Verification of the full consumption condition, (94), for all $\bar{h}$ large enough.* Given that (92) and (93) hold, we xfollow the same steps as in the proof of Result #5, Proposition 9 to construct the value function $W(z)$ for $z \geq z^\star$, and show that $\lambda(z) \in [0,1]$, ■

PROOF OF PROPOSITION 13 . Let us first recall the problem of the household in the pure-currency economy under a lump-sum transfer scheme, Υ^1:

$$rW(z) = \max \left\{ u(c, \bar{h} - h) + \alpha \left[U(y) + W(z-y) - W(z) \right] + W'(z)\dot{z} \right\}, \tag{96}$$

with respect to (c, h, y), subject to $c \geq 0$, $0 \leq h \leq \bar{h}$, $0 \leq y \leq z$, and $\dot{z} = h - c + \Upsilon^1 - \pi z$. Policy functions are denoted $y^1(z)$, $h^1(z)$, and $c^1(z)$. The distribution of real balances across households is $F^1(z)$. We compare equilibria of the pure currency economy to equilibria of an economy with money and bonds such that $\varrho = -\pi$, i.e., money and bonds have the same rate of return. The household problem, (39), becomes

$$rW(\omega) = \max_{c,h,y} \left\{ u(c, \bar{h} - h) + \alpha \left[U(y) + W(\omega - y) - W(\omega) \right] + W'(\omega)\dot{\omega} \right\}, \tag{97}$$

subject to $c \geq 0$, $0 \leq h \leq \bar{h}$, $0 \leq y \leq \omega$, and $\dot{\omega} = h - c - \pi\omega + \Upsilon^2$. The household problem (97) which is formally equivalent to (96) provided that $\Upsilon^2 = \Upsilon^1$. If this condition holds, $y^2(\omega) = y^1(\omega)$, $h^2(\omega) = h^1(\omega)$, $c^2(\omega) = c^1(\omega)$, and the distributions of wealth across the two economies are the same, $F^2(\omega) = F^1(\omega)$. In order to check that $\Upsilon^2 = \Upsilon^1$ we use the budget constraint of the government:

$$\begin{aligned} \Upsilon^2 &= \left[(\mu + \pi) q_0 - \mu \phi_0^2 \right] B_0 + \pi \phi_0^2 M_0 \\ &= \pi \left(\phi_0^2 B_0 + \phi_0^2 \right), \end{aligned}$$

where we used that $q_0 = \phi_0^2$, B_0 is the initial supply of bonds, and $M_0 = 1$ is the initial supply of money. By market clearing,

$$\begin{aligned} \pi \left(\phi_0^2 B_0 + \phi_0^2 M_0 \right) &= \pi \int_0^\infty \omega dF^2(\omega) \\ &= \pi \int_0^\infty z dF^1(z) \\ &= \pi \phi_0^1 M_0 = \pi \phi_0^1, \end{aligned}$$

which implies

$$\phi_0^2 = \frac{\phi_0^1}{1 + B_0}, \tag{98}$$

and $\Upsilon^2 = \Upsilon^1$. From (45), the clearing of the bonds market requires that there is a $z^2(\omega) \in \left[y^2(\omega), \omega \right]$ such that:

$$\phi_0^2 B_0 = \int_0^\infty \left[\omega - z^2(\omega) \right] dF^2(\omega).$$

Such a function exists provided that

$$\phi_0^2 B_0 \leq \int_0^\infty \left[\omega - y^2(\omega) \right] dF^2(\omega) = \int_0^\infty \left[z - y^1(z) \right] dF^1(z).$$

Substituting ϕ_0^2 by its expression given by (98), we rewritte this inequality as:

$$B_0 \leq \beta \equiv \frac{\int_0^\infty \left[z - y^1(z) \right] dF^1(z)}{\int_0^\infty y^1(z) dF^1(z)}.$$

The proof of the converse, namely, any liquidity-trap equilibrium of two-asset economy corresponds to an equilibrium of the pure-currency economy that features partial depletion, is analogous and is therefore omitted. ■

Appendix B: Numerical methods

Overview. In this section we provide a step-by-step numerical method to compute the stationary equilibrium with standard packages, for example Matlab. A detailed discussion of the numerical method is provided in the supplementary Appendix. To solve the system we need to start from some initial values close to the solution. Step 1 suggests an efficient method to compute initial values of λ_0 and Υ: the solution to an economy with zero inflation and full depletion, which is close to the equilibrium if the money growth rate is not very large but $\overline{h}$ is not very low. Given λ_0 and Υ, Step 2 (or 2' under linear preferences) computes the system of delay differential equations (DDE), which summarizes the household's optimal actions. Step 3 and 4 (or 4' under linear preferences) computes the Kolmogorov forward equation (KFE), which solves the stationary distribution. Step 5 solves λ_0 and Υ as fixed points.[29]

Step 1a. Fix $y(z) = z$ and $\pi = 0$. Solve the following values for initiation:

$$\begin{aligned} h(\lambda^\star) &= c(\lambda^\star) \\ z^\star &= (U')^{-1}\left[\left(\frac{r+\alpha}{\alpha}\right)\lambda^\star\right], \\ p &= \frac{\xi}{h'(\lambda^\star) - c'(\lambda^\star)}, \end{aligned}$$

where ξ is the negative eigenvalue of the Jacobian given by

$$\xi = -\frac{r+\alpha}{2}\left[\left[1 - \frac{4\alpha U''(z^\star)}{(r+\alpha)^2}\left[h'(\lambda^\star) - c'(\lambda^\star)\right]\right]^{1/2} - 1\right].$$

Under linear preferences, we have $h(\lambda^\star) = c(\lambda^\star) = 0$, $\lambda^\star = 1$, $z^\star = (U')^{-1}\left(\frac{r+\alpha}{\alpha}\right)$ and $p = \frac{\alpha}{r+\alpha}U''(z^\star)$.

Step 1b. Use ode45 routine of Matlab to integrate the following ODE of $\lambda(z)$ backward from $z = z^\star$ to $z = 0$:

$$\lambda'(z) = \frac{(r+\alpha)\lambda - \alpha U'(z)}{h(\lambda) - c(\lambda)},$$

where the initial values are given by $\lambda(z^\star) = \lambda^\star$ and $\lambda'(z^\star) = p$.

Step 1c. Having obtained $\lambda(z)$, use ode45 routine to integrate the following ODE of $f(z)$

[29] The common approach in the literature is to use an "upwind" finite-differences algorithm to iterate a system of PDEs composed of HJB and KFE. Instead, the equilibrium of our model defined as a system of DDEs can be solved efficiently with built-in Matlab routines with good control of error. For example, a laptop equipped with Intel i5 2.30GHz CPU and 8GM RAM takes 15 seconds to compute the calibrated model in Section 4.3, with an error tolerance of 10^{-6}. A finite-differences algorithm takes 27 times longer to converge, with an error tolerance of 5×10^{-3}.

forward from $z=0$ to $z=z^{\star}$:

$$f'(z)=-\frac{\alpha+\lambda'(z)\left[h'(\lambda)-c'(\lambda)\right]}{h(\lambda)-c(\lambda)}f(z)$$

where the initial value is given by $f(0)=1$. If $s\left(z_{-}^{\star}\right)>0$ (for example under the slack labor equilibrium of LRW models) then we construct the probability mass $1-F\left(z_{-}^{\star}\right)$ by the following KFE boundary condition

$$1-F\left(z_{-}^{\star}\right)=\frac{s\left(z_{-}^{\star}\right)f\left(z_{-}^{\star}\right)}{\alpha}.$$

It obtains $f(z)$.

Step 1d. The initial values of λ_0 and Υ are set to $\lambda_0=\lambda(0)$ and $\Upsilon=\pi\int_0^{z^{\star}}zf(z)\,dz/\int_0^{z^{\star}}f(z)\,dz$.

Step 2a. Jump to Step 2'a if under linear preferences. Given λ_0 and Υ (from Step 1 if it is the first time to run the iteration), use ddesd routine to integrate the following DDE system of $z(\zeta)$ and $\Omega(\zeta)$

$$z'(\zeta)=\frac{h(-\zeta)-c(-\zeta)-\pi z+\Upsilon}{(r+\alpha+\pi)\zeta+\alpha\Omega(\zeta)},$$

$$\Omega'(\zeta)=z'(\zeta)\left[U''\left[\left(U'\right)^{-1}\left[\Omega(\zeta)\right]\right]^{-1}+\mathbb{I}\left[\Omega(\zeta)<\lambda_0\right]z'\left[-\Omega(\zeta)\right]\right]^{-1},$$

where the initial values are given by $z(-\lambda_0)=0$ and $\Omega(-\lambda_0)=U'(0)$ (or some arbitrary large value if $U'(0)=\infty$). Stop integrating whenever $h(-\zeta)-c(-\zeta)-\pi z(\zeta)+\Upsilon=0$. Denote the stopping ζ and z as $\zeta^{\star}(\lambda_0,\Upsilon)$ and $z^{\star}(\lambda_0,\Upsilon)$. It obtains $z(\zeta)$ and $\Omega(\zeta)$.

Step 2b. Define

$$\begin{aligned}
y(z) &\equiv \left(U'\right)^{-1}\circ\Omega\circ z^{-1}(z),\\
z_d &\equiv z^{\star}-y(z^{\star}),\\
s(z) &\equiv h\left[-z^{-1}(z)\right]-c\left[-z^{-1}(z)\right]-\pi(z-\Upsilon).
\end{aligned}$$

Jump to Step 3.

Step 2'a. Given λ_0 and Υ (from Step 1 if it is the first time to run the iteration), use ddesd routine to integrate the following DDE of $y(z)$

$$y'=\begin{cases}1 \text{ if } z\leq\left(U'\right)^{-1}(\lambda_0),\\ \left[1+U''(y)\frac{h-\pi(z-y)+\Upsilon}{(r+\alpha+\pi)U'(y)-\alpha U'[y(z-y)]}\right]^{-1} \text{ if } z>\left(U'\right)^{-1}(\lambda_0),\end{cases}$$

where the initial value is given by $y(0)=0$. Stop integrating at either $z=(h+\Upsilon)/\pi$ or $U'[y(z)]=1+\frac{r+\pi}{\alpha}$. It obtains $y(z)$. Denote the stopping z as $z^{\star}(\lambda_0,\Upsilon)$. Define

$$\begin{aligned}
z_d &\equiv z^{\star}-y(z^{\star}),\\
s(z) &\equiv \overline{h}-\pi z+\Upsilon.
\end{aligned}$$

Step 2'b. Use ode45 routine to integrate the following ODE of $\lambda(z)$ forward from $z = 0$ to $z^\star$:

$$\lambda'(z) = \frac{(r + \alpha + \pi)\lambda(z) - \alpha U'[y(z)]}{\overline{h} - \pi(z - \Upsilon)},$$

with the initial condition $\lambda(0) = \lambda_0$. It obtains $\lambda(z)$. Define $\zeta^\star(\lambda_0, \Upsilon) = -\lambda(z^\star)$.

Step 3. Define $\varphi(z)$ as the solution to $\varphi - y(\varphi) = z$. Consider the region $[z_d, z^\star]$, where the density function $f(z)$ is simply an ODE solution of the following KFE:

$$f'(z) = -\frac{\alpha + s'(z)}{s(z)} f(z), \text{ for all } z \in (z_d, z^\star),$$

which has closed-form solution

$$f(z) = \left[\frac{s(z)}{s(z_d)}\right]^{\frac{\alpha}{\pi} - 1}, \text{ for all } z \in (z_d, z^\star)$$

under linear preferences (and we need to construct the probability mass $1 - F(z^\star_-)$ by the same KFE boundary condition in the Step 1c). Otherwise, use ode45 routine to solve $f(t)$ forward from $z = z_d$ to $z = z^\star$ with initial value $f(z_d) = 1$. It obtains $f(z)$ for all $z \in [z_d, z^\star]$.

Step 4. Jump to Step 4' if under linear preferences. Construct the "history" of $\phi(t)$ for all $t \in [-(z^\star - z_d), 0]$, by setting $\phi(t) = f(z_d - t)$, where $f(z)$ is given by Step 3. Use ddesd routine to integrate the following DDE of $\phi(t)$ forward from $t = 0$ to $t = z_d$:

$$\phi'(t) = \left[\frac{\alpha - s'(z_d - t)}{s(z_d - t)}\right]\phi(t) - \alpha\frac{s[\varphi(z_d - t)]}{s(z_d - t)^2}\phi[z_d - \varphi(z_d - t)], \text{ for all } t \in (0, z_d)$$

where the initial value $\phi(0)$ is given by

$$\phi(0) = 1 - \frac{s(z^\star_-)}{s(z_d)} f(z^\star_-).$$

Having obtained $\phi(t)$, set $f(z) = \phi(z_d - z)$ for all $z \in [0, z_d)$. Jump to Step 5.

Step 4'. Under linear preferences, use ode45 routine to integrate the following DDE of $\phi(t)$ forward from $t = 0$ to $t = z_d$:

$$\phi'(t) = \frac{\alpha + \pi}{s(z_d - t)}\phi(t) - \alpha s(z_d)^{1 - \frac{\alpha}{\pi}} \frac{s[\varphi(z_d - t)]^{\frac{\alpha}{\pi}}}{s(z_d - t)^2}, \text{ for all } t \in [0, z_d],$$

where the initial value $\phi(0)$ is given by

$$\phi(0) = 1 - \left[\frac{s(z^\star_-)}{s(z_d)}\right]^{\frac{\alpha}{\pi}}.$$

Having obtained $\phi(t)$, set $f(z) = \phi(z_d - z)$ for all $z \in [0, z_d)$.

Step 5. Define a function $\Gamma(\lambda_0, \Upsilon) : \mathbb{R}^2_+ \to \mathbb{R}^2$, where the first and second coordinates are given by

$$\begin{aligned}\Gamma^{(1)}(\lambda_0, \Upsilon) &= (r + \alpha + \pi)\,\zeta^\star + \alpha U'\left[y(z^\star)\right],\\ \Gamma^{(2)}(\lambda_0, \Upsilon) &= \pi\frac{\int_0^{z^\star} zf(z)\,dz}{\int_0^{z^\star} f(z)\,dz} - \Upsilon,\end{aligned}$$

where $\zeta^\star$, $z^\star$, y and f are constructed given λ_0 and Υ from previous steps. Use fsolve routine to solve $\lambda_0^\star$ and $\Upsilon^\star$ such that $\Gamma(\lambda_0^\star, \Upsilon^\star) = 0$.

Step 6. Finally, the stationary equilibrium is given by the marginal value function $W'(z) = -z^{-1}(z; \lambda_0^\star, \Upsilon^\star)$ and the density function $f(z; \lambda_0^\star, \Upsilon^\star)$. Agents accumulate real balances according to $\dot{z} = s(z; \lambda_0^\star, \Upsilon^\star)$, and the lumpy consumption is given by $y(z; \lambda_0^\star, \Upsilon^\star)$. The above numerical algorithm works whether the equilibrium features periodic full money depletion or periodic partial money depletion.