# Supplementary Appendix: Ad-valorem platform fees, indirect taxes and efficient price discrimination

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**Abstract**

In this supplementary appendix, we provide results and working for the welfare effects of price discrimination when there are either two goods or a continuum of goods. The results apply to the welfare effect of allowing a platform to use ad-valorem fees.

## 1 Introduction

In this supplementary appendix, we provide results and working for the welfare effects of price discrimination for the particular model of generalized Pareto demand developed in our paper "Ad-valorem platform fees, indirect taxes and efficient price discrimination." The demand a monopolist faces for a good $c$ is given by

$$Q_c(T_c) = \left(1 + \frac{\lambda(\sigma - 1) T_c}{c}\right)^{\frac{1}{1-\sigma}}, \tag{1}$$

where $T_c$ is the price set by the monopolist and $c$ is a demand shift parameter which vertically stretches demand across goods with different values of $c$ (i.e. inverse demand or buyers'

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willingness to pay is proportional to $c$). Note the parameters $\lambda$ and $\sigma$ must satisfy $\lambda > 0$ and $\sigma < 2$.

The monopolist chooses prices $T_c$ for each good $c$ to maximize its profit

$$\sum_{c \in C} g_c(T_c - d)Q_c(T_c).$$

If price discrimination is banned, it must set the same $T_c$ for each good.

We first consider a setting with just two goods $c_L$ and $c_H$, with $c_H > c_L$, so $g_c = 1$ for each good. We then consider the case with a continuum of goods drawn from the uniform distribution on $[c_L, c_H]$ so that profit becomes

$$(\frac{1}{c_H - c_L}) \int_{c_L}^{c_H} [(T_c - d)Q_c(T_c)]dc.$$

In each case, the results we obtained on price discrimination also apply for the welfare effects of allowing a platform to use ad-valorem fees.

## 2 Two-good case

Define $k = c_H/c_L > 1$ as the measure of dispersion of the two demand levels. We first establish a new result on the welfare effects of price discrimination.

**Proposition 1** ***(Welfare effects of banning price discrimination across two goods):*** *Assume that each good has demand given by (1) and the monopolist has zero marginal costs (i.e. $d = 0$). If there are two goods $c_L$ and $c_H$, then banning price discrimination across the two goods lowers welfare if demand is exponential ($\sigma = 1$) or log-convex ($1 < \sigma < 2$), and will also lower welfare if demand is log-concave ($\sigma < 1$) provided $k$ is sufficiently large (i.e. the demand for the two goods is sufficiently different).*

**Proof.** We consider three cases.

(i) Demand is log-concave so $\sigma < 1$. Then there is a choke price $T'_c = c/(\lambda(1-\sigma))$ at which demand becomes zero for good $c$. Let $c_L$ be fixed and consider increasing $k$ and so $c_H$. Let $z = \left(\frac{1}{2-\sigma}\right)\left(1 - \frac{1-\sigma}{2-\sigma}\right)^{\frac{1}{1-\sigma}}$ where $0 < z < e^{-1}$ given $\sigma < 1$. Under price discrimination, the

profit from the high-demand good is $c_H z/\lambda \to \infty$ as $k \to \infty$. The profit from the low-demand good is fixed at $c_L z/\lambda$. Total profit is unbounded as $k$ increases. On the other hand, with a uniform price the profit is bounded if both goods are to continue to be sold since the price cannot exceed the choke price for good $c_L$, which is $c_L/(\lambda(1-\sigma))$. Therefore, there exists a high enough $k$ such that the monopolist will give up on the low-demand good if it is forced to set a single price. The threshold $k_0$ such that the monopolist will no longer sell the low-demand good whenever $k \geq k_0$ is determined by

$$\left(\frac{1}{2-\sigma}\right)^{\frac{2-\sigma}{1-\sigma}} = \frac{1}{(k_0+1)}\left[\left(\frac{k_0+\sigma}{k_0+1}\right)^{\frac{1}{1-\sigma}} + \left(\frac{k_0\sigma+1}{k_0+1}\right)^{\frac{1}{1-\sigma}}\right], \tag{2}$$

which is obtained by comparing the monopolist's profit with and without dropping the low-demand good under uniform pricing. Note $k_0$ only depends on $\sigma$. For example, in the case of linear demand, solving (2) with $\sigma = 0$ implies $k_0 = 3$. With price discrimination, the monopolist will set the same price for the high-demand good as it would under uniform pricing, and a lower price for the low-demand good thereby generating additional profit, consumer surplus and welfare.

(ii) Demand is exponential so $\sigma = 1$. Then there is no choke price at which demand becomes zero. We compare welfare directly. Welfare for good $c$ is

$$W_c = \int_0^{Q_c} T_c(Q)\, dQ = \int_0^{Q_c} (-\frac{c}{\lambda}\ln Q) dQ,$$

so that $W_c(T_c^*) = 2ce^{-1}/\lambda$ under price discrimination given

$$Q_c(T_c) = e^{-\frac{\lambda T_c}{c}} \tag{3}$$

and

$$T_c^* = \frac{\lambda d + c}{\lambda(2-\sigma)}, \tag{4}$$

with $d = 0$ and $\sigma = 1$. Therefore, welfare across both goods under price discrimination is $W_{PD} = 2c_L(1+k)e^{-1}/\lambda$.

Now consider welfare without price discrimination. The monopolist will set the uniform price $T$ to maximize

$$\Pi = T\left(e^{-\frac{\lambda T}{c_L}} + e^{-\frac{\lambda T}{c_H}}\right).$$

The optimal uniform price $\widehat{T}$ solves the first-order condition

$$e^{-\frac{\lambda \widehat{T}}{c_L}}\left(1 - \frac{\lambda \widehat{T}}{c_L}\right) + e^{-\frac{\lambda \widehat{T}}{c_H}}\left(1 - \frac{\lambda \widehat{T}}{c_H}\right) = 0.$$

The solution can be written as $\widehat{T} = \rho c_L/\lambda$, where $\rho$ solves

$$(1-\rho)\, e^{-\rho} + \left(1 - \frac{\rho}{k}\right) e^{-\frac{\rho}{k}} = 0.$$

Note $\rho$ is just a function of $k$. Welfare under uniform pricing is

$$\begin{aligned} W_U &= \widehat{T}\left(e^{-\frac{\lambda \widehat{T}}{c_L}} + e^{-\frac{\lambda \widehat{T}}{c_H}}\right) + c_L\left(\frac{e^{-\frac{\lambda \widehat{T}}{c_L}}}{\lambda}\right) + c_H\left(\frac{e^{-\frac{\lambda \widehat{T}}{c_H}}}{\lambda}\right) \\ &= \frac{c_L}{\lambda}\left((k+\rho)\, e^{-\frac{\rho}{k}} + (1+\rho)\, e^{-\rho}\right). \end{aligned}$$

Therefore,

$$W_{PD} - W_U = \frac{c_L}{\lambda}\left(2\,(1+k)\, e^{-1} - (1+\rho)\, e^{-\rho} - (k+\rho)\, e^{-\frac{\rho}{k}}\right).$$

Since $\rho$ is just a function of $k$, and the term in brackets in $W_{PD} - W_U$ is just a function of $\rho$ and $k$, the sign of $W_{PD} - W_U$ just depends on $k$. It is straightforward to verify $W_{PD} - W_U > 0$ for all $k > 1$, and so welfare is higher under price discrimination.

The limit case as $k \to \infty$ provides some insight into what happens as the goods become dispersed. In the limit as $k \to \infty$, it can be shown $\rho \to k$. Then $W_{PD} - W_U \to 2c_L e^{-1}/\lambda > 0$. Moreover, as $k \to \infty$, $\widehat{T} \to kc_L/\lambda = c_H/\lambda = T^*_{c_H}$. In other words, for large $k$, the uniform price converges to the price that the monopolist would set to the high-demand good under price discrimination.

(iii) Demand is log-convex so $1 < \sigma < 2$. Welfare for good $c$ is

$$W_c = \int_0^{Q_c} T_c(Q)\, dQ = \int_0^{Q_c} \frac{c\left(1 - Q_c^{1-\sigma}\right)}{\lambda(1-\sigma)} dQ, \tag{5}$$

so that

$$W_c(T_c^*) = \frac{c}{\lambda(\sigma - 1)} \left[ \left( \frac{1}{2-\sigma} \right)^{2+\frac{1}{1-\sigma}} - \left( \frac{1}{2-\sigma} \right)^{\frac{1}{1-\sigma}} \right]$$

under price discrimination where demand is given in 1 and (4) has been substituted into (5). Therefore, welfare across both goods under price discrimination is

$$W_{PD} = \frac{c_L(1+k)}{\lambda(\sigma-1)} \left[ \left( \frac{1}{2-\sigma} \right)^{2+\frac{1}{1-\sigma}} - \left( \frac{1}{2-\sigma} \right)^{\frac{1}{1-\sigma}} \right]. \tag{6}$$

Now consider welfare without price discrimination. The monopolist will set the uniform price $T$ to maximize

$$\underset{T}{Max}\ \Pi = T \left[ \left( 1 + \frac{\lambda(\sigma-1)T}{c_L} \right)^{\frac{1}{1-\sigma}} + \left( 1 + \frac{\lambda(\sigma-1)T}{c_H} \right)^{\frac{1}{1-\sigma}} \right].$$

The optimal uniform price $\widehat{T}$ solves the first-order condition

$$\begin{aligned} & \left( 1 + \frac{\lambda(\sigma-1)\widehat{T}}{c_L} \right)^{\frac{1}{1-\sigma}} + \left( 1 + \frac{\lambda(\sigma-1)\widehat{T}}{c_H} \right)^{\frac{1}{1-\sigma}} \\ = \ & \frac{\lambda\widehat{T}}{c_L} \left( 1 + \frac{\lambda(\sigma-1)\widehat{T}}{c_L} \right)^{\frac{\sigma}{1-\sigma}} + \frac{\lambda\widehat{T}}{c_H} \left( 1 + \frac{\lambda(\sigma-1)\widehat{T}}{c_H} \right)^{\frac{\sigma}{1-\sigma}}. \end{aligned}$$

The solution can be written as $\widehat{T} = \rho c_L / (\lambda(\sigma-1))$, where for any given $\sigma$, the term $\rho$ is just a function of $k$ which solves

$$(1+\rho)^{\frac{1}{1-\sigma}} - \frac{\rho}{\sigma-1}(1+\rho)^{\frac{\sigma}{1-\sigma}} + \left(1 + \frac{\rho}{k}\right)^{\frac{1}{1-\sigma}} - \frac{\rho}{k(\sigma-1)} \left(1 + \frac{\rho}{k}\right)^{\frac{\sigma}{1-\sigma}} = 0.$$

Welfare under uniform pricing is

$$W_U = \frac{c_L}{\lambda(\sigma-1)}\left[\frac{(1+\rho)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma} - (1+\rho)^{\frac{1}{1-\sigma}} + \frac{k\left(1+\frac{\rho}{k}\right)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma} - k\left(1+\frac{\rho}{k}\right)^{\frac{1}{1-\sigma}}\right]. \tag{7}$$

Since $\rho$ is just a function of $k$ and the term in brackets in $W_U$ is just a function of $\rho$ and $k$ for any given $\sigma$, the sign of $W_{PD} - W_U$ just depends on $k$ for any particular $\sigma$. Evaluating (6) and (7) confirms $W_{PD} - W_U > 0$ for all $k > 1$ for any $1 < \sigma < 2$, so that welfare is higher under price discrimination.

Again, the limit case as $k \to \infty$ provides some insight into what happens as the goods become dispersed. In the limit as $k \to \infty$, it can be shown that $\rho \to k(\sigma-1)/(2-\sigma)$. Then $W_{PD} - W_U \to \frac{c_L}{\lambda(\sigma-1)}\left[\left(\frac{1}{2-\sigma}\right)^{2+\frac{1}{1-\sigma}} - \left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}}\right] > 0$. Moreover, as $k \to \infty$, $\widehat{T} \to kc_L/(\lambda(2-\sigma)) = c_H/(\lambda(2-\sigma)) = T^*_{c_H}$. In other words, for large $k$, the uniform price converges to the price that the monopolist would set for the high-demand good under price discrimination. ■

From the duality result in the main paper, Proposition 1 implies that for exponential and log-convex demand from the generalized Pareto family given by (1) and for log-concave demand given by (1) but with high enough $k$, banning a monopolist platform from setting ad-valorem fees will lower welfare.

Proposition 1 covers two types of cases. In the first case, demand is log-concave, so $\sigma < 1$. As an example, consider the case $\sigma = 0$ so that the monopolist faces linear demand for good $c$ given by

$$Q_c(T_c) = \left(1 - \frac{\lambda T_c}{c}\right). \tag{8}$$

Then there is a choke price at which demand becomes zero. As a result, provided $k$ is sufficiently high, a monopolist that can only set a single price will want to stop selling the good $c_L$ (i.e. the low-demand good) by setting its single profit maximizing price at the monopoly level for good $c_H$ (i.e. the high-demand good). This is because continuing to sell the low-demand good will sacrifice too much of the monopolist's profit from the high-demand good. Allowing it to price discriminate will not only increase the monopolist's profit but also consumer surplus and so welfare since sales of low-demand goods are enabled. The condition for this to arise in the linear demand example given in (8) is $k > 3$ when $d = 0$, so that the dispersion of demand

across goods does not have to be very high for the result to hold. The proposition shows the same logic holds for any log-concave demand. Figure 1 illustrates for the linear demand case. It also shows as $d$ increases, the critical value of $k$ declines, so our welfare finding continues to hold for $d > 0$.

For the exponential and log-convex case, the logic is actually similar even though there is no choke price at which demand becomes zero. When demand for the two goods is sufficiently dispersed, a monopolist that can only set a single price will set a price very close to the monopoly price for the high-demand good, thereby almost ruling out all sales of the low-demand good, and implying a welfare gain of allowing price discrimination. Figure 2 illustrates this property, showing the monopolist's optimal prices with and without price discrimination as $k$ varies in the particular case of exponential demand. In the proof of Proposition 1 we note this property formally by showing that as $k \to \infty$, the uniform price in the absence of price discrimination converges to the price that the monopolist would set for good $c_H$ under price discrimination, both when demand is exponential and when it is log-convex. This explains the result when the two goods are sufficiently dispersed. On the other hand, when the two goods are not dispersed very much, we know already from Aguirre et al. (2010) that price discrimination also raises welfare, which is consistent with our finding in Proposition 1.

The welfare effects of allowing price discrimination (i.e., an ad-valorem fee schedule) for the different cases captured by Proposition 1 are shown in Figure 1 in the main paper. In the log-concave case with $d = 0$, the critical level of $k$ for which welfare is higher under price discrimination than under uniform pricing is $k > 3.5$, so quantitatively we do not require unreasonably high dispersion in the demand for goods to get the welfare result.[1] In the exponential or log-convex case, Figure 1 in the main paper shows that welfare is always higher under price discrimination regardless of the level of dispersion $k$. For both cases, Figure 1 in the main paper also shows that the welfare finding extends to $d > 0$.

[1]Our numerical analysis suggests that the critical level of $k$ for which welfare becomes higher under price discrimination declines as $\sigma$ decreases below $-1$, so the sufficient condition $k > 3.5$ continues to hold.

Figure 1: Monopoly prices with linear demand (two good case)

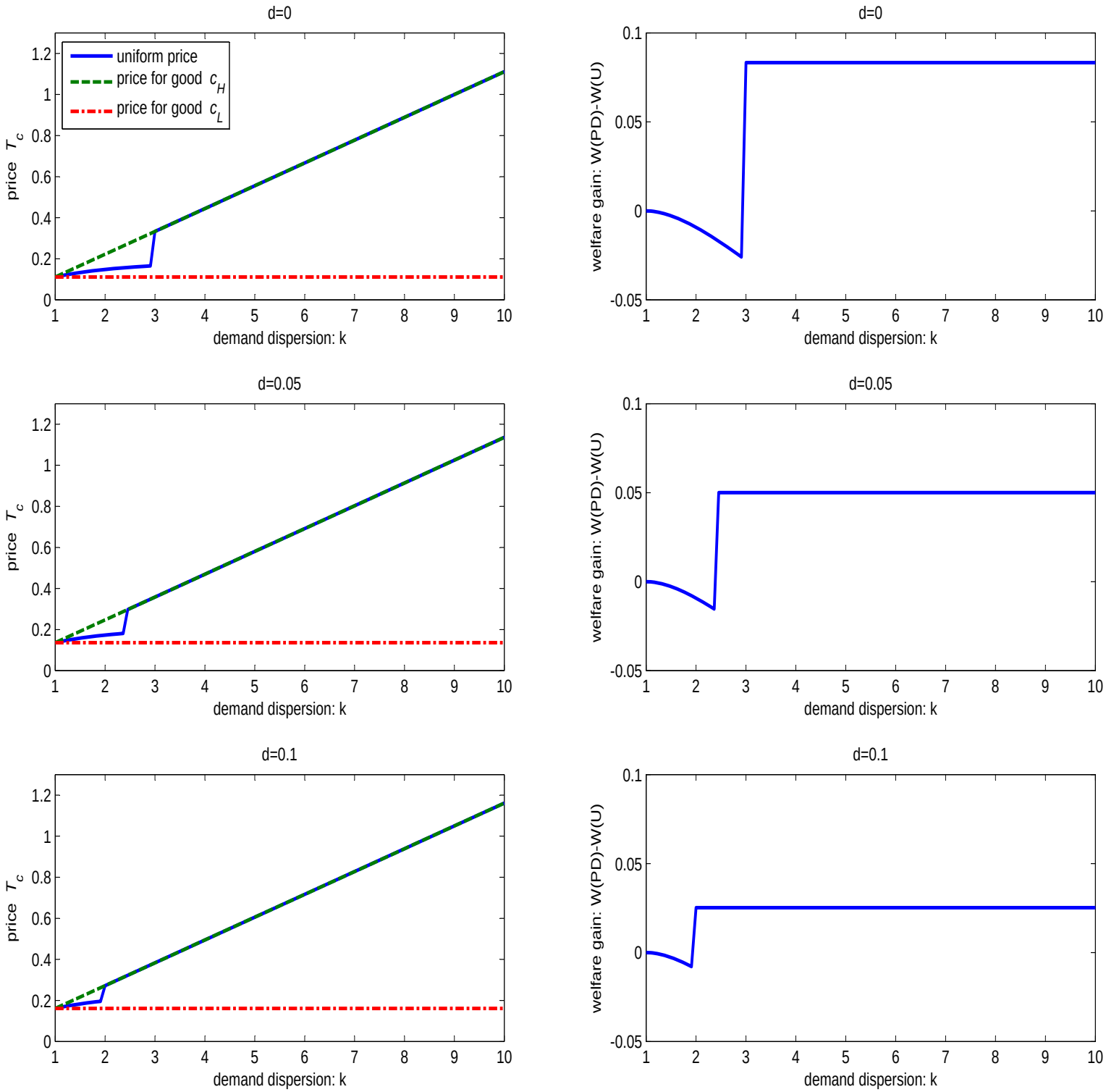
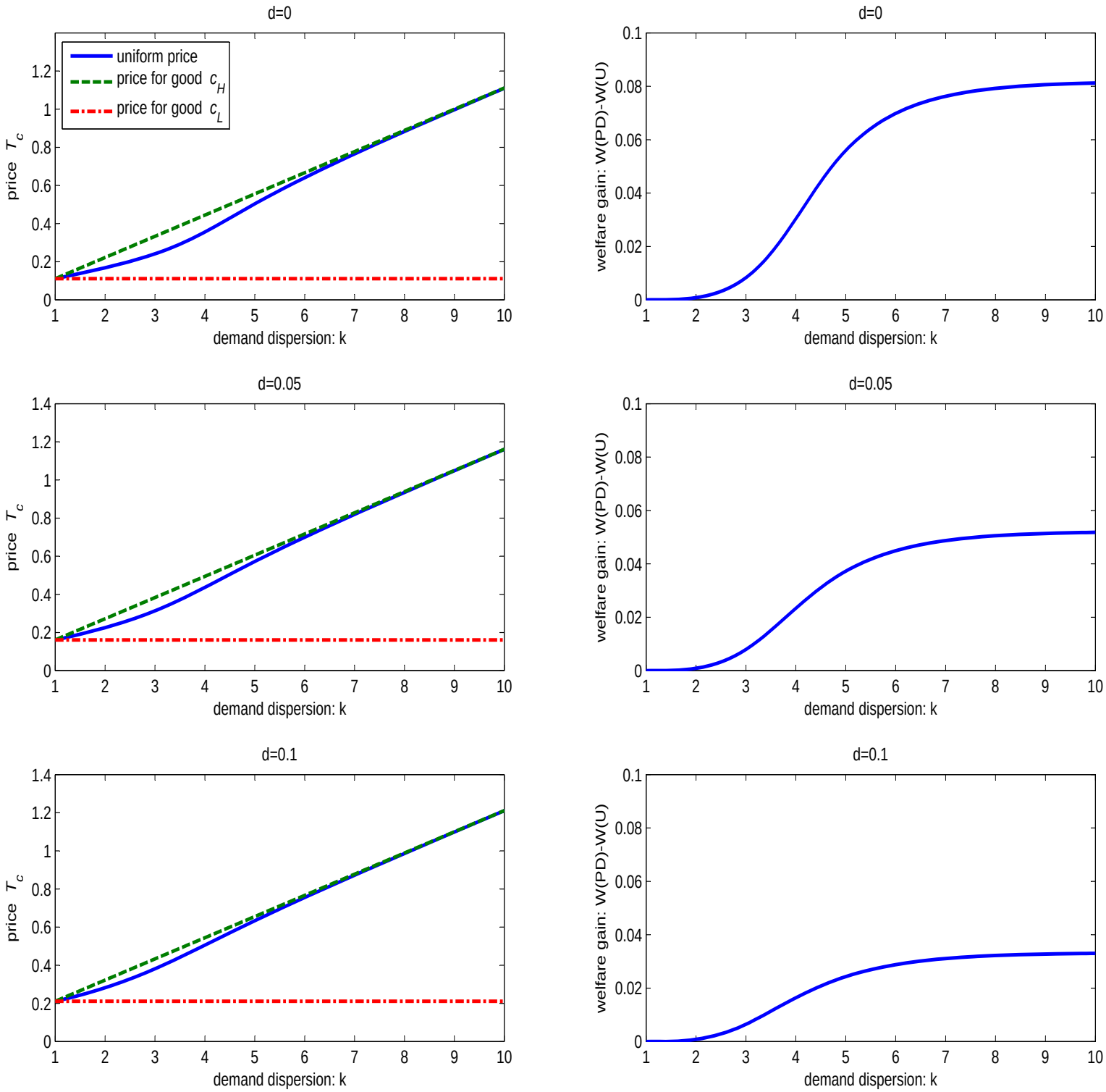


Figure 2: Monopoly prices with exponential demand (two good case)



# 3 Continuum of goods

As noted in the main paper, the qualitative conclusions on the welfare-gains of price discrimination (or equivalently, allowing ad-valorem fees) can hold when there are many goods rather than just two. In this section we will assume $c \in U[c_L, c_H]$ with $c_L > 0$, $c_H = kc_L$ and $k > 1$. Figures 3 and 4 below replicate Figures 1 and 2 for this case.

More generally, Figure 2 in the main paper shows that provided $k$ is large enough when demand is log-concave, and for all $k$ when demand is exponential or log-convex, welfare is higher under price discrimination (and so when a platform can use ad-valorem fees). In the log-concave case with $d = 0$, the critical level of $k$ for which welfare is higher under price discrimination than under uniform pricing is $k > 5$, and the critical value of $k$ declines as $d$ increases, so the dispersion of demand across goods does not have to be very high for price discrimination to generate higher welfare than uniform pricing.

In this supplementary appendix we provide some additional analysis to establish some analytical results. Throughout we assume $d = 0$. We first show some additional working for the welfare results noted in the main paper in the special case in which $\sigma = 0$ (so demand is linear). We then establish the result noted in the main paper that welfare is always higher under price discrimination whenever demand is log-concave provided there is enough dispersion of demand across goods. In particular, we show there is a cutoff level of $c$ equal to $xc_L$ (where $1 < x < k$) such that all goods with lower values of $c$ than this will be dropped by the monopolist, provided that the dispersion of demand across goods is large enough (i.e. $k > k_0$). We show that the threshold $k_0$ depends only on $\sigma$, and the cutoff value $x$ is a constant fraction of $k$ provided $k > k_0$.

## 3.1 Linear demand

Inverse demand faced by the monopolist for good $c$ is

$$T_c(Q_c) = \frac{c(1 - Q_c)}{\lambda}.$$

Figure 3: Monopoly prices with linear demand (continuum case)

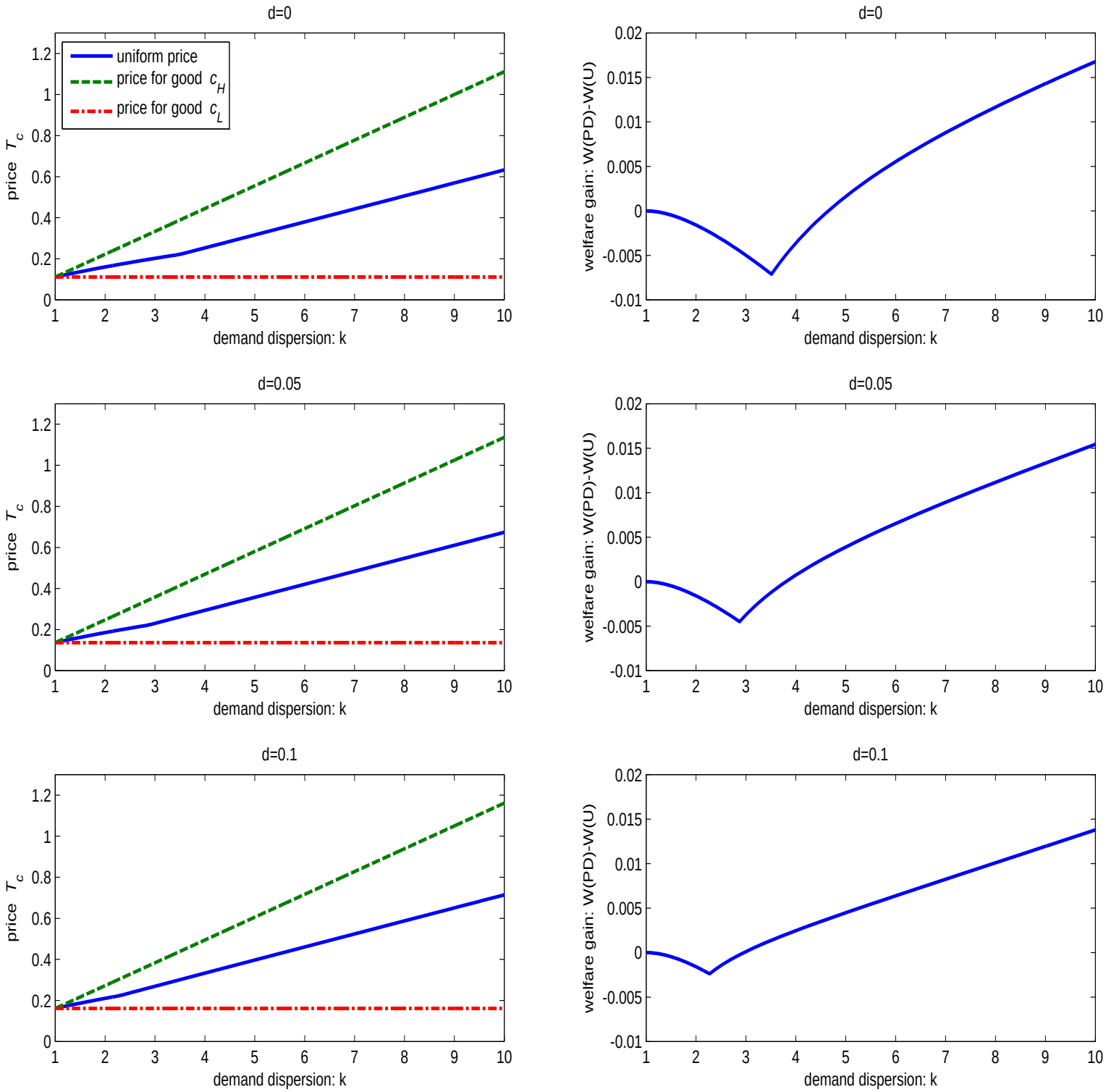
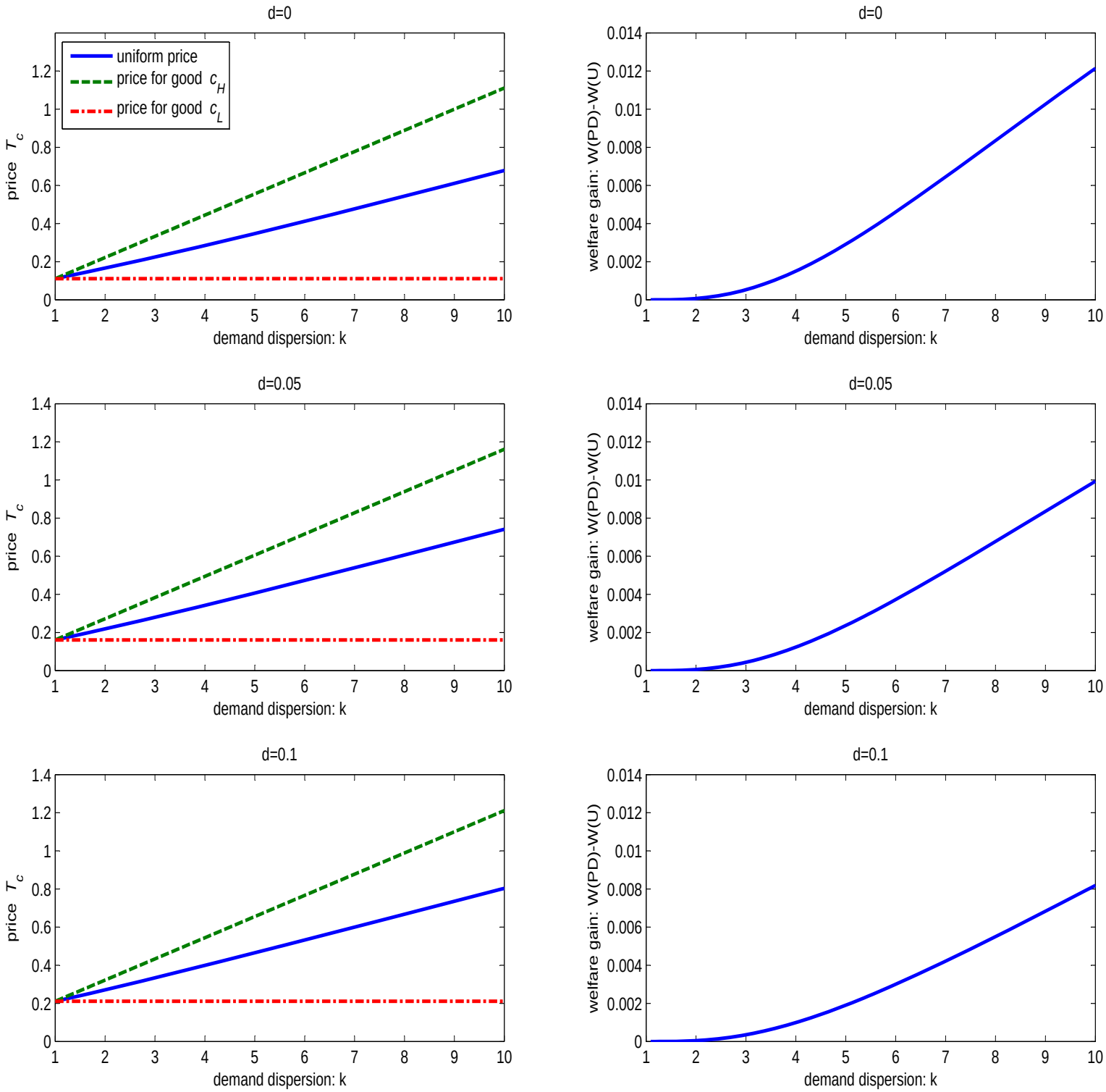


Figure 4: Monopoly prices with exponential demand (continuum case)



Then the problem is stated in exactly the same form as the third-degree price discrimination problem analyzed by Malueg and Schwartz (1994), except that we allow inverse demand to be multiplied by a constant positive parameter and we allow that the uniform distribution on $c$ be no longer centered at unity.[2] It turns out what matters for Malueg and Schwartz's results is the ratio of the highest to lowest value of $c$ in the support of the distribution, i.e. $k$. Therefore reinterpreting the relevant part of their Proposition 1 to our setting, it implies that for large enough dispersion $k > k_0$, some markets are dropped under uniform pricing; in this range, the ratio of welfare under price discrimination to welfare under uniform pricing increases monotonically with dispersion and exceeds 1 when dispersion is sufficiently large.

To calculate these points precisely, define $k_0 > 1$ which solves $1 + 2\ln k_0 = k_0$, so $k_0 \simeq 3.513$. Then the point at which dispersion is sufficiently large for welfare to increase under price discrimination arises when[3]

$$k > \frac{3k_0 - \sqrt{3k_0\,(4 - k_0)}}{k_0 - 4 + \sqrt{3k_0\,(4 - k_0)}} \simeq 4.651.$$

Thus, provided there is sufficient dispersion in costs, welfare is unambiguously higher with price discrimination.

## 3.2 Log-concave demand

The finding above for linear demand can be generalized. The demand for each good $c$ is given by (1) and inverse demand is

$$T_c\,(Q_c) = \frac{c\,(1 - Q_c^{1-\sigma})}{\lambda(1-\sigma)},$$

where $\sigma < 1$ given demand is log-concave. We will show that given these assumptions, the monopolist will set the price such that goods with $c$ lower than the cutoff level $xc_L$ will be dropped, provided that the dispersion of demand across goods is large enough (i.e. $k > k_0$). We show that the threshold $k_0$ depends only on $\sigma$, and the cutoff value $x$ is a constant fraction

[2] Their specification can be obtained by setting $\lambda = 1$, $c = a$, $c_L = 1 - x$ and $c_H = 1 + x$.

[3] There is a typo in Malueg and Schwartz's stated formula for this threshold (in their footnote 17) which does not generate the approximate numerical value they state in the footnote. However, their stated numerical value corresponds to ours, which we derived directly with our specification. I.e. if their threshold is denoted $x_e$ and ours is denoted $k_e$, then it can be checked that $k_e = (1 + x_e)\,/\,(1 - x_e)$.

of the dispersion $k$.

### 3.2.1 Allowing for price discrimination

If price discrimination is allowed, the monopolist solves the following problem for each good $c$:

$$\underset{T_c}{Max}\ \Pi_c = T_c \left(1 - \frac{\lambda\left(1-\sigma\right)T_c}{c}\right)^{\frac{1}{1-\sigma}}.$$

The first order condition yields the optimal price

$$T_c = \frac{c}{(2-\sigma)\lambda}.$$

Let the optimal price be denoted $T_c^*$. The corresponding demand for good $c$ is

$$Q_c\left(T_c^*\right) = \left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}},$$

and the monopolist's profit is

$$\Pi_c = \frac{c}{\lambda}\left(\frac{1}{2-\sigma}\right)^{\frac{2-\sigma}{1-\sigma}}.$$

The resulting welfare from good $c$ is

$$\begin{aligned} W_c &= \int_0^{Q_c} T_c\left(Q\right)dQ = \int_0^{Q_c} \frac{c\left(1-Q^{1-\sigma}\right)}{\lambda(1-\sigma)}dQ \\ &= \frac{c}{\lambda(1-\sigma)}\left[\left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}} - \left(\frac{1}{2-\sigma}\right)^{2+\frac{1}{1-\sigma}}\right]. \end{aligned}$$

Therefore, the monopolist's profit from all goods is

$$\Pi^{PD} = \left(\frac{1}{c_H - c_L}\right)\int_{c_L}^{c_H}\Pi_c dc = \left(\frac{1}{2-\sigma}\right)^{\frac{2-\sigma}{1-\sigma}}\frac{\left(c_H + c_L\right)}{2\lambda},$$

and the overall social welfare is

$$W^{PD} = \left(\frac{1}{c_H - c_L}\right)\int_{c_L}^{c_H} W_c dc = \frac{\left(c_H + c_L\right)}{2\lambda\left(1-\sigma\right)}\left[\left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}} - \left(\frac{1}{2-\sigma}\right)^{2+\frac{1}{1-\sigma}}\right].$$

### 3.2.2 Not allowing for price discrimination

If price discrimination is not allowed, the monopolist solves for the following problem:

$$\underset{x,T}{Max}\ \Pi^U = \left(\frac{T}{c_H - c_L}\right)\int_{xc_L}^{c_H}\left(1 - \frac{\lambda\,(1-\sigma)\,T}{c}\right)^{\frac{1}{1-\sigma}} dc$$

$$s.t.\ \ x \geq 1.$$

The Lagrangian is

$$L = \left(\frac{T}{c_H - c_L}\right)\int_{xc_L}^{c_H}\left(1 - \frac{\lambda\,(1-\sigma)\,T}{c}\right)^{\frac{1}{1-\sigma}} dc + \gamma(x-1),$$

where $\gamma$ is the Lagrangian multiplier.

The first order condition for $x$ when $x \geq 1$ is not binding is

$$\frac{\partial L}{\partial x} = 0 \Longrightarrow xc_L = (1-\sigma)\lambda T.$$

The first order condition for $T$ is

$$\frac{\partial L}{\partial T} = 0 \Longrightarrow \int_{(1-\sigma)\lambda T}^{c_H}\left(1 - \frac{(1-\sigma)\,\lambda T}{c}\right)^{\frac{1}{1-\sigma}}\left(\frac{c-(2-\sigma)\lambda T}{c-(1-\sigma)\,\lambda T}\right) dc = 0. \tag{9}$$

Define $c/(\lambda T) = t$. We can rewrite (9) as follows:

$$\lambda T\int_{1-\sigma}^{\frac{c_H}{\lambda T}}\left(1 - \frac{(1-\sigma)}{t}\right)^{\frac{1}{1-\sigma}}\left(\frac{t-(2-\sigma)}{t-(1-\sigma)}\right) dt = 0.$$

Let the optimal fee be denoted $\widehat{T}$. Accordingly, the optimal solution requires $c_H$ and $\widehat{T}$ always being proportional, i.e. $\widehat{T} = c_H/(z\lambda)$, where $z$ is a constant satisfying

$$\int_{1-\sigma}^{z}\left(1 - \frac{(1-\sigma)}{t}\right)^{\frac{1}{1-\sigma}}\left(\frac{t-(2-\sigma)}{t-(1-\sigma)}\right) dt = 0.$$

Therefore, the larger the $c_H$, the larger the $\widehat{T}$ and $x$. Define the threshold $k_0 = \frac{z}{(1-\sigma)}$ for a given

$\sigma$. When $k = \frac{c_H}{c_L} > k_0$, some low-$c$ goods get dropped because

$$xc_L = (1-\sigma)\lambda\widehat{T} > c_L. \tag{10}$$

Given $\widehat{T} = c_H/(z\lambda)$, (10) implies that the cutoff value $x$ is a constant fraction of $k$, i.e.

$$x = \frac{(1-\sigma)}{z}k, \tag{11}$$

which implies that $x$ is uniquely determined by $\sigma$ but not $\lambda$ (i.e. $\lambda$ is a scale parameter which does not affect $x$).

In the following discussion, we assume $k > k_0$, so some low-$c$ goods get dropped. The corresponding welfare for good $c$ is

$$W_c^U = \int_0^{Q_c\left(\widehat{T}\right)} \frac{c\left(1-Q^{1-\sigma}\right)}{\lambda(1-\sigma)} dQ = \frac{c}{\lambda(1-\sigma)}\left[Q_c\left(\widehat{T}\right) - \frac{Q_c\left(\widehat{T}\right)^{2-\sigma}}{2-\sigma}\right],$$

and the total welfare is

$$\begin{aligned} W^U &= \frac{1}{c_H - c_L}\int_{(1-\sigma)\lambda\widehat{T}}^{c_H} \frac{c}{\lambda(1-\sigma)}\left[Q_c\left(\widehat{T}\right) - \frac{Q_c\left(\widehat{T}\right)^{2-\sigma}}{2-\sigma}\right] dc \\ &= \frac{1}{c_H - c_L}\int_{(1-\sigma)\lambda\widehat{T}}^{c_H} \frac{c}{\lambda(1-\sigma)}\left[\left(\frac{c-(1-\sigma)\lambda\widehat{T}}{c}\right)^{\frac{1}{1-\sigma}} - \frac{\left(\frac{c-(1-\sigma)\lambda\widehat{T}}{c}\right)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma}\right] dc \\ &= \frac{1}{c_H - c_L}\int_{\frac{(1-\sigma)c_H}{z}}^{c_H} \frac{c}{\lambda(1-\sigma)}\left[\left(1-\frac{(1-\sigma)c_H}{cz}\right)^{\frac{1}{1-\sigma}} - \frac{\left(1-\frac{(1-\sigma)c_H}{cz}\right)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma}\right] dc. \end{aligned} \tag{12}$$

Define $c/c_H = s$. We can rewrite (12) as

$$W^U = \frac{Rc_H^2}{c_H - c_L},$$

where $R$ is a constant satisfying

$$R = \int_{\frac{1-\sigma}{z}}^{1} \frac{s}{\lambda(1-\sigma)} \left[ \left(1 - \frac{1-\sigma}{sz}\right)^{\frac{1}{1-\sigma}} - \frac{\left(1 - \frac{1-\sigma}{sz}\right)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma} \right] ds.$$

### 3.2.3 Welfare Comparison

As shown above, the welfare under price discrimination is

$$W^{PD} = ac_H + ac_L,$$

where

$$a = \frac{1}{2\lambda(1-\sigma)} \left[ \left(\frac{1}{2-\sigma}\right)^{\frac{1}{1-\sigma}} - \left(\frac{1}{2-\sigma}\right)^{2+\frac{1}{1-\sigma}} \right]. \tag{13}$$

In contrast, the welfare under uniform price is

$$W^U = \frac{Rc_H^2}{c_H - c_L},$$

where

$$R = \int_{\frac{1-\sigma}{z}}^{1} \frac{s}{\lambda(1-\sigma)} \left[ \left(1 - \frac{1-\sigma}{sz}\right)^{\frac{1}{1-\sigma}} - \frac{\left(1 - \frac{1-\sigma}{sz}\right)^{\frac{2-\sigma}{1-\sigma}}}{2-\sigma} \right] ds, \tag{14}$$

and $z$ is a constant satisfying

$$\int_{1-\sigma}^{z} \left(1 - \frac{(1-\sigma)}{t}\right)^{\frac{1}{1-\sigma}} \left(\frac{t-(2-\sigma)}{t-(1-\sigma)}\right) dt = 0. \tag{15}$$

Normalize $c_L = 1$, so $c_H = k$ and the welfare difference is

$$W^{PD} - W^U = ak + a - \frac{Rk^2}{k-1}.$$

Given that $a > R$ given $\sigma < 1$, we have

$$W^{PD} - W^U > 0 \Leftrightarrow k > \sqrt{\frac{a}{a-R}}.$$

Hence, welfare is always higher under price discrimination when there is enough cost dispersion; i.e. $k > \sqrt{\frac{a}{a-R}}$.

Note from above, when there is a continuum of uniformly distributed goods and demand is log-concave, we find the monopolist that is not allowed to price discriminate will set the price such that goods with $c$ below the cutoff level $xc_L$ are dropped, provided that the dispersion of demand across goods is large enough (i.e., $k > k_0$). As (11) suggests, the cutoff value $x$ is a constant fraction of the dispersion $k$ and is unique for each given $\sigma$, i.e.

$$x = \frac{(1-\sigma)}{z}k.$$

Accordingly, the fraction of goods dropped is

$$\frac{\frac{k(1-\sigma)}{z} - 1}{k-1},$$

which increases in $k$ given $(1-\sigma)/z$ is a fraction less than one.

Again, take the linear demand $\sigma = 0$ as an example. Equation (13) can be written as

$$a = \frac{1}{2\lambda}\left[\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)^3\right] = \frac{3}{16\lambda}. \tag{16}$$

Equation (14) can be rewritten as

$$R = \frac{1}{2\lambda}\int_{\frac{1}{z}}^{1}\left[s - \frac{1}{sz^2}\right]ds = \frac{1}{2\lambda}\left[\frac{1}{2} - \frac{1}{2z^2} + \frac{1}{z^2}\ln\left(\frac{1}{z}\right)\right]. \tag{17}$$

Note that $z$ is a constant satisfying (15):

$$\int_1^z \left(1 - \frac{2}{t}\right)dt = 0,$$

which implies

$$z - 1 - 2\ln z = 0,$$

so $z \simeq 3.513$, which corresponds to $k_0$ in the analysis of Section 2. For any $k > z$, there is a cutoff level $k/z$ such that any goods $c < k/z$ will be dropped.

Given $z \simeq 3.513$, we can also compare (16) and (17),

$$W^{PD} - W^{U} > 0 \Leftrightarrow k > \sqrt{\frac{a}{a-R}} = \sqrt{\frac{3}{4/3.513 - 1}} \simeq 4.651,$$

as we found for the linear demand case.

In conclusion, we have shown for the continuum case, that when demand given by (1) is log-concave, welfare is higher under price discrimination (and so when a platform can use ad-valorem fees) provided there is sufficient dispersion of demand across goods.

# References


[1] Malueg, D. and M. Schwartz (1994) "Parallel Imports, Demand Dispersion, and International Price Discrimination," *Journal of International Economics*, 37: 167-195.