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WP 09-06R

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# On the Implementation of Markov-Perfect Monetary Policy*

Michael Dotsey[†] and Andreas Hornstein[‡]

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**Abstract**

The literature on optimal monetary policy in New Keynesian models under both commitment and discretion usually solves for the optimal allocations that are consistent with a rational expectations market equilibrium, but it does not study whether the policy can be implemented given the available policy instruments. Recently, King and Wolman (2004) have provided an example for which a time-consistent policy cannot be implemented through the control of nominal money balances. In particular, they find that equilibria are not unique under a money stock regime and they attribute the non-uniqueness to strategic complementarities in the price-setting process. We clarify how the choice of monetary policy instrument contributes to the emergence of strategic complementarities in the King and Wolman (2004) example. In particular, we show that for an alternative monetary policy instrument, namely, the nominal interest rate, there exists a unique Markov-perfect equilibrium. We also discuss how a time-consistent planner can implement the optimal allocation by simply announcing his policy rule in a decentralized setting.

JEL Classification: E4, E5, E6

Keywords: Monetary policy, Markov-perfect, determinacy, interest rate rules, money supply rules

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# 1 Introduction

Currently there is a growing literature exploring the features of optimal monetary policy in New Keynesian models under both commitment and discretion. This work usually assumes that the optimal policy solves a constrained planning problem where the policymaker chooses among all allocations that are consistent with a market equilibrium. Recently, however, attention has been paid to how to implement the optimal policy through instrument rules. We believe that this is an important area of inquiry because the institutions responsible for setting policies rarely have direct control over allocations. It is therefore important to understand whether or not a planner's allocations are obtainable with a given institutional structure.

For the case of time-consistent policies that are Markov-perfect, King and Wolman (2004) have examined implementation issues when the monetary authority uses nominal money balances as the policy instrument in a sticky price environment. Surprisingly, they find that equilibria are no longer unique under a money-supply regime. Conditional on a given continuation allocation determined by the future policymaker, the current policymaker cannot implement a unique point-in-time equilibrium. These multiple equilibria are supported by strategic complementarities in the price-setting process. In particular, if a price-setting firm believes that all other price-adjusting firms will set relatively high (low) prices, then it will be optimal for the individual firm to set a relatively high (low) price.

In this paper we clarify how strategic complementarities that are inherent to the price-setting process interact with Markov-perfect policies. For the case of King and Wolman's (2004) money-supply rule, we show that multiple equilibria arise because the money-supply rule weakens the existing strategic complementarities in the price-setting process for low inflation outcomes. We then study the implementability of a Markov-perfect nominal interest rate policy, since actual monetary policy is usually implemented through interest rate policies. We find that a policy that uses the nominal interest rate as the policy instrument implements a unique point-in-time equilibrium. We obtain this result because contrary to the money-supply rule, the nominal interest rate instrument uniformly strengthens strategic complementarities and thereby eliminates multiple equilibria. Finally, we briefly discuss how a Markov-perfect nominal interest rate policy can also implement a unique rational expectations equilibrium.

The comment proceeds as follows. First, we briefly describe the standard New Keynesian economy used by King and Wolman (2004). Second, we describe the strategic complementarities in the price-setting process for firms. Third, we review the King and Wolman (2004) result that using a money-supply instrument generates multiple equilibria. Fourth, we show that using an interest rate instrument uniquely implements the Markov-perfect allocation. Finally, we discuss how a synthesis of the two instruments, the money supply and the nominal interest rate, uniquely implements the Markov-perfect allocation as a rational expectations equilibrium. A brief summary concludes.

## 2 The Economy

There is an infinitely lived representative household with preferences over consumption and leisure. The consumption good is produced using a constant-returns-to-scale technology with a continuum of differentiated intermediate goods. Each intermediate good is produced by a monopolistically competitive firm with labor as the only input. Intermediate goods firms set the nominal price for their products for two periods, and an equal share of intermediate firms adjusts their nominal price in any particular period. The economy we study is standard, and to save on space we provide only an outline of the economic environment and a summary of the equilibrium conditions. For a more detailed derivation, see, for example, King and Wolman (2004).

## 2.1 The environment

The representative household's utility is a function of consumption $c_t$, and the fraction of time spent working $n_t$,

$$\sum_{t=0}^{\infty} \beta^t \left[ \ln c_t - \chi n_t \right], \tag{1}$$

where $\chi \geq 0$, and $0 < \beta < 1$. The household's period budget constraint is

$$P_t c_t + B_t + M_t \leq W_t n_t + R_{t-1} B_{t-1} + M_{t-1} - P_{t-1} c_{t-1} + D_t + T_t, \tag{2}$$

where $P_t$ is the nominal price level, $W_t$ is the nominal wage rate, $B_t$ are the end-of-period holdings of nominal bonds, $T_t$ are lump-sum transfers, and $R_{t-1}$ is the gross nominal interest rate on bonds. The agent owns all firms in the economy, and $D_t$ is nominal profit income from firms. The household is assumed to hold money in order to pay for consumption purchases

$$P_t c_t \leq M_t, \tag{3}$$

and money holdings $M_t$ are adjusted at the beginning of the period. We will use the term "real" to denote nominal variables deflated by the nominal price level, which is the price of the aggregate consumption good, and we use lowercase letters to denote real variables. For example, real balances are $m_t \equiv M_t / P_t$.

The consumption good is produced using a continuum of differentiated intermediate goods as inputs to a constant-returns-to-scale technology. There is a measure one of intermediate goods, indexed $j \in [0, 1]$. Production of the consumption good $c_t$ as a function of intermediate goods $y_t(j)$ is

$$c_t = \left[ \int_0^1 y_t(j)^{(\varepsilon-1)/\varepsilon} dj \right]^{\varepsilon/(\varepsilon-1)}, \tag{4}$$

where $\varepsilon > 1$. Given nominal prices $P_t(j)$ for the intermediate goods, the nominal unit cost and price of the consumption good is

$$P_t = \left[\int_0^1 P_t(j)^{1-\varepsilon} dj\right]^{1/(1-\varepsilon)}, \tag{5}$$

and the relative price of good $j$ is $p_t(j) = P_t(j)/P_t$. Producers of the consumption good behave competitively in their markets.

Each intermediate good is produced by a single firm, and $j$ indexes both the firm and good. Firm $j$ produces its good using a constant-returns technology with labor as the only input,

$$y_t(j) = n_t(j). \tag{6}$$

In the labor market, each firm behaves competitively and takes wages as given, but since each intermediate good is unique, intermediate goods producers have some monopoly power for their product. Intermediate goods producers set their nominal price for two periods, and they maximize the discounted expected present value of current and future profits. Since the firm is owned by the representative household, the household's intertemporal marginal rate of substitution is used to discount future profits.

## 2.2 A symmetric equilibrium

We will study a symmetric equilibrium where all intermediate goods producers that face the same constraints behave the same. This means that in every period there will be two firm types: the firms that adjust their nominal price in the current period, type 0 firms with relative price $p_{0,t}$, and the firms that adjusted their price in the previous period, type 1 firms with current relative price $p_{1,t}$. Each period, half of all firms have the option to adjust their nominal price. The equilibrium of the economy is completely described by the sequence of real marginal cost, relative prices, inflation rates, nominal interest rates, and real balances,

$\{\psi_t, p_{0,t}, p_{1,t}, \pi_{t+1,} R_t, m_t\}$, such that

$$0 = (p_{0,t})^{-\varepsilon-1} (\mu\psi_t - p_{0,t}) + \beta \left(\frac{p_{0,t}}{\pi_{t+1}}\right)^{-\varepsilon-1} \left(\mu\psi_{t+1} - \frac{p_{0,t}}{\pi_{t+1}}\right) \frac{1}{\pi_{t+1}} \tag{7}$$

$$1 = 0.5 \left[p_{0,t}^{1-\varepsilon} + p_{1,t}^{1-\varepsilon}\right] \tag{8}$$

$$\pi_{t+1} = \frac{p_{0,t}}{p_{1,t+1}} \tag{9}$$

$$m_t = \psi_t / \chi \tag{10}$$

$$\psi_t = \frac{\pi_{t+1}}{\beta R_t} \psi_{t+1}. \tag{11}$$

Equation (7) represents the optimal pricing equation for a firm that can adjust its price in the current period. The first term on the right-hand side is the current period marginal profit, the second term is the discounted present value of next period's marginal profit, and $\mu = \varepsilon / (\varepsilon - 1)$ is the markup from the static profit maximization problem.[1] Equation (8) is the price index equation (5) in terms of relative prices. Equation (9) relates the inflation rate $\pi_{t+1} \equiv P_{t+1}/P_t$ to the ratio of a price-adjusting firm's optimal current relative price and next period's preset relative price. Equation (10) relates real balances to real marginal cost, using the household's optimal labor supply condition, together with the fact that real balances are equal to consumption. Equation (11) is the household Euler equation after substituting for the marginal utility of income. For ease of exposition, we will drop time subscripts from now on and denote next period's values by a prime.

Allocations in this economy are suboptimal because of two distortions. The first distortion results from the monopolistically competitive structure of intermediate goods production: the price of an intermediate good exceeds its marginal cost. The second distortion reflects inefficient production when relative prices are different from one.[2] The policymaker is assumed to maximize the lifetime utility of the representative agent, taking the competitive equilibrium conditions (7)-(11) as constraints. For a time-consistent Markov-perfect policy, the policymaker takes future policy choices as given and current policy choices are

[1] The current and discounted future profits are scaled by $1/(\varepsilon - 1)$.

[2] For a more detailed discussion of the distortions, see King and Wolman (2004).

functions of payoff-relevant state variables only. Because there are no state variables in our example, this amounts to the planner choosing a non-contingent allocation that maximizes the current period utility function of the representative agent. Taking future policy as given means that the planner has no control over future outcomes, such as future relative prices or allocations.

One usually states the problem in terms of the planner choosing the market allocation. In this case we can view the planner choosing a vector $x = (p_0, p_1, \pi', \psi)$ subject to constraints (7)-(11), and conditional on the choices of next period's policymaker, $x'$. The planner's choices determine the representative household's utility through their impact on allocational efficiency and the markup. Note that the statement of the planner's problem in terms of the market allocation does not involve any reference to the policy instrument, $z$, be it real balances or the nominal interest rate. To determine whether the Markov-perfect equilibrium can be implemented as a competitive equilibrium, we have to characterize the feasible set for market outcomes $x$ conditional on the policy instrument.

## 3 Implementation of Point-in-Time Equilibria

In most models of monetary economies, money-supply policies lead to a unique equilibrium with a determinate price level, whereas interest rate policies imply equilibrium indeterminacy. Exactly the opposite is true for the simple economy we have just described. As King and Wolman (2004) have shown, a Markov-perfect money-supply rule will imply non-uniqueness for the point-in-time equilibrium, and as we will show, a Markov-perfect interest rate policy will imply a unique point-in-time equilibrium. It turns out that (non)uniqueness of the equilibrium is related to the presence of strategic complementarities in the price-setting process and how the policy rule amplifies or weakens these complementarities.

Before we discuss the two policy rules, we want to demonstrate that strategic complementarities are inherent to the firms' price-setting problem. In the context of the model's monopolistic-competition framework, strategic complementarities are said to be present if it is optimal for an individual price-adjusting firm to increase its own relative price, $p_0$, if all other price-adjusting firms increase their relative price, $\bar{p}_0$. To study this issue we use a graphic representation of the individual firm's FOC for profit maximization, (7), which states that the sum of today's marginal profit, $MP(p_0, \psi)$, and tomorrow's discounted marginal profit, $\beta MP(p_0/\pi', \psi')/\pi'$, has to be zero. For the profit maximization problem to be well-defined, we need the profit function to be concave; that is, the marginal profit function $MP$ is decreasing in the relative price. In the Appendix we also show that

**Proposition 1** *With constant marginal cost, $\psi = \psi'$, tomorrow's marginal profit, $MP(p_0/\pi', \psi')/\pi'$, is increasing in the inflation rate $\pi'$ for a neighborhood around zero inflation, $\pi' = 1$.*

In Figure 1, we graph today's marginal profit (solid line) and tomorrow's marginal profit (dashed line) for an individual firm conditional on all other firms' relative price, $\bar{p}_0$, and a positive inflation rate. The positive inflation rate erodes the firm's relative price tomorrow and therefore the firm will set its optimal price, $p_0$, above the static profit-maximizing relative price, $\mu\psi$, such that it balances today's negative marginal profit against tomorrow's positive marginal profit. Now suppose that all other firms increase their relative price. It follows from expression (9) that tomorrow's inflation rate will increase, $\pi' = \bar{p}_0/p_1'$, and this will shift tomorrow's marginal profit curve up (dashed-dot line), leaving today's marginal profit curve unchanged. It is then optimal for the individual firm to also increase its own relative price. Thus, there is a source of strategic complementarities, independent of monetary policy. The choice of monetary policy instrument will modify strategic complementarities through its general equilibrium feedback effect on marginal cost.

## 3.1 A money supply policy

We now review King and Wolman's (2004) analysis of a Markov-perfect nominal money rule that sets the nominal money stock in proportion to the preset nominal price from the last period[3]

$$M = \tilde{m}P_1. \tag{12}$$

Normalizing the policy rule (12) with the price level and combining it with the equilibrium condition (10) determines marginal cost

$$\psi = \chi\tilde{m}p_1. \tag{13}$$

King and Wolman (2004) show that for most values of the money-supply policy parameter, $\tilde{m}$, the steady-state of the economy will not be unique. Since in a Markov-perfect equilibrium without state variables the expected future policy has to be a steady state, non-uniqueness of the steady state alone suggests that the monetary policy rule may result in indeterminacy of the point-in-time equilibrium. Suppose that we choose one of the possible steady states as a continuation of the economy in the next period. We now show that the choice of a money-supply instrument weakens strategic complementarities when the average firm chooses a low relative price, and that the complementarities persist when the average firm chooses a high relative price. The resulting change in shape of the optimal reaction function, that is, the mapping from the average firm's relative price to an individual firm's optimal relative price response, gives rise to multiple point-in-time equilibria.

Consider again the response of an individual firm to an increase in the relative price set by all other firms, but now allow for the feedback of these decisions to marginal cost coming through the money stock policy. When all other price-adjusting firms increase their relative price, it follows from the price index equation, (8), that the preset relative price, $\bar{p}_1$, declines. From equation (13) it then follows that today's marginal cost declines, which in

[3]Prior to studying the Markov-perfect money rule, King and Wolman (2004) briefly discuss a monetary policy rule that exogenously sets the nominal money stock at a constant value. For this policy a firm's pricing decision is not affected by other firms decisions and the equilibrium is unique. One can also show that for a constant money growth rule the pricing decisions are charcterized by strategic complementarity (substitutability) if the money stock is shrinking (growing). Nevertheless, with an exogenous money stock the current period outcome will depend on the preset nominal price, and since this price is not payoff relevant, this policy is not Markov-perfect.

turn shifts down today's marginal profit curve in Figure 1. Thus the policy-induced feedback effect reduces the individual firm's need to increase its own relative price in response to the general price increase; that is, it weakens the strategic complementarities.

It is easily shown that the impact of $\bar{p}_0$ on $\bar{p}_1$ declines with $\bar{p}_0$. Thus, strategic complementarities are weakened the most when the relative price of price-adjusting firms is the lowest. The resulting shape of a firm's optimal response function is depicted as the dashed line in Figure 2. The graphs displayed in Figure 2 are derived for parameter values $\beta = 0.99$, $\varepsilon = 11$, $\chi = 1/\mu$, and assuming that next period's policy generates a steady-state inflation rate $\pi = 1.05$. This parameterization is standard for sticky price models and implies a static markup of 10 percent, and an annual real interest rate of 4 percent if we interpret a period as a quarter. We can see that for low values of other firms' relative price choice, there are no strategic complementarities, and the reaction function is quite flat. If other firms start setting higher relative prices, then an individual firm's own optimal relative price starts to increase and the rate at which it responds also increases. Thus, the reaction function becomes steeper than the 45-degree line and multiple equilibria due to self-fulfilling expectations are possible. In the Appendix we prove the following Proposition.

**Proposition 2** *Suppose current and future policymakers use the same money stock rule $\tilde{m}$. If $\tilde{m} \in (\tilde{m}_1, \tilde{m}_2)$, then, in general, at least two point-in-time equilibria exist. If $\tilde{m} = \tilde{m}_1$ then the point-in-time equilibrium is unique.*

## 3.2 An interest rate policy

In this section we evaluate the benefits of using an interest rate instrument to implement Markov-perfect policies. We find that steady states and point-in-time equilibria are unique, despite the fact that the reaction function remains characterized by strategic complementarities. In what follows, we solve for the current equilibrium, $x$, conditional on current policy $z = R$ and future equilibrium outcomes $x'$. With a fixed nominal interest rate, policy affects marginal cost through the Euler equation,

$$\psi = \frac{\psi'}{\beta R p_1'} p_0, \tag{14}$$

which combines (9) and (11).

The existence of a unique steady state for a given nominal interest rate is straightforward to show; see the Appendix.

**Proposition 3** *Conditional on the nominal interest rate $R \geqslant 1$, there exists a unique steady state $(p_0^*, p_1^*, \psi^*)$.*

A point-in-time equilibrium also exists and it is unique despite the continued presence of strategic complementarities. Indeed, the interest rate rule strengthens existing strategic complementarities. Consider again the response of an individual firm to an increase in the relative price set by all other firms, but now allow for the feedback coming through the interest rate policy. From equation (14) it now follows that today's marginal cost increases, which in turn shifts up today's marginal profit curve in Figure 1. Thus, the policy-induced feedback effect increases the individual firm's need to increase its own relative price in response to the general price increase; that is, it strengthens the strategic complementarities.[4] The dash-dot line in Figure 2 displays the reaction function for the interest rate policy conditional on the same parameter values used for the money stock rule. In the following proposition, proved in the Appendix, we state that as long as tomorrow's policy does not try to implement price stability, there will always exist a unique point-in-time equilibrium for the current period.

**Proposition 4** *(A) If next period's policy choice attains an inflationary or deflationary steady-state outcome, then (1) for any nominal interest rate for which a current period equilibrium exists it is unique, and (2) there always exists a nominal interest rate for which*

[4] We note that the unique equilibrium is not obtained because the interest rate instrument introduces commitment to the policymakers choice set. Indeed, the Markov-perfect solution of the planning problem is obtained without even considering how the optimal allocation can be implemented, be it through a money rule or an interest rate rule.

*an equilibrium exists. (B) If next period's policy choice attains a steady-state outcome with stable prices, then (1) the current period equilibrium is indeterminate if current policy also tries to attain the stable-price steady state $\beta R = 1$; (2) no current period equilibrium exists if $\beta R \neq 1$.*

Finally, the monetary policymaker can implement the Markov-perfect equilibrium as a globally unique rational expectations equilibrium through a policy that jointly determines the nominal interest rate and the money stock as in Carlstrom and Fuerst (2001) and Adão, Correia, and Teles (2003). The choice of a nominal interest rate eliminates the potential for multiple point-in-time equilibria, whereas the money rule picks the Markov-perfect equilibrium allocation among the possible solutions to the system of dynamic equations. Formally, a choice of $z = (\tilde{m}, R)$ that is consistent with the Markov-perfect equilibrium determines a unique rational expectations equilibrium as follows. From (13) we observe that choosing $\tilde{m} = \psi/(\chi p_1)$ yields a monetary policy that is consistent with the Markov-perfect equilibrium. Using this choice we can substitute for marginal cost in the household Euler equation (11) and together with the definition of inflation (9) and the choice of $R$ this yields the following linear restriction on current relative prices.

$$\frac{\chi \tilde{m} p_1}{p_0} = \frac{\chi \tilde{m} p_1'}{\beta R p_1'} \Rightarrow \frac{p_1}{p_0} = \frac{1}{\beta R}.$$

This restriction together with the price index equation (8) uniquely determines relative prices. Given the unique relative prices, one obtains unique solutions for real balances and marginal cost.[5]

The Markov-perfect equilibrium can only be implemented through the joint determination of the money stock and the interest rate, since, for the usual reasons, a non-contingent

[5] Standard characterizations of monetary policy have the policymaker set a price (quantity) and have the quantity (price) be determined as an equilibrium outcome. The proposed combination policy has the policymaker choosing both, price and quantity, and therefore requires that the policymaker have complete information on the state of the world. It is not obvious if a policymaker can implement this kind of combination policy in an environment with incomplete information.

nominal interest rate only policy leads to locally indeterminate equilibria. Early work of Sargent and Wallace (1975) has shown how a non-contingent nominal interest rate policy leads to nominal indeterminacy in flexible price models, and Carlstrom and Fuerst (2001) and Adão, Correia, and Teles (2003) have pointed out that such a policy leads to real indeterminacy in sticky price models. The usual procedure to eliminate dynamic indeterminacies arising from a fixed nominal interest rate policy — making the interest rate decision contingent on other endogenous variables; see, e.g., McCallum (1986), Boyd and Dotsey (1994), or Carlstrom and Fuerst (1998), cannot be used to implement Markov-perfect equilibria. This approach is not applicable, since, by definition, decisions in Markov-perfect equilibria can depend only on payoff-relevant state variables and not other endogenous variables, be they lagged or contemporaneous. A feasible way to obtain a locally unique rational expectations equilibrium for the interest rate rule is to restrict the solution to be in accord with McCallum's (1983) minimum state variable solution. Since there are no state variables, the minimum state variable solution must be the steady state, which we have shown to be unique for the interest rate policy, both in real and nominal terms. We also note that in an economy like ours with flexible prices, it is well known that the minimum state variable solution still displays nominal indeterminacy. This difference indicates another important distinction between flexible and sticky price environments.

# 4 Conclusion

In this comment we have analyzed the importance of the monetary policy instrument in decentralizing a time-consistent planner's optimal policy. In that regard, our work is part of a growing literature investigating the implementation of optimal plans. We have shown that whether a planner uses a money instrument or an interest rate instrument is crucial for determining if optimal Markov-perfect allocations can be attained via the appropriate setting of the instrument. King and Wolman (2004) were the first to alert us to the non-trivial ramifications of decentralization. They produced a surprising result of significant impact, namely, that decentralization is a non-trivial problem. With regard to using a money instrument, implementation of the optimal allocation is unattainable. A time-consistent planner using a money instrument could not implement the allocations chosen by a planner who was able to directly pick allocations. In fact, they showed that steady states and equilibria were not unique at the optimal inflation rate. Since, in reality, no central bank picks allocations, this result presents a challenge for understanding just how a time-consistent central bank might operate. Here we have shown that it does not. A planner using an interest rate instrument can achieve the Markov-perfect allocations of the standard time-consistent planning problem. The result occurs for two key reasons. The interest rate instrument pins down future inflation in ways unobtainable using a money instrument and, in so doing, increases the degree of strategic complementarity that arises from the monopolistically competitive price-setting problem itself.

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# Appendix

## A Proof of Prop 1. Strategic Complementarities

The optimal relative price of a price-setting firm satisfies the FOC for profit maximization (7). With constant marginal cost, $\psi = \psi'$, and positive inflation this implies

$$p_0 \geq \mu\psi \geq 1 \geq p_0/\pi' \tag{A.1}$$

since the marginal profit function is decreasing in $p_0$. The derivative of the firm's marginal profit tomorrow with respect to inflation is

$$\frac{\partial MP\left(p_0/\pi', \psi\right)/\pi'}{\partial \pi'} = (\varepsilon - 1)\left(\frac{p_0}{\pi'}\right)^{-\varepsilon-1}\left(\mu^2\psi - \frac{p_0}{\pi'}\right)\frac{1}{\pi'^2} \tag{A.2}$$

Thus, tomorrow's marginal profit is increasing in inflation if and only if

$$\mu^2\psi > \frac{p_0}{\pi'}. \tag{A.3}$$

Note that with zero inflation the optimal relative price satisfies $p_0 = \mu\psi$. Since we have a positive markup, $\mu > 1$, we get

$$\mu^2\psi > \mu\psi = p_0. \tag{A.4}$$

By continuity condition (A.3) is satisfied for a neighborhood around zero inflation.■

## B Proof of Prop 2. Non-uniqueness of PITE with Money Rule

Suppose that today's and tomorrow's policymakers choose the same policy rule within the set of feasible policy rules, $\tilde{m} = \tilde{m}' \in (\tilde{m}_1, \tilde{m}_2)$. It is straightforward to show that this policy is consistent with the existence of two steady-state equilibria (King and Wolman (2004)). We now show that even conditional on choosing future behavior to be in accord with one of the two possible steady states, $p_1' = p_1^*$ and $\psi' = \psi^*$, there exist two point-in-time equilibria in the current period. An individual firm's optimal relative price is determined by the profit maximization condition, (7),

$$p_0 = \mu\frac{\psi + \beta\psi^*\pi'^{\varepsilon}}{1 + \beta\pi'^{\varepsilon-1}}, \tag{B.1}$$

conditional on today's marginal cost and tomorrow's marginal cost and inflation rate. Together with the policy rule (13) and the definition of the inflation rate (9), the reaction function simplifies to

$$\frac{1}{\mu\chi\tilde{m}}p_0 = p_0\frac{(p_1/p_0) + \beta\left(p_0/p_1^*\right)^{\varepsilon-1}}{1 + \beta\left(p_0/p_1^*\right)^{\varepsilon-1}} = g\left(p_0, p_1^*\right). \tag{B.2}$$

In equation (B.2) the left-hand side price $p_0$ is interpreted as an individual firm's optimal relative price in response to the expected aggregate relative prices, $p_0$ and $p_1^*$, on the right-hand side. Note that the price index equation (8) implies that $p_1$ is a decreasing function of $p_0$. For parameter values and policy choice such that $\mu\chi\tilde{m} = 1$, we can interpret $g$ as the reaction function and Figure 3 can be used to visualize the argument below.

One can show that the "reaction" function $g$ in terms of the relative price $p_0$ intersects the 45-degree line at $p_0 = 1$ and is above (below) the 45-degree line when $p_0$ is less than (greater than) one,

$$g\left(p_0, p_1^*\right) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} p_0 \text{ for } p_0 \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 1. \tag{B.3}$$

As $p_0$ becomes large, $g(p_0, p_1^*)$ converges to the 45-degree line from below,

$$\lim_{p_0 \to \infty} g\left(p_0, p_1^*\right) = p_0. \tag{B.4}$$

With some some additional algebra, one can show that the derivative of the $g$ function at $p_0 = 1$ is

$$\frac{\partial g\left(p_0, p_1^*\right)}{\partial p_0}|_{p_0=1} = -\frac{1 - \beta\left(p_1^*\right)^{1-\varepsilon}}{1 + \beta\left(p_1^*\right)^{1-\varepsilon}}. \tag{B.5}$$

We can now show that for $\tilde{m} \in (\tilde{m}_1, \tilde{m}_2)$ the LHS and the RHS of expression (B.2) will in general intersect twice. On the one hand, from the properties of the steady state it follows that since $\tilde{m} > \tilde{m}_1$, that is, $\mu\chi\tilde{m} > 1$, the slope coefficient of the LHS linear expression in $p_0$ is less than one. Thus the LHS defines a line through the origin below the 45-degree line. On the other hand, the RHS of (B.2) intersects the 45-degree line at $p_0 = 1$ and stays above (below) the 45-degree line whenever $p_0$ is less than (greater than) one. Furthermore, as $p_0$ becomes arbitrarily large the RHS of (B.2) converges to the 45-degree line from below.

Since the LHS is strictly below the RHS for $p_0 \leq 1$, the two curves do not intersect in this range. We know that at least one intersection point exists, since we consider policy rules that are consistent with the existence of a steady state, and the steady-state price is a solution to the reaction function (B.2). Thus, there must be an intersection point for $p_0 > 1$.

If $\tilde{m} = \tilde{m}_1$, then we know that a unique non-inflationary steady state with $p_0 = 1$ exists, and this steady state also satisfies (B.2). For this case, the LHS is the 45-degree line and the RHS has a unique intersection with the 45-degree line at $p_0 = 1$. Furthermore, from (B.5) it follows that the slope of the RHS at $p_0 = 1$ is negative. With a marginally larger value of $\tilde{m}$, the slope of the LHS becomes less than one, and there will be at least two intersections with the RHS to the right of $p_0 = 1$.■

# C Proof of Prop 3. Uniqueness of Steady State with Interest Rate Rule

Equations (14) and (9) determine the unique steady-state inflation rate

$$\pi^* = \beta R. \tag{C.1}$$

Equations (8), (9), and (C.1) uniquely determine the steady-state relative prices

$$p_0^{*\varepsilon-1} = 0.5\left(1+\pi^{*\varepsilon-1}\right) \text{ and } p_1^* = p_0^*/\pi^*. \tag{C.2}$$

From equation (7) we obtain the steady-state marginal cost

$$\psi^* = \frac{1}{\mu}\frac{1+\beta\pi^{*\varepsilon-1}}{1+\beta\pi^{*\varepsilon}}p_0^*. \tag{C.3}$$

■

# D Proof of Prop 4. (Non)uniqueness of PITE with Interest Rate Rule

The current equilibrium is defined by the two equations (14) and (7), which map the current period relative price $p_0$ to current period marginal cost $\psi$. Rewriting (7), we have

$$\psi = f_1(p_0) = \left(\frac{1}{\beta R}\frac{\psi'}{p_1'}\right)p_0 \tag{D.1}$$

$$\psi = f_2(p_0) = \frac{1}{\mu}\left(p_0 + \beta A' p_0^{\varepsilon}\right), \tag{D.2}$$

where $A' = (p_1')^{1-\varepsilon}\left(1-\mu\frac{\psi'}{p_1'}\right)$, and next period's variables are evaluated at their steady-state values, $p_1^*$ and $\psi^*$ as determined by (C.1), (C.2) and (C.3). An intersection of the two functions represents a potential equilibrium.

The two functions always intersect at $p_0 = 0$, but $p_0 = 0$ is not a feasible outcome since the price index equation (8) together with $p_1$ positive implies a lower bound $\underline{p}_0$ for the optimal relative price. Both functions are strictly increasing at $p_0 = 0$,

$$\frac{\partial f_1}{\partial p_0} = \frac{1}{\mu\beta R}\frac{\pi^* + \beta(\pi^*)^{\varepsilon}}{1+\beta(\pi^*)^{\varepsilon}} \tag{D.3}$$

$$\frac{\partial f_2}{\partial p_0} = \frac{1}{\mu}\left(1+\beta A'\varepsilon p_0^{\varepsilon-1}\right). \tag{D.4}$$

The function $f_2$ is strictly concave (linear, strictly convex) if $A' < 0$ ($A' = 0$, $A' > 0$),

$$\frac{\partial^2 f_2}{\partial p_0^2} = \frac{1}{\mu}\beta A'\varepsilon(\varepsilon-1)p_0^{\varepsilon-2}. \tag{D.5}$$

The sign of the term $A'$ depends on the inflationary stance of next period's steady-state policy. From (7) we get

$$\begin{aligned}\beta A' &= \beta\left(p_1^*\right)^{1-\varepsilon}\left\{1-\mu\left[\frac{1}{\mu}\frac{1+\beta\left(\pi^*\right)^{\varepsilon-1}}{1+\beta\left(\pi^*\right)^{\varepsilon}}p_0^*\right]\frac{1}{p_1^*}\right\} \\ &= \beta\left(p_1^*\right)^{1-\varepsilon}\left\{1-\pi'\frac{1+\beta\left(\pi^*\right)^{\varepsilon-1}}{1+\beta\left(\pi^*\right)^{\varepsilon}}\right\} \\ &= \beta\left(p_1^*\right)^{1-\varepsilon}\left\{\frac{1-\pi^*}{1+\beta\left(\pi^*\right)^{\varepsilon}}\right\}. \qquad \text{(D.6)}\end{aligned}$$

The first equality uses the steady-state expression for next period's marginal cost (C.3), and the second equality uses the steady-state expression for next period's inflation rate (C.2). Thus, $A'$ is negative (positive) if next period's policy is inflationary, $\pi^* > 1$ (deflationary, $\pi^* < 1$), and $A' = 0$ if next period's policy implements price stability, $\pi^* = 1$.

If next period's policy is inflationary and an intersection between $f_1$ and $f_2$ exists for positive values of $p_0$, the intersection point is unique since the function $f_1$ is linear and the function $f_2$ is strictly concave. The two functions intersect for positive $p_0$ if at $p_0 = 0$ the function $f_2$ is steeper than $f_1$,

$$\frac{\partial f_1}{\partial p_0} = \frac{1}{\mu}\frac{1}{\beta R}\frac{\pi^*+\beta\left(\pi^*\right)^{\varepsilon}}{1+\beta\left(\pi^*\right)^{\varepsilon}} < \frac{1}{\mu} = \left.\frac{\partial f_2}{\partial p_0}\right|_{p_0=0}. \qquad \text{(D.7)}$$

This condition can always be satisfied for a sufficiently large nominal interest rate $R \geq 1$. In other words the policymaker can always find an interest rate for which the functions intersect. Recall that there is a lower bound for feasible relative prices $\underline{p}_0$, so the policymaker has to choose an interest rate that implies a sufficiently large value for the relative price $p_0$. A policymaker can always find such an interest rate, since he can always replicate the steady state by choosing $R = R^*$. Thus there exists a choice for $R$ such that an equilibrium exists and it is unique. An analogous argument applies if next period's policy is deflationary.

If next period's policy implements price stability, that is, $\psi^* = 1/\mu$ and $p_1^* = 1$, then the only policy for today that is consistent with the existence of an equilibrium is a nominal interest rate such that $\beta R = 1$. But then equations (D.1) and (D.2) are satisfied for any feasible combination of $(p_0, \psi)$ such that

$$p_0 > \underline{p}_0 \text{ and } \psi = p_0/\mu.$$

If current policy is inflationary or deflationary, $\beta R \neq 1$, then the only solution to equations (D.1) and (D.2) is $p_0 = 0$. But $p_0 = 0$ is not a feasible outcome, so no equilibrium exists. ■

Figure 1: Strategic Complementarities

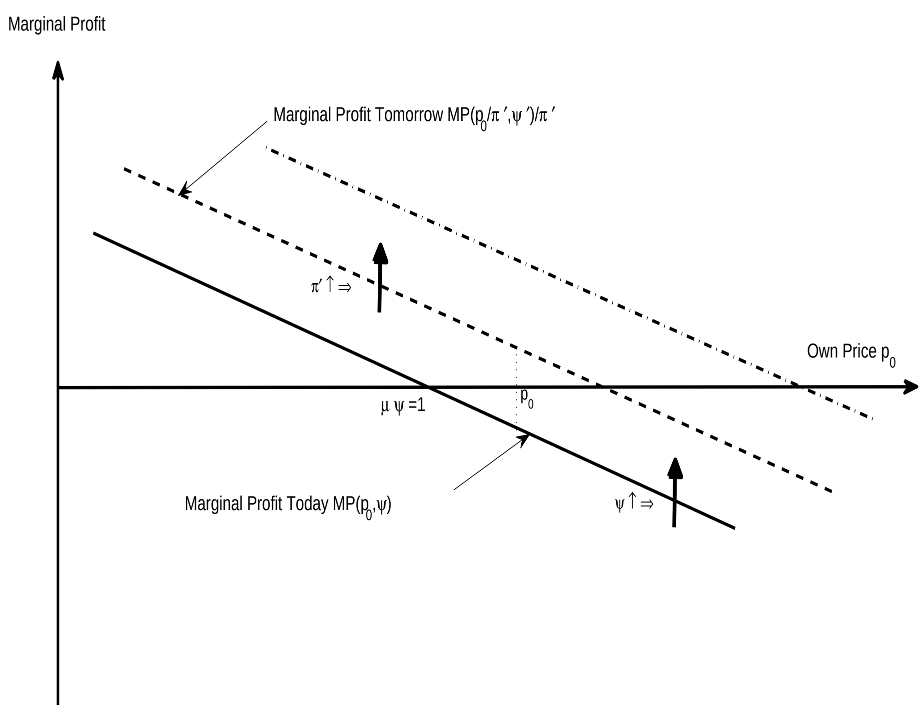

Figure 2: Reaction Functions for Money Stock and Interest Rate Rules