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Abstract

Using private information and club theories, this paper develops a theory of firms in general equilibrium. Firms are defined to be assignments of technologies and agents to clubs. In equilibrium, firms form endogenously and multiple types may co-exist. We formulate the general equilibrium problem as both a Pareto program and as a competitive equilibrium. Welfare and existence theorems are provided. In the competitive equilibrium, club memberships are priced and purchased, so the market determines which organizations exist as well as who is a member. Pareto optima and competitive equilibria of several examples are computed. In our examples, elements of limited commitment and monitoring capabilities are important factors that affect both the internal organization of firms and their equilibrium distribution. Furthermore, equilibrium organizational structure varies with the aggregate endowment of capital and the distribution of wealth.

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1 Introduction

This paper incorporates multi-agent private information models into classical general equilibrium theory. We do this by studying a model where there is risk and moral hazard in production, but where multi-agent firms may form to mitigate these incentive problems through supervision. In our theory, not only does the market set the prices of credit, insurance, and production inputs, it also determines which kinds of firms form, their internal organization, their compensation structure, and who joins them in what capacity. A byproduct of our theory is that the distribution of wealth affects the industrial organization of the economy, as well as the economy-wide distribution of labor effort and consumption.

Agency, or private information, theory has been enormously influential in the study of the firm. In early work, Alchian and Demsetz (1972) used it to develop a theory of the firm based on the supervision of team production.[1] Other work has used agency theory to study internal features of the firm. For example, Holmström and Milgrom (1990, 1991) used it to study job design and task assignment. In labor economics, Lazear and Rosen (1981) used it to study tournaments, while Mirrlees (1976) and Calvo and Wellicz (1978) used it to study incentives in hierarchies.[2] In growth and development, Bannerjee and Newman (1993) and Aghion and Bolton (1997) used agency models to study organizational structure, occupational choice, and growth. In corporate finance, agency theory has been extensively used to study capital structure by analyzing the strategic interaction between managers, equity holders, and debtors.[3]

In these examples there are two common features that we want to emphasize. First, more than one person participates in each arrangement, and second, private information causes an incentive problem. Because of these features, contracts in these environments are characterized by joint consumption and joint production. For example, in a career tournament, promotion depends on relative performance. In team production and in a hierarchy, production depends on all of the member's efforts.

Arrangements with joint production and joint consumption are precisely what club theory, developed originally by Buchanan (1965), was designed to study. A club's members,

[1]A more formal treatment is contained in Holmström (1982).

[2]Also, see the survey in Lazear (1996).

[3]See the survey in Harris and Raviv (1992).

be it through their characteristics or activities, can affect the production of the club good or their enjoyment of it. Common examples of clubs include swimming pools and marriages.

Our strategy in this paper is to treat firms as clubs. We work with a private information model where firms form to supervise production. These multi-agent supervisory arrangements require multi-agent contracts with the joint consumption and joint production features of a club good. Like Alchian and Demsetz (1972), we identify these supervisory production arrangements as firms.

We decentralize the economy using recent developments in general equilibrium club theory by Cole and E.C. Prescott (1997) and Ellickson, Grodal, Scotchmer, and Zame (1999). Unlike partial-equilibrium treatments of multi-agent principal-agent theory, our decentralization does not take reservation utility levels, input prices, or even assignments of individuals to principal-agent relationships as exogenous. Instead, club memberships (the multi-agent contracts or firms) and other goods are priced and traded in markets and the industrial organization of the economy is solely a function of the primitives.[4]

Our identification of firms as clubs departs from the classical treatment of the firm in competitive equilibrium, where a firm is an exogenously specified entity that maximizes profits.[5] Instead, our identification is closer in spirit to that of McKenzie (1959, 1981). In McKenzie's formulation, the aggregate production set is a convex cone and firms are identified with entrepreneurial factors supplied by individuals. In this paper, the aggregate production set is also a convex cone but instead of individuals supplying an entrepreneurial factor they supply their participation in a contract, participation that we identify as membership in a firm. The formation of firms thus appears as another linear activity in the aggregate production set.[6]

Section 2 contains a prototype that is used to illustrate our theory and techniques. Production requires capital and labor effort. Individuals may work alone, as in a sole proprietorship, or they may work in a two-person firm. A sole proprietor's output is public

[4]Legros and Newman (1996) is an important paper on extending multi-agent principal-agent models to general equilibrium. They studied incentive-based organizations in a general equilibrium economy with risk neutral agents. Instead of using competitive analysis, they used a core equilibrium concept and took the price of the capital input as exogenous.

[5]See Arrow and Debreu (1954).

[6]For more on this perspective, see the discussions contained in McKenzie (1981) and Hornstein and E.C. Prescott (1993).

but his effort is private information so there is a moral hazard problem. The two-person firm consists of a worker and a supervisor. The worker operates a technology identical to that of a sole proprietor. All the supervisor does is monitor the worker's effort; the supervisor's effort is not an input into production. The organizational tradeoff is between a sole proprietorship limited by moral hazard versus a supervisor-worker firm that requires costly labor monitoring.

Following Cole and E.C. Prescott (1997) and Ellickson, *et al.* (1999), we work in a continuum economy with a finite number of types of agents, where only clubs that consist of a finite number of agents may form. We make the minor assumption that points of relevant sets like consumption and output lie in finite sets and the important economic assumption that allocations are lotteries over these sets.[7] Our commodity space contains the set of incentive-compatible contracts and the corresponding jobs. Agents then choose a probability distribution, which may be deterministic, over these assignments.[8] This commodity space allows us to properly add up memberships in firms, ensuring that in equilibrium there is one person for each job in each firm.[9]

After describing the environment and defining the contracts as club goods, we formulate the general equilibrium problem as a Pareto program. In addition to the usual resource constraints on consumption and capital usage, there are also resource constraints on firm formation. These constraints ensure that for each firm created, there is one person assigned to each job in the firm. The Pareto program is linear but its dimensions are too large to practically solve using standard linear programming methods. Instead, we describe how the Pareto program can be computed using the Dantzig-Wolfe decomposition algorithm.

[7]Private information environments naturally utilize such lotteries. Consumption sets constrained by incentive compatibility conditions may not be convex, or even if they are, commodity spaces without lotteries may preclude possible gains to trade. See Cole (1989).

[8]This assignment feature of our models is related to occupational choice models. Sattinger (1993) discusses these models in his survey on assignment models of the earnings distribution. Like our paper, assignment models in the tradition of Koopmans and Beckmann (1957) formulate the allocation problem as a linear program and use the dual to generate prices. Unlike our paper, this literature does not use lotteries, restricting solutions to be integer valued.

[9]This decentralization uses a different commodity space then the one used by E.C. Prescott and Townsend (1984a, 1984b) to decentralize their private information economies, which corresponded to single agent principal-agent theory. Instead of directly incorporating incentive and related constraints in the set of feasible commodities, they placed these constraints in agents consumption sets and used as a commodity space the space of contract characteristics. We did not use this commodity space because it cannot be used to keep track of the numbers of clubs.

Details of this implementation are contained in Appendix A. The last subsection of Section 2 provides a numerical example for illustrative purposes.

Section 3 contains the decentralization. A competitive equilibrium is defined. The Welfare theorems and an existence theorem are provided. The proofs are based on Negishi's mapping from the space of Pareto weights to the space of transfers. These proofs are contained in Appendix B.

Section 4 reports solutions to several additional numerical examples for the prototype. In the examples, wealth distribution and the aggregate capital stock determine the equilibrium industrial organization of the economy. Section 5 demonstrates how our approach can be used to study other incentive problems. It contains the following extensions to the prototype: limited commitment, incentive problems internal to a multi-agent firm, agents who differ intrinsically in their preferences and abilities, and hierarchies with incentives. Finally, Section 6 provides some concluding comments.

2 A Prototype

In this prototype, we consider a continuum economy where agents can be assigned to two different types of firms. The first type of firm is sole production, or self-employment. Output of this firm is a stochastic function of the capital input and effort. While output and the capital input are publicly observed, effort is not, thus creating a version of the classic moral hazard problem. The second type of firm is a supervisor-worker pair. Like the self-employment firm, there is a capital input and an effort input into production. The supervisor monitors and observes the worker's effort. To avoid difficult issues involving collusion we assume that since effort is observed by more than one person, it can costlessly be made public, as in Harris and Townsend (1981). Supervision is not costless because we require the supervisor to extend the same amount of effort as the worker he monitors, though we do assume that supervisory effort is less onerous. Furthermore, in several of the economies that we study, capital is relatively abundant and labor is relatively scarce. Consequently, assigning a person to supervise rather than to work alone can lead to foregone output.

Our strategy is first to describe the environment and then to describe each type of possible firm. For each type of firm, there is a corresponding contract as in the agency literature. The contract will specify objects such as the capital input, effort taken, and output dependent consumption. In the two-agent firm, the contract will specify these objects for both members of the firm. This multi-agent contract is the club good. After specifying the contract for a firm, we will represent a contract as a single good. Utilities and resource constraints will be expressed in terms of the contracts rather than the components of the contracts. This representation is an essential step in our decentralization of these club economies with private information, though in subsequent analysis, we retrace our steps and analyze prices in terms of the items in the contract.

2.1 Environment and notation

There is a continuum of agents of measure one. All agents have the same preferences over consumption $c \in C$, effort $a \in A$, and job $j \in J = \{w, s\}$. Job $j = w$ means the agent is a worker and job $j = s$ means the agent is a supervisor. When referring to effort and consumption of the worker and supervisor in a two-person firm, we will index them by the job. We assume that a member of a self-employed club is a worker for utility and production purposes. We express the utility function over consumption, effort, and job as $U(c, a, j)$. Even though we do so in our examples, it is not necessary for us to assume separability or concavity of this function. We will be choosing probabilities over consumption and the other variables and expected utility is linear in lotteries.

There is a production technology freely available to all agents that produces output $q \in Q$ as a stochastic function of worker's effort a and a capital input $k \in K$. We represent this function as a conditional probability distribution $p(q|a, k)$. For convenience, we assume that $p(q|a, k) > 0$, for all q, a, k. Furthermore, production shocks are identically and independently distributed over technologies.

We assume that the sets C, Q, A, and K contain finite numbers of elements. This assumption will greatly facilitate computation and proofs. For some variables, like consumption, this assumption should be viewed as an approximation to a continuum. For other variables, like the capital input, this assumption can be viewed as fundamental and not an approximation. In practice, we often want to study technologies with an indivisibility in inputs such as with a plant or on a smaller scale a machine.

There is a finite number, I, of agent types in the economy. We denote an agent's type by $i \in \{1, ..., I\}$. The number of each type i of agents is a positive fraction, $\alpha_i > 0$, of the population. In the Pareto program, types will only differ in their Pareto weights $\lambda_i \geq 0$. In the decentralization, types will only differ in their non-negative endowment of capital κ_i. The total endowment of capital is $\kappa = \sum_i \alpha_i \kappa_i$. This capital is divisible and is the fundamental ingredient into creating the capital input k. We assume that it can be converted at a constant rate of return of one.[10]

2.1.1 Self-employment firms

The contract

A self-employment firm or club consists of one unsupervised agent. The agent's effort a is private information but his output q and consumption c are public information. The contract between this agent and the rest of the economy is a probability distribution over an assigned capital input k, a recommended effort a, an output q, and consumption c. The probability distribution is defined over the set $C \times Q \times A \times K$. We assume that there are n_1 elements in this grid. Given that an agent is in a self-employment firm, we denote $\pi(c, q, a, k)$ as the probability of receiving a (c, q, a, k) grid point. There is no explicit reference in self-employment clubs to the job of the agent since by construction the agent is the worker. It should be noted that this distribution does not preclude a deterministic assignment of capital, effort, and output-contingent consumption. Many of our examples will contain this feature. An example of such a contract would be

probability	c_w	q	a	k
0.20	0.00	0.00	0.40	2.00
0.80	0.45	1.00	0.40	2.00

In this contract, the agent uses 2.00 units of the capital input and supplies 0.40 units of labor effort, both with probability one. He produces the low output ($q = 0.00$) twenty percent of the time and the high output ($q = 2.0$) eighty percent of the time. If he produces the low output he receives 0.00 units of consumption, and if he produces the high output he receives 0.45 units of consumption.

[10]We distinguish between the capital endowment and the capital input in order to not restrict potential values of the distribution of wealth and to allow transactions in the credit market of arbitrary size.

The expected utility an agent receives from being in a self-employment club is

$$\sum_{c,q,a,k} \pi(c,q,a,k)U(c,a,w).$$

Not all probability distributions over the grid are feasible. A feasible π must be incentive compatible, be consistent with the technology $p(q|a,k)$, and be a probability measure. Incentive constraints take the following form,

$$\sum_{c,q} \pi(c,q,a,k)U(c,a,w) \geq \sum_{c,q} \pi(c,q,a,k)\frac{p(q|\hat{a},k)}{p(q|a,k)}U(c,\hat{a},w), \quad \forall k,\hat{a},a. \tag{1}$$

The next set of constraints ensure that π be consistent with the exogenous probability $p(q|a,k)$. These "mother nature" constraints are

$$\forall \bar{q},\bar{a},\bar{k}, \quad \sum_{c} \pi(c,\bar{q},\bar{a},\bar{k}) = p(\bar{q}|\bar{a},\bar{k})\sum_{c,q} \pi(c,q,\bar{a},\bar{k}). \tag{2}$$

Finally, the probability measure constraint is that π be non-negative and that

$$\sum_{c,q,a,k} \pi(c,q,a,k) = 1. \tag{3}$$

The set of feasible contracts is $\Pi_1 = \{\pi \in \Re_+^{n_1} | \pi \text{ satisfies } (1),(2),(3)\}$. We will refer to each element of Π_1 as a different type of self-employment firm.

The contract as a commodity

If we were to restrict ourselves to self-employment firms, we could follow E.C. Prescott and Townsend (1984) and decentralize this economy by making the commodity space Euclidean with the same number of dimensions as there are grid points. Agents would choose quantities of each grid point and the incentive, mother nature, and probability measure constraints would be placed in agents' consumption sets.

As we will see in our discussion of the supervisor-agent firm, that strategy does not work in our club environment. Instead, we use a commodity space that is based on an alternative representation of the contracts. Rather than letting agents choose components of the contract, we let agents choose quantities of feasible contracts and base our commodity space on this alternative representation.

Let $B_1 \subset \Pi_1$ be the set of *basic* feasible solutions to the linear inequalities defined by Π_1, with the notation $b_1 \in B_1$. The cardinality of B_1 is m_1. As is well known, this set is the set of extreme points of the convex polyhedron defined by Π_1. This set is bounded because Π_1 is bounded. Furthermore, because there are a finite number of variables and a finite number of linear inequalities, the set B_1 contains a finite number of elements.

Our strategy is to allow agents to choose a lottery only over the set of contracts B_1. There is no loss in generality from this assumption since the lotteries make the convex hull of B_1 feasible, and the convex hull of B_1 is Π_1. To implement our strategy, we first define utility in terms of these contracts. Preferences are

$$u(b_1) = \sum_{c,q,a,k} b_1(c,q,a,k)U(c,a,w).$$

Next, we define the expected amount of resources used by a type b_1 self-employment firm. Consumption use is the excess of consumption over output,

$$r_{(c-q)}(b_1) \quad = \quad \sum_{c,q,a,k} b_1(c,q,a,k)(c-q),$$

and capital usage is

$$r_k(b_1) \quad = \quad \sum_{c,q,a,k} b_1(c,q,a,k)k.$$

2.1.2 Multi-agent firms as supervisor-worker pairs

The contract

There are two agents in this type of firm or club. The contract between one of these firms and the rest of the economy is a probability distribution over an assigned capital input k, an effort level for the worker a_w, an effort level for the supervisor a_s, an output q, consumption for the supervisor c_s, and consumption for the worker c_w. We write the grid as $C \times C \times Q \times A \times A \times K$ and denote n_2 as the number of elements in this grid. Given a supervisor and worker in a supervisor-worker firm, let $\pi(c_w, c_s, q, a_w, a_s, k)$ be the probability of the members receiving point $(c_w, c_s, q, a_w, a_s, k)$. At this point, we do not identify who is the worker and who is the supervisor. That decision will be incorporated later. An example of one of these supervisor-worker contracts is

probability	c_w	c_s	q	$a_w = a_s$	k
0.50	0.05	0.35	0.00	0.40	1.00
0.50	0.05	0.35	1.00	0.40	1.00

With probability one, the supervisor-worker firm uses 1.00 units of the capital input and both the worker and supervisor supply 0.40 units of effort. The low (0.00) and high (1.00) outputs are produced with equal probability, while the worker receives a constant wage of 0.05 and the supervisor receives a constant wage of 0.35.

The expected utility that a worker receives from being in a supervisor-worker firm is

$$\sum_{c_w,c_s,q,a_w,a_s,k} \pi(c_w, c_s, q, a_w, a_s, k)U(c_w, a_w, w),$$

while the expected utility of a supervisor is

$$\sum_{c_w,c_s,q,a_w,a_s,k} \pi(c_w, c_s, q, a_w, a_s, k)U(c_s, a_s, s).$$

As we said earlier, we assume that the use of a supervisor makes the worker's effort public. Consequently, there are no incentive constraints. The mother nature constraints are

$$\forall \bar{q}, \bar{a}_w, \bar{a}_s, \bar{k}, \quad \sum_{c_w,c_s} \pi(c_w, c_s, \bar{q}, \bar{a}_w, \bar{a}_s, \bar{k}) = p(\bar{q}|\bar{a}_w, \bar{k}) \sum_{c_w,c_s,q} \pi(c_w, c_s, q, \bar{a}_w, \bar{a}_s, \bar{k}). \tag{4}$$

We also assume that to supervise a worker, a supervisor must spend an equal amount of effort as the worker. For example, if a worker works an 8-hour shift, the supervisor needs to be there the whole time. We write these constraints by putting zero probability on other effort configurations,

$$\forall (c_w, c_s, q, a_w, a_s, k) \ni a_w \neq a_s, \quad \pi(c_w, c_s, q, a_w, a_s, k) = 0. \tag{5}$$

Finally, the probability measure constraint is that π be non-negative and that

$$\sum_{c_w,c_s,q,a_w,a_s,k} \pi(c_w, c_s, q, a_w, a_s, k) = 1. \tag{6}$$

The set of feasible contracts is $\Pi_2 = \{\pi \in \Re_+^{n_2} | \pi \text{ satisfies } (4), (5), (6)\}$. Again, we refer to each element of this set as a particular type of supervisor-worker firm.

In this two-agent firm, the object π is a contract that specifies joint usage of capital, each member's effort, and each member's output dependent consumption. The contract contains joint production features as well as joint consumption features. For these reasons, the contract is a club good.

The contract as a commodity

As with the self-employment firm, we express the contract in units rather than in terms of components. Let B_2 be the set of basic feasible solutions to Π_2, with notation $b_2 \in B_2$. The cardinality of B_2 is m_2. Preferences defined over these contracts are

$$u(b_2, j) = \sum_{c_w, c_s, q, a_w, a_s, k} b_2(c_w, c_s, q, a_w, a_s, k) U(c_j, a_j, j), \quad j = w, s,$$

while the expected resource usage for each $b_2 \in B_2$ is

$$\begin{aligned} r_{(c-q)}(b_2) &= \sum_{c_w, c_s, q, a_w, a_s, k} b_2(c_w, c_s, q, a_w, a_s, k)(c_w + c_s - q), \quad \text{and} \\ r_k(b_2) &= \sum_{c_w, c_s, q, a_w, a_s, k} b_2(c_w, c_s, q, a_w, a_s, k) k. \end{aligned}$$

2.2 The Pareto program

Our strategy is to choose lotteries over the grid of basic feasible solutions (which themselves are lotteries). As we mentioned earlier, there is no loss in generality from this strategy. The sets of feasible contracts, Π_1 and Π_2 are both convex sets, and any element in a convex set can be represented by a convex combination of its extreme points. The advantage of this strategy is that our problem remains in Euclidean space, greatly facilitating proofs. Furthermore, we will interpret the lotteries as fractions of the population in equilibrium.

The commodity space is $\Re^{m_1+2m_2+1}$, where m_i is the number of elements in B_i, $i = 1, 2$, where m_2 is multiplied by two because there are two jobs in supervisor-worker clubs, and where the last dimension is for the capital input. One of the choice objects is the fraction of each agent type i assigned to each firm type, that is over B_1 and B_2, and assigned to each possible job within a firm. We refer to these probabilities as π_i, but the support is the types of firms, not the grid used earlier. It is useful to explicitly express some constraints in terms of the number of clubs. Denote $\delta(b_1)$ as the number of b_1 clubs in the economy and $\delta(b_2)$ as the number of b_2 clubs in the economy.

We can now proceed to write out the Pareto program.

Program 1

$$\max_{\pi_i, \delta \geq 0, k} \sum_i \lambda_i \alpha_i \left(\sum_{b_1} \pi_i(b_1) u(b_1) + \sum_{b_2, j} \pi_i(b_2, j) u(b_2, j) \right)$$

subject to the probability measure constraints

$$\forall i, \ \sum_{b_1} \pi_i(b_1) + \sum_{b_2, j} \pi_i(b_2, j) = 1, \tag{7}$$

the club or matching constraints

$$\forall b_1, \ \ \delta(b_1) = \sum_i \alpha_i \pi_i(b_1), \tag{8}$$

$$\forall b_2, \ \ \delta(b_2) = \sum_i \alpha_i \pi_i(b_2, 1) = \sum_i \alpha_i \pi_i(b_2, 2), \tag{9}$$

and the resource constraints

$$\sum_{b_1} \delta(b_1) r_{(c-q)}(b_1) + \sum_{b_2} \delta(b_2) r_{(c-q)}(b_2) \leq 0, \tag{10}$$

$$\sum_{b_1} \delta(b_1) r_k(b_1) + \sum_{b_2} \delta(b_2) r_k(b_2) - k \leq 0, \ \text{ and} \tag{11}$$

$$k \leq \kappa. \tag{12}$$

The program assigns a fraction $\pi_i(b_1)$ of each agent type i to each b_1-type of self-employment firm and, similarly, it assigns a fraction $\pi_i(b_2, j)$ to each job j in each b_2-type of supervisor-worker firm. Incentive compatibility and mother nature constraints have already been incorporated through our definitions of B_1 and B_2. Constraints (7) are the probability measure constraints, one for each agent type i. The resource constraints are (10), (11), and (12), with the coefficients r defined earlier and the number of firms δ defined as above. The consumption resource constraint (10) requires that no more consumption is consumed than produced by the economy. Constraints (11) and (12) ensure that the use and production of the capital input does not exceed the capital endowment κ.

The club or matching constraints are (8) and (9). The first set of these constraint (8), for the self-employment firms, simply defines the number of these firms in the economy. The second set of club constraints (9) defines the number of supervisor-worker firms, as well, but it also ensures that for each supervisor-worker club in the economy with contract $b_2 \in B_2$, there is one agent assigned to be a supervisor of a b_2 club and one agent assigned to be a worker of that same type of club.[11]

This program is linear. Despite the apparent simplicity of the notation, however, it is impractical to directly compute solutions to it. The sets B_1 and B_2 contain astronomical numbers of elements, so the number of variables and the number of club constraints (8) and (9) are huge.[12] Enumerating this set is clearly not practical. To compute solutions, we instead rewrote the program in terms of the components, that is, the grids of consumption, output, effort, and capital. The details are contained in Appendix A.

2.3 An example

In this subsection, we provide a simple example that illustrates the Pareto Program. We use the following grids

$$
\begin{aligned}
C &= \{0.00, 0.05, 0.10, ..., 1.20\}, \\
Q &= \{0, 1\}, \\
A &= \{0.0, 0.4\}, \\
K &= \{0, 1, 2\}.
\end{aligned}
$$

There are two types of agents with $\lambda_1 = 0.2$ and $\lambda_2 = 0.8$. The aggregate capital endowment is $\kappa = 0.4$. (Later we will vary the Pareto weights and the capital endowment in several experiments.) The two types of agents are equal fractions of the population, so $\alpha_1 = \alpha_2 = 0.5$.

[11]If we had followed E.C. Prescott and Townsend (1984) and used components of contracts as our commodity space, it is not clear how we could have designed analogous constraints that would ensure that the club memberships properly add up.

[12]For example, if there are 10 constraints and 100 variables in Π_1 then there are at most 100 choose 10 possible candidates for the set of basic feasible solutions (over 13 trillion).

Preferences

Preferences are

$$
\begin{aligned}
U(c,a,w) &= c^{0.5} + (1-a)^{0.5}, \text{ and} \\
U(c,a,s) &= c^{0.5} + (1-0.25a)^{0.5}.
\end{aligned}
$$

Agents are risk averse and dislike effort but supervisory effort is less painful than worker effort.

Technology

There are two inputs into production, the capital input and the worker's effort. When capital is 0, effort is unproductive and output is virtually zero.[13] Formally,

$$
p(q|a,k=0.0) = \left\{ \begin{array}{ccc} a & q=0 & q=1 \\ \hline 0.0 & .99 & .01 \\ 0.4 & .99 & .01 \end{array} \right.
$$

When capital is at its intermediate value of one, effort becomes productive. The production function is

$$
p(q|a,k=1.0) = \left\{ \begin{array}{ccc} a & q=0 & q=1 \\ \hline 0.0 & 0.8 & 0.2 \\ 0.4 & 0.5 & 0.5 \end{array} \right.
$$

Finally, when capital is at its highest level, effort is even more productive. The production function is

$$
p(q|a,k=2.0) = \left\{ \begin{array}{ccc} a & q=0 & q=1 \\ \hline 0.0 & 0.6 & 0.4 \\ 0.4 & 0.2 & 0.8 \end{array} \right.
$$

In expected terms, the marginal product of capital for the low effort is practically linear while it is diminishing for the high effort.

Experiment: $\kappa = 0.4$, $\lambda_1 = 0.2$

We report a solution to the case where the aggregate capital stock is $\kappa = 0.4$ and the Pareto weight on type-1 agents is $\lambda_1 = 0.2$ (so $\lambda_2 = 0.8$). In the Pareto optimum we found two types of supervisor-worker firms and two types of self-employment firms. The levels of b_2 for the two supervisor-worker firms are as follows. The first one consists of two low Pareto

[13]For programming convenience we did not set any of these probabilities to zero or one.

weight type-1's. There are number $\delta(b_2) = 0.10$ of these clubs in the economy, and these clubs employ forty percent of the type-1's ($\pi_1(b_2) = 0.40$), which is twenty percent of the population. The contract is

probability	c_w	c_s	q	$a_w = a_s$	k
0.50	0.05	0.05	0.00	0.40	1.00
0.50	0.05	0.05	1.00	0.40	1.00

Because there is no moral hazard in supervisor-worker firms and agents are risk averse, consumption is constant over output.

The second type of supervisor-worker firm consists of a type-1, who is supervised by a type-2. There are number $\delta(b_2) = 0.30$ of these firms in the economy, employing 60 percent of the population. They employ the remaining 60 percent of type-1's ($\pi_1(b_2) = 0.60$) and they also employ 60 percent of the type-2's ($\pi_2(b_2) = 0.60$). This contract is

probability	c_w	c_s	q	$a_w = a_s$	k
0.50	0.05	0.35	0.00	0.40	1.00
0.50	0.05	0.35	1.00	0.40	1.00

Supervisors are paid more in this type of firm than in the previous type because these supervisors are type-2's, who have the higher Pareto weight. Also, notice that there is full-risk sharing for the entire type-1 population since there is no incentive problem for any of their efforts, that is, none of them are self-employed.

While the remaining two firms are technically self-employment firms, they really correspond to inactivity for the residual 40 percent of the type-2 population. In these firms, both zero units of capital and effort, $a = k = 0.0$, are supplied. Members of these firms are essentially the idle rich. There are number $\delta(b_1) = 0.16$ clubs in which the member receives 0.35 units of consumption with conditional probability one and there are number $\delta(b_1) = 0.04$ clubs in which 0.40 units of consumption are received.[14] These firm distributions imply that type-2 agents are randomly assigned with probability 0.40 to inactive self-employment b_1 firms. Finally, note that the total number of firms in this economy is

[14] Despite the curvature of the utility function, type-2 agents did not always receive the same consumption. This is not the result of any inherent non-convexity in the economy but, instead, it is solely a function of the coarseness of the consumption grid. Since preferences are separable and utility from consumption is concave, if we made the consumption sufficiently fine the program would put mass one on some consumption point between 0.35 and 0.40.

0.60 but because supervisor-worker firms use two people, the entire population of size 1.0 is assigned to a firm.

3 Decentralization

The commodity space is Euclidean with $L = \Re^{m_1+2m_2+1}$. Consequently, prices p lie in this set as well.

Consumer's problem

The consumption set for type-i agents is

$$X_i = \{x_i \in \Re^{m_1+2m_2} \subset L_+ | \sum_{b_1} x_i(b_1) + \sum_{b_2,j} x_i(b_2,j) = 1\}. \tag{13}$$

The endowment of a type-i agent is an $\xi_i \in L$. Expected utility for an agent is $\sum_{b_1} x(b_1)u(b_1) + \sum_{b_2,j} x(b_2,j)u(b_2,j)$. Sometimes we write it more compactly as $u_i x_i$. A type-i's problem is to maximize expected utility,

$$\max_{x_i} \sum_{b_1} x(b_1)u(b_1) + \sum_{b_2,j} x(b_2,j)u(b_2,j)$$

subject to $x_i \in X_i$ and a budget constraint

$$\sum_{b_1} x_i(b_1)p(b_1) + \sum_{b_2,j} x_i(b_2,j)p(b_2,j) \leq \xi_i p.$$

In our economies, the natural endowments are those where agents of type-i have a non-negative amount of capital, κ_i and zero quantities of the other goods. In these cases we will write the endowment as $\xi_i = (0, 0, ..., 0, \kappa_i)$ so income will equal $p_k \kappa_i$. Income from their capital endowment is used by agents to purchase a firm type, a job, and a distribution over the capital input, output, and consumption.

Before continuing with the decentralization, it might be helpful to examine the prices and income that support the Pareto optimum reported in the previous example as a competitive equilibrium. In this experiment we found that the type-1 consumers had negative income. It will be shown later that the price of capital is non-negative so this optimum cannot be supported solely by a distribution of the capital good. Instead, some taxes and transfers are needed as well. Normalizing the price of capital to be one, per capita incomes needed to support this optimum were -0.217 for type-1 agents and 1.017 for type-2 agents. This could correspond to a capital distribution of 0.0 to type-1's with a tax on them of -0.217 (all expressed in per capita terms) and a distribution of 0.8 to type-2's with a transfer to them of 0.217.

The prices of a job and its associated contract b_2 in the two types of supervisor-agent firms, that is, the $p(b_2, j)$, are

	Worker	Supervisor
first type of b_2 firm	-0.253	-0.076
second type of b_2 firm	-0.253	0.921

Note that the price of being a worker in either firm is identical. If the price was not identical, the type-1 agent would choose the cheaper firm since both firms give him identical consumptions and efforts, that is, identical utilities from membership. Note again, that the type-1 worker does not care about the type of his partner. The negative price for the workers (and a supervisor in the first firm) means that the worker or the supervisor is being paid to join the firm. For type-1 agents, being paid to join a firm is how they earn "income" to overcome the deficit in their budget constraint. Note that the payment for being a supervisor is less than it is for being a worker because the supervisor gets more utility.

The prices of the two types of self-employment firms, $p(b_1)$, are

	price
first type of b_1 firm	1.129
second type of b_1 firm	1.295

The price paid to be in the first type of firm is slightly lower than the price paid to be in the second type because members consume a lower amount in the first one, while producing the same expected amount in both.

Intermediary's problem

Returning to the decentralization, we label the Arrow-Debreu firm of standard competitive analysis as an intermediary, partly because it trades in insurance and credit contracts, but also to avoid confusion with the self-employment and supervisor-worker firms that are the focus of this paper. Because there is constant returns to scale, we assume without loss of generality that there is only one intermediary. The intermediary produces a $y \in L$ that is in the intermediary's production set. For convenience, we denote $\delta(b_1)$ as the number of b_1 clubs produced by the intermediary and $\delta(b_2)$ as the number of b_2 clubs produced by the intermediary.

The production set consists of several constraints. The first one is a resource constraint on the capital input. The intermediary purchases capital y_κ from consumers and then uses it to create as its output the capital input k used in creating goods (b_1) and (b_2, j). As is conventional, the input y_k is negative in sign and outputs are positive in sign. This constraint is

$$\sum_{b_1} \delta(b_1) r_k(b_1) + \sum_{b_2} \delta(b_2) r_k(b_2) + y_\kappa \leq 0, \tag{14}$$

so capital sold is less than or equal to capital purchased.

The intermediary also produces insurance contracts that are contingent consumption transfers. It collects premia and distributes indemnities, and these need to balance to zero. This constraint is

$$\sum_{b_1} \delta(b_1) r_{(c-q)}(b_1) + \sum_{b_2} \delta(b_2) r_{(c-q)}(b_2) \leq 0. \tag{15}$$

In selling supervisor worker firms, b_2, the intermediary is also staffing positions in them. Each one of these b_2 firms requires two members so for each level of b_2 produced there needs to be one worker for each supervisor. We write this matching constraint (and the trivial one for the self-employment firm) as

$$\forall b_2, \ \ \delta(b_2) \ = \ y(b_2, w) = y(b_2, s), \tag{16}$$

$$\forall b_1, \ \ \delta(b_1) \ = \ y(b_1). \tag{17}$$

The intermediary's production set is $Y = \{y \in L, \delta(b_1) \in \Re^{m_1}, \delta(b_2) \in \Re^{m_2} | (14), (15), (16), (17) \text{ hold}\}$.

The intermediary's maximization problem is to solve

$$\max_{y, \delta(b_1), \delta(b_2)} \sum_{b_1} p(b_1) y(b_1) + \sum_{b_2, j} p(b_2, j) y(b_2, j) + p_k y_\kappa$$

subject to $(y, \delta(b_1), \delta(b_2)) \in Y$.

Market clearing

Market clearing is as follows

$$\forall b_1, \ \ \sum_i \alpha_i (x_i(b_1) - \xi_i(b_1)) \ = \ y(b_1), \tag{18}$$

$$
\begin{aligned}
\forall b_2, j, \ \sum_i \alpha_i(x_i(b_2,j) - \xi_i(b_2,j)) &= y(b_2,j), &(19)\\
-\sum_i \alpha_i \kappa_i &= y_\kappa. &(20)
\end{aligned}
$$

As we said earlier, in our economies we will usually set $\xi_i(b_1) = 0$, and $\xi_i(b_2, j) = 0$.

As is apparent, we have mapped our example into the standard formulation. Unlike our example, however, in the definitions and theorems that follow we allow heterogeneity in preferences and consumption sets. We provide this generality because we are interested in extending our economies in these directions and these extensions do not require any substantive modifications to the proofs.

Definition 1 *Let x be the vector of consumptions across the I types of agents. A compensated equilibrium in this economy is a (x^*, y^*, p^*) such that*

1. *$\forall i$, x_i^* minimizes px_i subject to $x_i \in X_i$ and $u_i x_i \geq u_i x_i^*$.*
2. *y^* maximizes py s.t. $y \in Y$.*
3. *$\sum_i \alpha_i(x_i^* - \xi_i) = y^*$.*

Definition 2 *A competitive equilibrium in this economy is the same as a compensated equilibrium except that 1. above is replaced by*

1. *$\forall i$, x_i^* maximizes $u_i x_i$ s.t. $x_i \in X$ and $px_i \leq p\xi_i$.*

Notice that substitution of the market clearing constraints, 3 in Definition 1, into the intermediary's production set delivers the resource constraints in the Program 1. Alternatively, we can write Program 1 in terms of x_i and y if we substitute the production set and market clearing constraints for the resource constraints.

3.1 Theorems

In this section we state the Welfare Theorems as well as an existence theorem. Aside from the bounded linear structure of the economy, the only assumption we use is the following

Assumption 1 (Non-satiation) *For any feasible allocation x, there exists for all i, $x_i' \in X_i$ such that $u_i x_i' > u_i x_i$.*

In our economies non-satiation is relatively easy to satisfy. For example, one can include a very high utility consumption grid point that cannot be offered to any agent type with probability one.

Our proofs are based on Negishi (1960) and are contained in the Appendix B. The Negishi mapping finds a fixed point in the space of Pareto weights at which consumers' budget constraints are satisfied. The mapping solves the Pareto program for each set of Pareto weights so the fixed point is also a solution to the Pareto program. We adopted Negishi's strategy because his programming approach is relatively straightforward in our linear economies. Furthermore, the proofs will demonstrate that as long as we restrict ourselves to analyzing Pareto optima generated by non-zero Pareto weights, we are guaranteed that a competitive equilibria exists that supports that Pareto optimum. This is a useful result since we will be computing Pareto optima and then finding prices and incomes that support them as competitive equilibria.

Theorem 1 (First Welfare Theorem) *Given Assumption 1, every competitive equilibrium is Pareto optimal.*

As usual, the proof is straightforward so we do not include it in Appendix B. However, note that a competitive equilibrium is a fixed point of Negishi's mapping at $\lambda > 0$, that a fixed point of his mapping is a solution to the Pareto program with $\lambda > 0$, and thus is necessarily Pareto optimal.

The Second Welfare Theorem is usually stated in terms of all Pareto optima. We now state a slightly stronger version of it.

Theorem 2 (Second Welfare Theorem) *Any solution to the Pareto program for some Pareto weights $\lambda \geq 0$ can be supported as a compensated equilibrium.*

This theorem is slightly stronger than the usual formulation because all Pareto optima are solutions to the Pareto program but not all solutions to the Pareto program – those where $\lambda_i = 0$, for some i – need be Pareto optimal.

The next theorem provides a sufficient condition that can be checked to see if a Pareto optimum can be supported as a competitive equilibrium

Theorem 3 *Any solution to the Pareto program for some Pareto weights $\lambda > 0$ can be supported as a competitive equilibrium.*

The standard approach for showing that a compensated equilibrium is a competitive equilibrium is to assume there exists a cheaper point. The next theorem demonstrates the connection between this strategy (Arrow's Remark) and Theorem 3. It shows that a cheaper point at a compensated equilibrium implies that the corresponding Pareto weights are non-zero so a competitive equilibrium exists. In a sense, then, the Pareto weight condition is more general.

Theorem 4 *Take a solution to the Pareto program for $\lambda \geq 0$. If, at the corresponding compensated equilibrium, there exists for all i, a cheaper point satisfying $x_i \in X_i$, then $\lambda > 0$.*

We now prove existence of a competitive equilibrium using the Negishi mapping.

Theorem 5 (Existence) *For any given distribution of endowments, if the Pareto weights at a fixed point of Negishi's mapping are non-zero, then a competitive equilibrium exists.*

In the next subsection, we discuss different assumptions that may be used to guarantee that a competitive equilibrium exists.

3.2 Analysis of equilibrium prices

Let $\mu_{(c-q)}$ be the dual variable on the consumption resource constraint and μ_k be the dual variable on the capital input resource constraint in Program 1. At a competitive equilibrium, these variables are equal to the dual variables in the intermediary's maximization problem (see the proofs to Theorems 2 and 3). Using this substitution and the first-order conditions to the intermediary's problem, prices are

$$\begin{aligned} p(b_1) &= \mu_{(c-q)} r_{(c-q)}(b_1) + \mu_k r_k(b_1), \\ p(b_2, 1) + p(b_2, 2) &= \mu_{(c-q)} r_{(c-q)}(b_2) + \mu_k r_k(b_2), \\ p_k &= \mu_k. \end{aligned}$$

The first equation is the price of purchasing a unit of the b_1 point, which is a probability distribution over the (c, q, a, k) grid. The term $r_{(c-q)}$ is the expected compensation to the self-employed person above and beyond what he produces and r_k is the expected amount of the capital input used in production. The price of a b_1 good is simply the shadow price of net consumption times the quantity of consumption plus the shadow price of capital times the quantity of capital used. In order for an agent to purchase a self-employment firm at level b_1, he must pay the resource cost required to operate the firm at that level.

The second equation is the total sum of purchase prices paid by a worker and a supervisor in a b_2 firm. Just as with the self-employment firm, the total cost of a b_2 supervisor-agent firm is the value of the resources used by that level of b_2.

Remark 1 *Different strategies can be used to guarantee existence of a competitive equilibrium. One strategy that can be used when agents are endowed with non-negative quantities of the capital endowment and zero units of other goods is the following. Assume that the zero consumption point is in the consumption grid and that expected output is positive for all levels of effort and the capital input. We know from the resource constraint that* $\mu_k \geq 0$ *so agents will have at least zero income and we know that* $\mu_{(c-q)} > 0$ *by non-satiation. For many standard utility functions, a contract that assigns zero capital, the lowest effort, and distributes the zero consumption grid point is incentive compatible and thus in our commodity space. Assigning this contract with probability one is in the agent's consumption set. By assumption, such an allocation will produce positive output and therefore have a negative price. Since with non-negative endowments of the capital input, agents' incomes are non-negative, a cheaper point exists in their consumption set and a competitive equilibrium exists.*

To give a more specific interpretation of the pricing functions $p(b_1)$ and $p(b_2, j)$, we substitute in the r's to obtain

$$p(b_1) = \mu_{(c-q)} \sum_{c,q,a,k} b_1(c,q,a,k)(c-q) + \mu_k \sum_{c,q,a,k} b_1(c,q,a,k)k, \tag{21}$$

$$\begin{aligned} p(b_2,1) + p(b_2,2) = {} & \mu_{(c-q)} \sum_{c_w,c_s,q,a_w,a_s,k} b_2(c_w,c_s,q,a_w,a_s,k)(c_w+c_s-q) \\ & + \mu_k \sum_{c_w,c_s,q,a_w,a_s,k} b_2(c_w,c_s,q,a_w,a_s,k)k. \end{aligned} \tag{22}$$

By marginally varying the components of b_1 and (b_2, j) one at a time, we can see how these pricing functions behave.

For example, suppose that we vary b_1, the probability of a self-employment firm receiving (c, q, a, k), in such a way as to keep the determination of the inputs a and k in the firm constant (though these maybe deterministic or randomly determined) but allow consumption compensation c to vary. In this contract, the joint probability of consumption and output is $\sum_{a,k} b_1(c, q, a, k) \equiv prob(c, q)$, so the first term on the right-hand side of the pricing equation (21) reduces to

$$\mu_{(c-q)} \sum_{c,q} prob(c, q)(c - q) = \mu_{(c-q)} E(c - q), \tag{23}$$

where the expectation is defined under the contract $b_1(c, q, a, k)$. This expectation is the expected deficit of compensation over output stipulated by the contract, multiplied by the shadow price of consumption. Note that if consumption were a constant c under the contract, then (23) simplifies even further to

$$\mu_{(c-q)} E(c - q) = \mu_{(c-q)} c - \mu_{(c-q)} E(q).$$

In this case the consumption compensation to the owner is purchased at price $\mu_{(c-q)}$, and the output of the firm is sold at this same price. More generally, of course, c will vary with q for incentive reasons, but the price of contract $c(q)$ is still its actuarial fair value.

Similarly, we can vary marginally the capital input k. Here the easiest case for interpretation is a comparison of two contracts $b_1(c, q, a, k)$ which vary with a discrete, non-random choice of capital k. In this case, again reverting to probabilities, $b_1(c, q, a, k) \equiv prob(c, q, a, k)$, so if a particular input combination (a, k) receives weight or probability one and all other input combinations receive probability zero, then for that (a, k) combination

$$prob(c, q, a, k) \equiv prob(c, q|a, k) prob(a, k) \equiv prob(c, q|a, k)$$

Taking the derivative with respect to k in the first term in (21) delivers

$$\mu_{(c-q)} \sum_{c,q} (c - q) \frac{\partial prob(c, q|a, k)}{\partial k}. \tag{24}$$

Condition (24), of course, is the marginal revenue product effect, here on the expected net deficit. Again, in the special case of constant, output-invariant consumption c, (24) reduces to

$$-\mu_{(c-q)} \sum_{q} \frac{q \partial prob(q|a, k)}{\partial k},$$

which is the expected marginal product of k in production times minus the shadow price of consumption, as an increase in the expected surplus reduces the expected net deficit.

Likewise, a small marginal change in the recommended effort delivers

$$\mu_{(c-q)} \sum_{c,q} (c-q) \frac{\partial prob(c,q|a,k)}{\partial a}$$

with the same interpretation.

We can gain additional insight into the behavior of prices by turning to the problem confronting the household at these prices, and again considering special cases. Suppose that $x_i(b_1)$ puts mass one on at most one b_1 and no other. Then the simplified problem confronting the household is

$$\max \sum_{c,q,a,k} U(c,a,w) b_1(c,q,a,k)$$

by choice of b_1 in B_1 satisfying the budget constraint

$$p(b_1) \leq p_k \kappa_i. \tag{25}$$

Writing out that budget constraint more fully yields, as in (21),

$$\mu_{(c-q)} \sum_{c,q,a,k} b_1(c,q,a,k)(c-q) + \mu_k \sum_{c,q,a,k} b_1(c,q,a,k) k \leq p_k \kappa_i. \tag{26}$$

Proceeding as before, suppose the household is choosing a contract or self-employment firm which holds the (a,k) input pair fixed but varies in the consumption compensation $c(q)$, that is in $prob(c,q)$. Then this special problem reduces still further to

$$\max \sum_{c,q} U(c,a,w) prob(c,q) \tag{27}$$

subject to the first term in (26)

$$\mu_{(c-q)} \sum_{c,q,a,k} prob(c,q)(c-q) \leq I_r, \tag{28}$$

where I_r is residual of the household's budget constraint left to purchase consumption, both its level and its variability. Without incentive constraints the household would purchase some level with no variation, that is, full insurance. For example, with two outputs q_1 and

q_2 and hence two states of nature, with exogenous probabilities determined by the input combination (a, k), here held fixed, and with the choice of state contingent consumptions c_1 and c_2, the ratio of probabilities in the budget line in (28) matches the ratio of probabilities in the objective function (27). Thus the solution is full insurance, $c_1 = c_2$ unless one hits a binding incentive constraint.

Likewise, a small change in the input choice k for the self-employment firm that the household is buying implies both a negative income effect in the budget, in the term $\mu_k(\kappa_i - k)$, and a change in the market odds line above, in the prices, as embodied in $prob(q|a, k)$.

Turning at last to the supervisor-worker firm, all of the above analysis has a straightforward extension. We focus only then on the decision as to whether to be a worker or a supervisor in a particular b_2 firm. Formally, let $x_i(b_2, w)$ denote the chosen probability of being a worker and $x_i(b_2, s)$ be the probability of being a supervisor. Then the choice problem of the consumer is

$$\max x_i(b_2, w)u(b_2, w) + x_i(b_2, s)u(b_2, s)$$

subject to $x_i(b_2, w) + x_i(b_2, s) = 1$, and

$$x_i(b_2, w)p(b_2, w) + x_i(b_2, s)p(b_2, s) \leq p_k \kappa_i.$$

If for household i probability x_i is to be interior, that is neither zero nor one, then the first-order condition holds at equality for household i and

$$u(b_2, s) - u(b_2, w) = \delta_i(p(b_2, s) - p(b_2, w)). \tag{29}$$

Define $\Delta U \equiv u(b_2, s) - u(b_2, w)$, and $\Delta P \equiv p(b_2, s) - p(b_2, w)$. As established in Appendix B, δ_i is the Lagrange multiplier on household i's budget constraint and it must equal $1/\lambda_i$ in a competitive equilibrium. Thus if household i is indifferent to being a worker or a supervisor, then

$$\lambda_i \Delta U = \Delta P.$$

Put differently, since household type i is indifferent in equilibrium to being a worker or a supervisor, then he is willing to randomize, so the price differential is precisely the utility differential, scaled by the Pareto weight.

More generally, there will not be an interior solution and equation (29) will not hold at equality. In this case, with two types of agents and Pareto weights $\lambda_1 \leq \lambda_2$, for any b_2 chosen with positive probability in equilibrium such that $\Delta U \geq 0$, we get the inequalities

$$\lambda_1 \Delta U \leq \Delta P \leq \lambda_2 \Delta U,$$

and thus the price differential ΔP is bounded by the utility differences scaled by the Pareto weights. For these inequalities to hold in equilibrium, the high Pareto weight agent must be assigned the supervisory job, which is the higher utility job when $\Delta U \geq 0$. If for some reason the equilibrium chooses a b_2 where the high utility job is the worker's job then $\Delta U \leq 0$, and the above inequalities would be reversed.

4 More Numerical Examples

In this section, we report solutions to different parameterization of the earlier example. The only parameters we vary are the Pareto weights and the aggregate capital endowment. All other features of the environment, the utility function, the grids, or the production technology, were left unchanged. However, in Section 5.1 we add limited commitment to the model. For each example, we calculated Pareto optimum by solving Program 1a in Appendix A. Then, using this solution we solved the dual to Program 1 to calculate prices and a distribution of the capital endowment that supported the Pareto optimum.

Experiment: $\kappa = 0.4$ and $\lambda_1 = 0.3$ (Same aggregate capital but more equal Pareto weights)

In this experiment, we leave the aggregate capital stock unchanged from the earlier example but make the Pareto weights more equal. With this change, the distribution of clubs changes. In this allocation there are 0.15 clubs consisting of two type-1's that use 2 units of capital and 0.10 clubs consisting of two type-1's that use 1 unit of capital. Type-1's are gaining because now some of them get to do the less onerous supervisory effort while at the earlier $\lambda_1 = 0.2$ more of them were supplying the worker's effort. All type-2's are now idle. This may appear surprising at first because type-2's supply no effort, thus effort disutility declines relative to what it was in the $\lambda_1 = 0.2$ experiment. However, type-2's consumption also declines so that overall their total utility declines as is expected with a declining Pareto weight. Again, supporting prices and incomes entail taxes on type-1's and transfers to type-2's. However, at $\lambda_1 = 0.4$ the allocation is similar to the $\lambda_1 = 0.3$ one, but incomes for the type-1's arrive at zero so this allocation can be supported with a capital distribution that gives zero capital to type-1's and all of it to the type-2's.

Experiment: Lower aggregate capital levels

If we lower the economy-wide aggregate level capital to $\kappa = 0.2$ then for all $\lambda_1 < 0.5$ membership in supervisor-worker clubs consists of two type-1 agents. Because capital is scarce in these economies, there is no shortage of low-Pareto weight agents to supply supervisory effort.

Experiment: Higher aggregate capital levels

In the other direction, as economy-wide capital increases we see the persistence of $(1, 2)$ clubs, for relatively low λ_1. The reason for this persistence is that as capital becomes abundant, supervisory effort becomes scarce and it is worth having high Pareto weight individuals suffer a disutility of effort from supervision because their monitoring produces a lot of consumption that they can consume.

We also observe an increased frequency of non-trivial self-employment clubs as κ increases from 0.4 and as long as λ_1 is not too low. The next experiment contains these characteristics.

Experiment: $\kappa = 1.75$, $\lambda_1 = 0.4$ (Much higher aggregate capital)

In the Pareto optimum there are no supervisor-worker clubs. All production is being done through self-employment, some of it trivial in a sense that will be explained below.

The first self-employment club consists of a type-1. There were 0.50 of these clubs in the economy, which is the entire type-1 population. They are fully capitalized and effort is being supplied at its maximum. As a consequence, moral hazard constraints bind and consumption is a non-constant function of output.

probability	c_w	q	a	k
0.1040	0.00	0.00	0.40	2.00
0.0960	0.05	0.00	0.40	2.00
0.8000	0.45	1.00	0.40	2.00

The type-2's are split among three different self-employment clubs. There are 0.125 of the first type of club. In these clubs, the agents use zero units of capital, supply zero units of effort and receive 0.75 units of consumption. There are 0.252 of the second type of club. In these clubs, the agent uses 2.0 units of capital, supplies no effort, and receives 0.75 units of consumption. The last club is identical to the previous one except that the agent receives 0.70 units of consumption. There are 0.123 of these clubs. These latter two clubs have an expected level of output of 0.4 even though no effort is supplied.

The prices of the four type of self-employment clubs are

	price
Type-1	-0.232
First type-2	3.795
Second type-2	3.795
Third type-2	3.538

The self-employment firm labeled Type-1 is the only club that type-1's chose. The price of this club is negative because net resource flows out are higher than the cost of inputed capital. The other clubs are the three self-employment clubs chosen by type-2's. For all three of these clubs, the prices of membership are positive because net consumption inflows far outweigh the capital input costs. Notice that the price of the first and second type-2 clubs are the same, despite using different levels of capital. They are the same because both clubs supply the same consumption and effort. An agent's utility is identical from these two goods so if the price differed he would only choose the one that gave a higher utility. Finally, the price of the third type-2 club is lower than the price of the other two type-2 clubs since less consumption is paid out to members of this club.

4.1 Uniqueness

In our numerical experiments, we found that unequal Pareto weights and a smaller capital stock led to more supervisor-worker firms. One concern with this statement, and similar statements summarizing numerical examples, is that there may be multiple equilibria supported by different industrial organizational structures, that is, the mapping from Pareto weights to industrial organization is a correspondence and the numerical solutions are picking a particular set of points from this correspondence.

In general, this is a potential problem but it did not seem to be a factor in the examples. As we vary Pareto weights, we move along the Pareto frontier, from vertex to vertex of the piecewise linear Pareto frontier, unless our hyperplane is coincident with a linear portion of the Pareto frontier.[15] At each vertex of the frontier, more than one separating hyperplane, that is, set of Pareto weights, may support a particular Pareto optimal distribution of utilities. As we discussed earlier, the Pareto weights equal the inverse of the marginal utility of income so a Pareto optimum may be supported by more than one price system and distribution of income.[16] Indeed, we computed examples where this was the case but the industrial organization of the economy was unchanged at these vertices. For this reason, we do not think multiple equilibria affect our conclusions based on the numerical examples.[17] Finally, it is also worth mentioning that the piecewise linearity of the Pareto frontier is not just a function of the consumption and action grids. Even with a continuum of consumption and action levels and well-behaved preferences, incentive constraints or an indivisibility in capital can create linear portions of Pareto frontiers.

5 Extensions

This section contains four extensions to the prototype. The first one adds a limited commitment problem. The second one restricts the supervisor to observe a signal that is only partially correlated with the agent's effort. The third extension adds heterogeneity in agents abilities and preferences. The final one adds to the prototype by incorporating hierarchies with incentives problems.

5.1 Limited commitment

In this section, we extend the previous example to incorporate limited commitment. We keep the same supervisor-worker environment but with one modification. When a firm receives the capital input, the supervisor may abscond with the capital and convert it into

[15]Since we used a simplex-based algorithm to solve the linear programs, the numerical solutions were always vertices of the Pareto frontier.

[16]It is also possible that more than one set of dual vectors could support a solution to a particular Pareto program. As was evident in Section 3.2, prices are unique up to the values of the dual variables. Uniqueness of the dual variables is assured if the solution to the linear program is non-degenerate.

[17]Multiple competitive equilibria are also possible for a given distribution of endowments and the general equilibrium literature has long been concerned with this issue. In performing our Second Welfare Theorem calculations, we did find an example where two different Pareto optima could be reached with the same initial distribution of the capital stock. But, again, the industrial organization of the economy was not affected, even though consumption levels were.

own consumption at some constant rate of return. If a supervisor does this he cannot be punished or rewarded except to the extent that he receives utility from consuming the capital. In other words, no consumption transfers to or from him may be made. In a supervisor-worker firm, we assume that only the supervisor may run off with the capital. We also allow the agent in the self-employment club to run off with any capital assigned to him.

Incorporating this limited commitment feature requires an additional variable in the grid spaces plus some additional constraints in the sets Π_1 and Π_2. We let $d \in D = \{0, 1\}$, where $d = 0$ means the agent stays in the firm and does not run off with the capital. Conversely, $d = 1$ means the agent runs off with the capital. If an agent runs off with the capital, he converts it into consumption at some exogenous rate r with no effort supplied. We modify the utility function to handle the departure decision. In particular, we write $V(k, d = 1) = U(rk, 0, j)$, while utility from staying is unchanged from before.

Self-employment firms

The grid space for the self-employment club is now $C \times Q \times A \times D \times K$. The addition of limited commitment requires constraints that guarantees that it is optimal for an agent to stay in the firm when recommended to do so. (Since the departure decision is public, we will not require a limited commitment constraint on agents recommended to leave.)

The limited commitment constraint is

$$\forall k, \sum_{c,q,a} \pi(c, q, a, d = 0, k)U(c, a, w) \geq \sum_{c,q,a} \pi(c, q, a, d = 0, k)V(k, d = 1). \tag{30}$$

On the left-hand-side of the constraint is the self-employed agent's utility from staying and on the right-hand-side is his utility from running off.

The incentive constraints are almost identical to the ones in the earlier problem. They only apply if the agent stays, that is, if $d = 0$. The constraints are

$$\sum_{c,q} \pi(c, q, a, d = 0, k)U(c, a, w) \geq \sum_{c,q} \pi(c, q, a, d = 0, k)\frac{p(q|\hat{a}, k)}{p(q|a, k)}U(c, \hat{a}, w), \ \forall k, \hat{a}, a. \tag{31}$$

The next set of constraints ensure that π is consistent with the exogenous probability $p(q|a, k)$. These "mother nature" constraints are straightforward if the agent stays. They are simply

$$\forall \bar{q}, \bar{a}, \bar{k}, \ \sum_{c} \pi(c, \bar{q}, \bar{a}, d = 0, \bar{k}) = p(\bar{q}|\bar{a}, \bar{k}) \sum_{c,q} \pi(c, q, \bar{a}, d = 0, \bar{k}). \tag{32}$$

However, if the agent runs off, we assume that production is the same as it would have been with no capital and no effort. Consequently, the mother nature constraints are

$$\forall \bar{q}, \bar{a}, \bar{k}, \quad \sum_{c} \pi(c, \bar{q}, \bar{a}, d=1, \bar{k}) = p(\bar{q}|a=0, k=0) \sum_{c,q} \pi(c, q, \bar{a}, d=1, \bar{k}). \tag{33}$$

Note that when $d = 1$, the only relevant specification of the production function is the one with zero effort and zero capital. The grid allows non-zero specifications of effort along this path but in equilibrium these grid points will receive zero weight since they give the agent disutility without affecting production.

Finally there is the probability measure constraint, namely that π be non-negative and

$$\sum_{c,q,a,d,k} \pi(c, q, a, d, k) = 1. \tag{34}$$

The set of feasible contracts in the self-employment firm with limited commitment is $\Pi_1 = \{\pi \in \Re^{2n_1} | (30), (31), (32), (33), (34)\}$. The set of commodities is then the set of basic feasible solutions, B_1, to Π_1.

Supervisor-worker firms

For supervisor-worker firms, we proceed much like we did for self-employment firms. We assume for programming simplicity that if the supervisor runs off with the capital then the remaining worker provides zero effort along with the zero capital input. In general, this is not a desirable assumption since for some production functions effort may be productive even for zero capital inputs. However, in the examples we study, effort is not productive for a zero capital input so there are no adverse consequences from this assumption.

The grid space for the supervisor-worker firm is now $C \times C \times Q \times A \times A \times D \times K$. The addition of the limited commitment requires a limited commitment constraint that guarantees that it is optimal for an agent to stay in the firm when recommended to do so. Again, since the departure decision is public we will not require a limited commitment constraint on agents recommended to leave.

In the supervisor-worker firm, the limited commitment constraint applies only to the supervisor. Therefore, the constraint is

$$\begin{aligned} \forall k, \quad & \sum_{c_w, c_s, q, a_w, a_s} \pi(c_w, c_s, q, a_w, a_s, d=0, k) U(c_s, a_s, s) \\ \geq \quad & \sum_{c_w, c_s, q, a_w, a_s} \pi(c_w, c_s, q, a_w, a_s, d=0, k) V(k, d=1). \end{aligned} \tag{35}$$

The next set of constraints ensures that π be consistent with the exogenous probability $p(q|a_w, k)$. These "mother nature" constraints are straightforward if the agent stays. They are simply

$$\forall \bar{q}, \bar{a}_w, \bar{a}_s, \bar{k}, \quad \sum_{c_w, c_s} \pi(c_w, c_s, \bar{q}, \bar{a}_w, \bar{a}_s, d=0, \bar{k}) = p(\bar{q}|\bar{a}_w, \bar{k}) \sum_{c_w, c_s, q} \pi(c_w, c_s, q, \bar{a}_w, \bar{a}_s, d=0, \bar{k}). \tag{36}$$

As before, if the agent runs off, we assume that production is the same as it would have been with no capital and no effort. Consequently, the mother nature constraints are

$$\begin{aligned} \forall \bar{q}, \bar{a}_w, \bar{a}_s, \bar{k}, \quad & \sum_{c_w, c_s} \pi(c_w, c_s, \bar{q}, \bar{a}_w, \bar{a}_s, d=1, \bar{k}) \\ & = p(\bar{q}|a_w=0, k=0) \sum_{c_w, c_s, q} \pi(c_w, c_s, q, \bar{a}_w, \bar{a}_s, d=1, \bar{k}). \end{aligned} \tag{37}$$

As before, we need constraints that ensure $a_w = a_s$. We do this by

$$\forall (c_w, c_s, q, a_w, a_s, d, k) \ni a_w \neq a_s, \quad \pi(c_w, c_s, a_w, a_s, d, k) = 0. \tag{38}$$

Finally there is the probability measure constraint, namely that π be non-negative and

$$\sum_{c_w, c_s, q, a_w, a_s, d, k} \pi(c_w, c_s, q, a_w, a_s, d, k) = 1. \tag{39}$$

The set of feasible contracts in the supervisor-worker firm with limited commitment is $\Pi_2 = \{\pi \in \Re^{2n_2} | (35), (36), (37), (38), (39)\}$. The set of commodities is then the set of basic feasible solutions, B_2, to Π_2.

5.1.1 Numerical examples

The parameterization of our limited commitment examples are the same as in the previous examples. In particular, if the limited commitment problem were formulated so that the commitment constraints never bound, e.g., if $r = 0$, then the solution to the limited commitment problem would be the same as the previous example. Again, we varied the aggregate capital stock κ and the Pareto weights. The only additional parameter required to run these experiments is the return on capital received by an agent who runs off, that is, r. In the following examples, we set $r = 0.25$.

Experiment: $\kappa = 0.4,\ \lambda_1 = 0.2$

In the Pareto optimum we found one supervisor-worker firms and three self-employment firms. The supervisor-worker firm consisted of a type-1 worker supervised by a type-2. There were 0.40 of these clubs in the economy, which employed eighty percent of the type-1's and eighty percent of the type-2's. The b_2 for these clubs was

probability	c_w	c_s	q	$a_w = a_s$	k
0.50	0.05	0.35	0.00	0.40	1.00
0.50	0.05	0.35	1.00	0.40	1.00

The remaining three clubs are inactive self-employment clubs. Two of them are clubs for type-2's and the other one is a club for type-1's. The only thing to note about them is that consumption of type-1's is 0.05 and consumption of type-2's is around 0.35 (it is not always 0.35 for the consumption grid reasons discussed earlier.)

Comparison of this solution with the corresponding example in the first model illuminates the effect of limited commitment. In these examples, limited commitment does not change the firm size distribution. In both cases, there are 0.4 numbers of supervisor-worker firms. Limited commitment, however, does change firm membership. In the previous example, there were 0.1 numbers of $(1, 1)$ clubs, where the notation (i, j) refers to a type-i agent being the worker and a type-j agent being the supervisor. In these clubs, type-1's both worked and supervised. The advantage of this for type-2's was that they did not have to expend very much effort. All the effort they expended was in the 0.3 numbers of $(1, 2)$ clubs in which they supervised. With limited commitment, $(1, 1)$ clubs of the previous specification are not feasible. In that example, type-1 agents received 0.05 units of consumption, regardless of whether they were a supervisor or an agent. With limited commitment the supervisor in one of those clubs would run off with the one unit of capital, giving him 0.25 units of consumption and a lower effort level. Consequently, the program decides to eliminate $(1, 1)$ clubs and replace them with $(1, 2)$ clubs. In these club, higher consumption levels are given to the supervisors to keep them from running off with the capital as well as because they have a high Pareto weight.

A supporting competitive equilibrium with transfers

Below we report prices that support the above Pareto optimum. In this experiment we found that the type-1 consumers had negative income. Consequently, this optimum cannot be supported solely by a distribution of the capital good. Instead, some taxes and transfers are needed as well. Normalizing the price of capital to be one, per capita incomes needed to support this optimum were -0.044 for type-1 agents and 0.844 for type-2 agents. This could correspond to a capital distribution of 0.0 to type-1's with a tax on them of -0.044 (all expressed in per capita terms). Similarly, type-2's would be endowed with 0.8 units of capital and receive a transfer 0.044. Note that taxes and transfers are less than in the environment without limited commitment. Here there is more equality.

The price of the supervisor-agent club is as follows

Worker	Supervisor	Cost of club inputs
-0.083	0.796	0.713

Notice that the price of membership sums to the cost of club inputs. Prices of the self-employment clubs are not reported but since they are inactive, that is, use 0.0 units of capital, the price is simply the expected net resource flow of the consumption good.

Other experiments

We find similar patterns for nearby levels of capital. For lower $\kappa = 0.2$, all non-trivial production is done by each $(1, 2)$ supervisor-worker firms. Interestingly, in the less unequal range of the Pareto weights, we find that the Pareto optima can be supported with positive distributions of capital, that is, there is no need for tax and transfers.

As capital increases, we continue to see the $(1, 2)$ firms. Of course, there are more of them, because capital is less scarce. Interestingly, at $\kappa = 0.6$ we observe the existence of some $(1, 2)$ supervisor-worker firms that use 2.0 units of capital, even while some type-1's and type-2's are idle (and others are in $(1, 2)$ firms that use 1.0 units of capital). This occurs despite the assumed diminishing marginal returns to production. The reason for this pattern is that idle workers suffer less disutility. Apparently, the program is balancing this advantage with the loss of output.

At higher $\kappa = 1.0$ we observe extensive use of highly capitalized $(1, 2)$ firms. At unequal Pareto weights, *all active* production is done by this type of club with a capital input of 2.0. At $\lambda_1 = .4$, however, we see the arrival of another kind of inactive self-employment firm. These firms with a high-weight individual are being given two units of capital and that individual is told to run off with the capital. Apparently, an allocation with more numbers of $(1, 2)$ clubs would give type-2 agents too much disutility from supervising. Interestingly, the consumption of type-2's who run off is close to the level of type-2's who do not run off but supervise in $(1, 2)$ firms. In general, if the program assigns a fraction of agents of a given type to stay and the rest to runoff, the program wants to fully insure these agent against consumption variability but cannot because an agent who runs off with the capital is barred from making or receiving ex post transfers. For this parameterization, consumption from running off is close enough to consumption for those who are not assigned to runoff that the extra consumption variability has a relatively minor effect on the value of the objective function.

Solutions to examples with higher levels of aggregate capital look much like the $\kappa = 1.0$ case. At unequal weights, $(1, 2)$ firms predominate and at more unequal weights we see self-employment firms in which the agent is told to run off with the capital. The only difference between the experiments is that as the capital level increases, the self-employment firms show up at more unequal Pareto weights.

5.2 Incentive problems internal to a multi-agent firm

In this section, we demonstrate how the problems can be modified to include incentive concerns internal to the firm. We modify the first example by allowing the supervisor to observe a signal that is correlated with the agent's effort, as in Holmström (1979). The first model can be viewed as a special case of this example in which the principal observes a perfectly correlated signal of the worker's effort. For simplicity, we drop the limited commitment portion of the problem, but it is easy enough to retain.

The grid space for the self-employment club is $C \times Q \times A \times K$ as in the original formulation. Just as in that model we are assuming that a supervisor needs to be physically present to observe the signal. Therefore there is no supervision in a self-employment firm. Consequently, the sets Π_1 as well as B_1 are unchanged from the first model.

Let Z be the set of signals. We assume that like output, the signal is publicly observed and is determined by a combination of effort, the capital input, and a hidden shock. The grid space for the supervisor-worker firm is now $C \times C \times Q \times Z \times A \times A \times K$. The addition of the signal means that we need to add the signal to the production function. We allow the supervisor's effort to effect the informativeness of a signal. We write the joint probability of the output and the signal as a function of effort and the capital input as

$p(q,z|a_w,a_s,k)$. We no longer assume that the worker's and supervisor's effort must be the same. We also assume that all (q,z) pairs are possible for all (a_w,a_s,k) triplets. Of course, the probability distribution of output alone is then $p(q|a_w,k) = \sum_z p(q,z|a_w,a_s,k)$, $\forall a_s$, and as the notation suggests the output does not depend on the supervisory effort.

With the addition of a correlated signal there is now an incentive constraint on the worker's effort. We still assume that the supervisor's effort is public. The worker's incentive constraints are

$$\sum_{c_w,c_s,q,z} \pi(c_w,c_s,q,z,a_w,a_s,k)U(c_w,a_w,w)$$
$$\geq \sum_{c_w,c_s,q,z} \pi(c_w,c_s,q,z,a_w,a_s,k)\frac{p(q,z|\hat{a}_w,a_s,k)}{p(q,z|a_w,a_s,k)}U(c_w,\hat{a}_w,w), \quad \forall k,a_s,a_w,\hat{a}_w, \tag{40}$$

where the worker takes an action knowing the supervisor's effort. The addition of a correlated signal means the worker's consumption will depend on the signal as well as the output. Because there is no supervisor to observe effort in a self-employment firm, consumption of an agent in that firm will depend on output alone.[18]

The next set of constraints ensures that π be consistent with the exogenous probability $p(q,z|a_w,a_s,k)$. These "mother nature" constraints are straightforward. They are simply

$$\forall \bar{q},\bar{z},\bar{a}_w,\bar{a}_s,\bar{k}, \quad \sum_{c_w,c_s} \pi(c_w,c_s,\bar{q},\bar{z},\bar{a}_w,\bar{a}_s,\bar{k}) = p(\bar{q},\bar{z}|\bar{a}_w,\bar{a}_s,\bar{k}) \sum_{c_w,c_s,q,z} \pi(c_w,c_s,q,z,\bar{a}_w,\bar{a}_s,\bar{k}). \tag{41}$$

Finally, there is the probability measure constraint, namely that π be non-negative and

$$\sum_{c_w,c_s,q,z,a_w,a_s,k} \pi(c_w,c_s,q,z,a_w,a_s,k) = 1. \tag{42}$$

The set of feasible contracts in the supervisor-worker firm with limited supervision is $\Pi_2 = \{\pi \in \Re^{n_2 * n_z} | \pi \text{ satisfying } (40),(41),(42)\}$, where n_2 was the number of elements in the grid of the original problem and n_z is the number of signals. Again, B_2 is the set of basic feasible solutions to Π_2 and the problem is thus mapped into the framework of Program 1.

[18] Supervisory incentives can be examined as well by assuming that the supervisor's effort is private information. In this case, an incentive constraint on the supervisor's effort would be needed, but as in the multi-agent firms of E.S. Prescott and Townsend (1999), this can be easily done.

5.3 Heterogeneity

If agents are heterogenous in ability or preferences, it may be necessary to expand the commodity space by indexing club memberships by the member's intrinsic type. A type is intrinsic if it affects the set of feasible allocations. To incorporate this kind of heterogeneity we proceed as follows. There are still $i = 1, ..., I$ types of agents, but now each type is also identified with an intrinsic type $t(i) \in \Re_+$. Two types i_1 and i_2 may be of the same intrinsic type, that is, $t(i_1) = t(i_2)$. Indeed, the earlier examples can be thought as a special case, where there was only one intrinsic type. In those examples, utility did not depend on type nor did the production function.

The utility function is $U_t(c, a, j)$. Heterogenous ability is incorporated by indexing the production function by the intrinsic type of the worker, t_w, and the intrinsic type of the supervisor, t_s, (if any). The production function is $p_{(t_w,t_s)}(q|a_w, a_s, k)$. For a self-employment firm, we write $p_{(t_w)}(q|a_w, k)$.

We now index the firms not only by type of organization but also by the composition of intrinsic types. If, for example, there are two intrinsic types, then the set of self-employment firms, N_{se} is

$$N_{se} = \{(t_1), (t_2)\},$$

and the set of supervisor-agent firms, N_{sa}, is

$$N_{sa} = \{(t_1, t_1), (t_1, t_2), (t_2, t_1), (t_2, t_2)\},$$

with the total set of firms being $N = N_{se} \cup N_{sa}$. For each $n \in N$, there is a set of basic feasible solutions that we write as B^n with b^n as an element of this set. Sometimes we write an element of this set as $b^{(t)}$ or $b^{(t,t')}$. The sets B^n are constructed exactly as before except that the appropriate type-dependent utility and production functions are now used for each firm. As before, we use $j = \{s, w\}$ to index an individual's job within a firm. Finally, because utility may be type dependent we write the utility function over basic feasible solutions b^n and the job as $u_t(b^n, j)$.

In Program 2, we now index the choice variable by the type of the firm, that is, we choose $\pi_i^n(b^n)$, $\pi_i^n(b^n, j)$ and $\delta^n(b^n)$. The Pareto program is

Program 2

$$\max_{\pi_i^n,\delta^n\geq 0,k}\sum_i \lambda_i\alpha_i\left(\sum_{n\in N_{se}}\sum_{b^n}\pi_i^n(b^n)u_{t(i)}(b^n,w)+\sum_{n\in N_{sa}}\sum_{b^n,j}\pi_i^n(b^n,j)u_{t(i)}(b^n,j)\right)$$

subject to the probability measure constraints

$$\forall i,\quad \sum_{n\in N_{se}}\sum_{b^n}\pi_i^n(b^n)+\sum_{n\in N_{sa}}\sum_{b^n,j}\pi_i^n(b^n,j)=1; \tag{43}$$

the matching constraints for self-employment firms

$$\forall n\in N_{se},\forall b^n,\quad \delta^n(b^n)=\sum_i \alpha_i\pi_i^n(b^n), \tag{44}$$

and the matching constraints for supervisor-agent firms

$$\forall n\in N_{sa},\forall b^n,\quad \delta^n(b^n)=\sum_i \alpha_i\pi_i^n(b^n,w)=\sum_i \alpha_i\pi_i^n(b^n,s), \tag{45}$$

the resource constraints

$$\sum_{n,b^n}\delta^n(b^n)r_{(c-q)}(b^n)\leq 0, \tag{46}$$

$$\sum_{n,b^n}\delta^n(b^n)r_k(b^n)-k\leq 0, \tag{47}$$

$$k\leq\kappa; \tag{48}$$

and constraints that restrict agents to choose membership in firms that are feasible for their basic type. We do this by requiring that zero probability be allocated to assignments that put an intrinsic type in a firm and job that the intrinsic type cannot be assigned to. For self-employment firms, the restrictions are

$$\forall i,\ \forall t'\neq t(i),\ \ \forall b^{(t')},\ \ \pi_i^{(t')}(b^{(t')})=0. \tag{49}$$

For supervisor-worker firms, the restrictions are

$$\begin{aligned}\forall i,\ \forall(t,t')\ni t\neq t(i),\ \ \forall b^{(t,t')},\ \ \pi_i^{(t,t')}(b^{(t,t')},w) &= 0,\\ \forall i,\ \forall(t,t')\ni t'\neq t(i),\ \ \forall b^{(t,t')},\ \ \pi_i^{(t,t')}(b^{(t,t')},s) &= 0.\end{aligned} \tag{50}$$

To decentralize this economy we use the commodity space and add to agent's consumption sets the restrictions that they cannot choose membership in a firm that does not contain their type. This condition is represented in the program by constraints (49) and (50), though of course, it would be written in x notation.

5.4 Hierarchies

More complicated organizational structures can be incorporated into our framework as well. Incentives can be added to the Rosen (1982) hierarchy model.[19] To incorporate hierarchies, we would proceed as follows.

Production may be done by a variety of firms. Each firm consists of z levels of hierarchy. A level in a hierarchy is denoted by h. A firm is indexed by the number of people at each level in its H levels of hierarchy. Denote the type of firm by $n = (n_1, n_2, ..., n_z)$, where n_h (a non-negative integer) is the number of people working at the hth level in the hierarchy. The number of people working a given level in a hierarchy may be 0. For example, a self-employment firm would be identified by $n = (1, 0, 0, ..., 0)$, while a supervisor-worker pair would be identified with $n = (1, 1, 0, 0, ..., 0)$. It will usually make sense to assume that if there is no one assigned to a given level of a hierarchy then no one can be assigned to a higher level in the hierarchy. Finally, we define J^n as the set of (h, l) hierarchy-position assignments associated with an n-type firm.

For simplicity, we do not allow types to differ in any intrinsic way. Preferences are defined as $U(c, a, h)$, so utility depends on the level within the hierarchy but not the position within that level. Let $a_{h,l}$ be the action of an agent in the lth position at level h in the hierarchy. Consumption may differ across positions since we want to allow it to depend on performance measurements. Let $c_{h,l}$ refer to the consumption of the lth agent in the hth level of a hierarchy. Sometimes a_h will refer to the vector of efforts at level h, while c and a will refer to the vector of consumptions and efforts, respectively, across and within hierarchical levels. Also, define a variable m that is a vector of performance measurements. These measurements may be portions of output attributable to an individual or the result of internal evaluations. We denote by q the joint output of the hierarchy. The technology is then written as $p(m, q | a_{1,1}, a_{1,2}, ..., a_{1,n_1}, a_{2,1}, ..., a_{z,n_z}, k)$. A contract for an n-type hierarchy is

$$\pi^n(c, m, q, a) = \pi^n(c_{1,1}, c_{1,2}, ..., c_{1,n_1}, c_{2,1}, ..., c_{z,n_z}, m, q, a_{1,1}, a_{1,2}, ..., a_{1,n_1}, a_{2,1}, ..., a_{z,n_z}, k)$$

[19] Early papers that looked at incentives in hierarchies include Mirrlees (1976) and Calvo and Wellisz (1978).

Incentive constraints would look something like

$$\forall n, \forall h, l, \ \forall a_{h,l}, \sum_{c,m,q,a_{-(h,l)},k} \pi^n(c,m,q,a,k) U(c_{h,l}, a_{h,l})$$

$$\geq \sum_{c,m,q,a_{-(h,l)},k} \pi^n(c,m,q,a,k) \frac{p(m,q|a_{-(h,l)}, \hat{a}_{h,l}, k)}{p(m,q|a_{-(h,l)}, a_{h,l}, k)} U(c_{h,l}, \hat{a}_{h,l}), \quad \forall k, \hat{a}_{h,l}. \tag{51}$$

Depending on the monitoring technology these constraints would need to be modified. For example, if actions taken by a higher level in the hierarchy make lower level actions public then the set of deviating actions might need to be altered.

Technology constraints are

$$\forall n, \forall \bar{m}, \bar{q}, \bar{a}, \bar{k}, \ \sum_c \pi^n(c, \bar{m}, \bar{q}, \bar{a}, \bar{k}) = p(\bar{m}, \bar{q}|\bar{a}, \bar{k}) \sum_{c,m,q} \pi^n(c, m, q, \bar{a}, \bar{k}). \tag{52}$$

and the probability measure constraint is

$$\sum_{c,m,q,a,k} \pi^n(c,m,q,a,k) = 1. \tag{53}$$

Thus we can define the set

$$\Pi^n = \{\pi^n \in \Re^l | (51), (52), (53), \text{ hold}\},$$

and let B^n being the set of basic feasible solutions to this set. We write the utility of someone in the lth position of the hth level of a hierarchy as $u(b^n, (h,l))$. Resource usage is written $r_{(c-q)}(b^n)$ and $r_k(b^n)$.

The choice variables are $\pi_i^n(b^n, (h,l))$ and $\delta^n(b^n)$. The first term is the probability that a type i is in a type-n firm, assigned to level h of the hierarchy, assigned position l in that level of the hierarchy, and the firm's grand contract is b^n.

The Pareto program is

Program 3

$$\max \sum_i \alpha_i \lambda_i \sum_n \sum_{b^n, (h,l) \in J^n} \pi_i^n(b^n, (h,l)) u(b^n, h))$$

$$\text{s.t. } \forall i, \ \sum_n \sum_{b^n, (h,l) \in J^n} \pi_i^n(b^n, (h,l)) = 1,$$

$$
\begin{aligned}
&\forall n, \forall (h,l), (h',l') \in J^n, \quad \forall b^n, \\
&\qquad \delta(b^n) = \sum_i \alpha_i \pi_i^n(b^n, (h,l)) = \sum_i \alpha_i \pi_i^n(b^n, (h',l')),
\end{aligned}
$$

$$\sum_n \sum_{b^n} \delta^n(b^n) r_{(c-q)}(b^n) \leq 0,$$

$$\sum_n \sum_{b^n} \delta^n(b^n) r_k(b^n) - k \leq 0,$$

$$k \leq \kappa.$$

The matching constraints ensure that for each occupied position at each level of a type-n firm hierarchy, there is one and only one person assigned to it.

To decentralize this economy we proceed as before.

6 Conclusion

Our framework can address other questions in organizational theory. For example, the models studied in E.S. Prescott and Townsend (1999) can be adapted. One of their models considered the tradeoff between relative performance and collusion (as also studied by Holmström and Milgrom (1990), Itoh (1993), and Ramakrishnan and Thakor (1991)). To map that problem into this paper's framework would require allowing two types of clubs to form: one would be a relative performance club and the other would be a collusion club. Agents would choose which type of club to enter. Collusion models are particularly interesting, challenging, and a relatively unexplored area of the theory of the firm, not to mention incentive theory. For more on these models see Tirole (1992).

E.S. Prescott and Townsend (1999) also studied a model in which two workers are assigned to work a set of technologies. This problem is related to the task assignment problem studied by Holmström and Milgrom (1991) and more generally to the multi-agent principal agent models studied in papers like Demski and Sappington (1984) and Mookherjee (1984). For a given set of technologies, the workers assigned to them may work each of them jointly or individually. Efforts on individually worked technologies are private information while efforts on jointly worked technologies are public, though there are some costs to this type of assignment. We found examples of individually and jointly worked technologies. To map this model into the framework of this paper, agents would match into pairs that simultaneously decide which technologies to operate jointly and which separately. If joint operation is itself interpreted as an organization then in the sense of this paper, we have allocations that resemble organizations within clubs.

For both models in the earlier paper we solved a version of Program 1 in which pairs of agents were a priori matched together. Within these matches, however, the type of organization was endogenous. We found that the internal distribution of Pareto weights was a critical factor in determining organizational structure. The extension in this paper makes endogenous both the internal distribution of Pareto weights as well as the net resource flow from the rest of the economy to a matched pair of agents.

A Computing

Because of the large number of variables and constraints, Program 1 requires too much memory to compute using standard linear programming algorithms. In this appendix, we describe an algorithm that computes solutions to Program 1, despite its large dimensions. It is essentially a version of the Dantzig-Wolfe decomposition algorithm as developed in Dantzig and Wolfe (1961). We will provide only the briefest description of their algorithm. More information on it can be found in advanced linear programming textbooks like Bertsimas and Tsitsiklis (1997).

First, we take advantage of the structure of the matching constraints to change Program 1 from a linear program with many variables and many constraints to one with many variables but only a few constraints. Consider the trivial club constraint in (8) associated with a particular b_1 contract. The only variables with non-zero coefficients for this constraint are $\pi_1(b_1)$, $\pi_2(b_1)$, and $\delta(b_1)$. We can think of these variables and the club constraint as blocks of non-zero coefficients in the constraint matrix. The values of these variables do not affect the club constraints for other b_1 and b_2, just as the values of variables involving other b_1 and b_2 do not affect the feasibility of this b_1 constraint. Writing this block in isolation gives

$$\begin{bmatrix} \alpha_1 & \alpha_2 & -1 \end{bmatrix} \begin{bmatrix} \pi_1(b_1) \\ \pi_2(b_1) \\ \delta(b_2) \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}. \tag{54}$$

When combined with the non-negativity constraints on $\pi_1(b_1)$, $\pi_2(b_1)$, and $\delta(b_1)$, this system of equations defines a polyhedral cone with its vertex at the origin. As such, it can be represented as the set of all non-negative linear combinations of its extreme rays. The two extreme rays of this cone are

$$\begin{array}{lcccl} [& 1/\alpha_1 & 0 & 1 &], \text{ and} \\ [& 0 & 1/\alpha_2 & 1 &]. \end{array} \tag{55}$$

The scale of these rays is indeterminate so we scale them to $\delta(b_1) = 1$. The interpretation of the first ray (the first row in (55)) is that it takes $1/\alpha_1$ units of $\pi_1(b_1)$ to make one unit of $\delta(b_1)$. In particular, one unit of the first ray corresponds to one self-employment firm with a b_1 contract and a type-1 as the member. The second ray is interpreted similarly for type-2 agents. Because the rays have this type-specific interpretation we label the quantity of them with the notation $\delta^{(i)}(b_1)$, $i = 1, 2$. In the program, these quantities will be constrained by the non-club constraints.

The exact same steps can be done for each of the b_2 matching constraints in (9). For a given b_2 contract, the only non-zero variables for its matching constraints are $\pi_1(b_2, w)$, $\pi_1(b_2, s)$, $\pi_2(b_2, w)$, $\pi_2(b_2, s)$, and $\delta(b_2)$. Again, we think of these matching constraints and corresponding variables as an isolated block because the values of these variables do not affect the feasibility of any other matching constraints, and the values of other variables do not affect the feasibility of these matching constraints. Writing this block in isolation gives

$$\begin{bmatrix} \alpha_1 & 0 & \alpha_2 & 0 & -1 \\ 0 & \alpha_1 & 0 & \alpha_2 & -1 \end{bmatrix} \begin{bmatrix} \pi_1(b_2, w) \\ \pi_1(b_2, s) \\ \pi_2(b_2, w) \\ \pi_2(b_2, s) \\ \delta(b_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{56}$$

Along with the non-negativity constraints, this system of equations is a polyhedral cone with a vertex at the origin. As before, it can be represented by the set of all non-negative linear combinations of its extreme rays. The four extreme rays of this cone are

$$\begin{array}{l} [\ 1/\alpha_1 \quad 1/\alpha_1 \quad 0 \quad 0 \quad 1\ ], \\ [\ 1/\alpha_1 \quad 0 \quad 0 \quad 1/\alpha_2 \quad 1\ ], \\ [\ 0 \quad 1/\alpha_1 \quad 1/\alpha_2 \quad 0 \quad 1\ ], \text{ and} \\ [\ 0 \quad 0 \quad 1/\alpha_2 \quad 1/\alpha_2 \quad 1\ ]. \end{array} \tag{57}$$

The scale of rays is indeterminate and, as before, we scale them to $\delta(b_2) = 1$ so that each ray can be interpreted as one unit of a type-pair specific match. For example, the second row of (57) corresponds to one firm with a b_2 contract that consists of a type-1 as the worker and a type-2 as the supervisor. Because the rays represent type-pair specific matches we introduce the notation $\delta^{(i,i')}(b_2)$ to indicate the number of b_2 firms that consist of a type-i worker and a type-i' supervisor.

Our strategy is to rewrite Program 1 in terms of these extreme rays, the type-specific club goods. Each ray will correspond to a variable in the new linear program and the program will choose non-negative quantities of these variables, which means that the matching constraints will be automatically satisfied. The objective function and the non-matching constraints of Program 1 will be expressed in terms of the new variables $\delta^{(i)}(b_1)$ and

$\delta^{(i,i')}(b_2)$. More specifically, the objective function and the constraints that connect the blocks, that is (7), (10), (11), and (12), will be evaluated at (55) and (57) in order to determine the coefficients for the new variables. This means, for example, that $\alpha_i \pi_i(b_1) = \delta^{(i)}(b_1)$, $\alpha_i \pi_i(b_2, w) = \sum_{i'} \delta^{(i,i')}(b_2)$, and $\alpha_i \pi_i(b_2, s) = \sum_{i'} \delta^{(i',i)}(b_2)$.

The new version of Program 1 takes the following form.

Program 1a

$$\max_{\delta^{(i)}, \delta^{(i,i')} \geq 0, k} \sum_i \lambda_i \left(\sum_{b_1} \delta^{(i)}(b_1) u(b_1) + \sum_{i', b_2} \delta^{(i,i')}(b_2) u(b_2, w) + \sum_{i', b_2} \delta^{(i',i)}(b_2) u(b_2, s) \right)$$

subject to the probability measure constraints

$$\forall i, \ \sum_{b_1} \delta^{(i)}(b_1) + \sum_{i', b_2} \delta^{(i,i')}(b_2) + \sum_{i', b_2} \delta^{(i',i)}(b_2) = \alpha_i,$$

the consumption resource constraint

$$\sum_{(i)} \sum_{b_1} \delta^{(i)}(b_1) r_{(c-q)}(b_1) + \sum_{(i,i')} \sum_{b_2} \delta^{(i,i')}(b_2) r_{(c-q)}(b_2) \leq 0,$$

and the capital resource constraints

$$\sum_{(i)} \sum_{b_1} \delta^{(i)}(b_1) r_k(b_1) + \sum_{(i,i')} \sum_{b_2} \delta^{(i,i')}(b_2) r_k(b_2) - k \leq 0, \ \text{ and}$$

$$k \leq \kappa.$$

With two types, Program 1a contains only five constraints. However, it still has an enormous number of variables, even more than in Program 1.

Despite the large number of variables in Program 1a, it is still practical to compute it. The strategy is to work with a subset of the variables and then generate new variables as they are needed. In linear programming terms, this is called a column-generating method. The Dantzig-Wolfe algorithm is based on this idea.

We start by solving Program 1a restricted to a small set of the variables, namely, $\delta^{(i)}(b_1)$ and $\delta^{(i,i')}(b_2)$ that correspond to a proper subsets of B_1 and B_2, but which can still deliver a feasible solution. The solution to this restricted program delivers a candidate solution which is optimal with respect to the small set of variables, but not necessarily with respect to the entire unrestricted set of variables. This restricted program also delivers dual variables on the constraints in Program 1a, which can be interpreted as the value of relaxing these constraints.

The next step in the algorithm, and the reason column-generating methods work for this program, is based on our definition of b_1 and b_2 as the extreme points of the sets Π_1 and Π_2, respectively. Recall that Π_1 is defined over the consumption, output, action, and capital inputs grids. Its dimensions are relatively small. For each (i)-type firm, we solve a subprogram that uses as its constraint matrix, the set Π_1. The objective function of the subprogram consists of the effect of the subprogram variables on the sum of the objective function of Program 1a and the constraints of Program 1a, the latter weighted by the dual variables from the restricted master program.

Just like in the simplex algorithm, the idea of this step is to see if there is another $\delta^{(i)}(b_1)$, that is, another basic feasible solution to Π_1, which can be introduced into the restricted Program 1a in order to improve upon the candidate solution. If the optimum to the subprogram does not improve upon it then none of the other basic feasible solutions to the subprogram, that is, none of the other $\delta^{(i)}(b_1)$, will improve upon it either. Since in practice linear programming algorithms usually visit only a small set of vertices, solving the subprogram is an inexpensive way of checking all the $\delta^{(i)}(b_1)$.

The same step is performed over Π_2 for each (i, i')-type of supervisor-worker firm. Again, the goal is to see if another variable – here a $\delta^{(i,i')}(b_2)$ – can be introduced which will improve on the candidate solution generated by the restricted version of Program 1a. If no such $\delta^{(i)}(b_1)$ or $\delta^{(i,i')}(b_2)$ exists then the candidate solution is indeed optimal. Otherwise, the $\delta^{(i)}(b_1)$ and $\delta^{(i,i')}(b_2)$ calculated from the subprograms are added to the set of variables in the restricted Program 1a and the algorithm repeats. Once a solution to Program 1a has been found, the corresponding solution to Program 1 is easily found.

The advantage of the algorithm is that the entire sets of B_1 and B_2 do not need to be stored in memory. Instead, new elements are generated only as needed. As long as not too many iterations are required, memory requirements and speed will not be an issue. In our numerical examples, the algorithm converged very quickly, so the only memory limitation was the size of the subprograms. Finally, it is worth pointing out that the subprograms based on the Π_i decompose with respect to action, capital-input pairs. If needed, this property can be used to increase the size of the subprograms that can be computed.[20]

B Proofs

B.1 Overview

The proofs in this appendix are based on Negishi (1960). Negishi proves existence by finding a fixed point in the space of Pareto weights that is a competitive equilibrium with zero transfers. While not explicitly stated in his paper, his mapping is based on the second Welfare Theorem.[21] Aside from our linear structure, there are two substantial differences between Negishi's economy and ours. Because of these differences, some modifications to his proof are required.

The first difference between our economies and his is that we do not assume free disposal. Free disposal makes prices positive, and in our economies, with its hedonic properties, this is entirely inappropriate. Negishi's proof, as it stands, cannot handle negative prices. In his mapping, the n prices are mapped into an $n-1$ dimensional simplex. With negative prices this mapping does not work. We modify his approach to avoid this problem in the following way. We divide prices into positive and negative components. We then map the positive and negative portions into a $2n-1$ dimensional simplex, and then subtract the negative component from the positive one to recover the actual prices.

The second difference between Negishi's economy and ours is that Negishi assumes that endowments are interior to consumption sets. There are two reasons he makes this assumption. The first reason is that this assumption satisfies Slater's condition, a sufficient condition for the application of the Kuhn-Tucker theorem. Assuming an interior endowment is clearly inappropriate in our economies. Instead, we obtain the existence of Kuhn-Tucker vectors directly from the linear structure of our economy. A Kuhn-Tucker vector, the dual variables in a linear program, will exist if a solution to the linear program is finite.

The second reason that Negishi assumes that endowments are interior to consumption sets is that when combined with positive prices (implied by free disposal and a non-satiation assumption) there exists a cheaper consumption point for each agent. The existence of a

[20] See E.S. Prescott (2000) on how to use the Dantzig-Wolfe decomposition algorithm to compute solutions to moral-hazard programs.

[21] Takayama (1985) does emphasize this.

cheaper consumption point implies that the Pareto's weights at the fixed point are strictly positive. Strictly positive Pareto weights are important because they imply that the fixed point of Negishi's mapping is a competitive equilibrium rather than just a compensated equilibrium.

This proof should be viewed as an existence proof for bounded linear economies in Euclidean space without free disposal and without satiation.

We start by stating without proof several needed theorems.

Theorem 6 (Goldman-Tucker) *There exists an optimal solution x^* to a linear program and an optimal solution μ^* to its dual if and only if (x^*, μ^*) is a saddle point of the Lagrangian $L(x, \mu)$.*

See Takayama (1985) for example. The vector of dual variables μ^* is a Kuhn-Tucker vector.

We also need a theorem connecting the Kuhn-Tucker vector to the saddle-point of the Lagrangian plus the necessary and sufficient first-order conditions.

Theorem 7 *In a concave program, for a vector to be a Kuhn-Tucker vector, it is necessary and sufficient that it be part of a saddle point of the Lagrangian. Furthermore, this condition holds if and only if the usual necessary and sufficient conditions hold.*

See Rockafellar (1970) for example.

The remaining mathematical result we need is Kakutani's fixed point theorem.

Theorem 8 (Kakutani's Fixed Point Theorem) *Let K be a compact convex set in $\Re^n$ and let $f(x)$ be a non-empty, compact-valued, convex-valued, upper hemi-continuous correspondence mapping K into K. Then, there is a fixed point $\hat{x}$ such that $\hat{x} \in f(\hat{x})$.*

The Pareto program for this economy is

Program B

$$\max_{\{x_i\} \geq 0, y} \sum_i \lambda_i \alpha_i u_i x_i$$

$$\text{s.t.} \quad \sum_i \alpha_i (x_i - \xi_i) \leq y,$$

$$
\begin{aligned}
-\sum_i \alpha_i(x_i - \xi_i) &\le -y, \\
fy &\le 0, \\
\forall i, \;\; g_i x_i &\le b_i.
\end{aligned}
$$

In terms of the problem in the text, the x_i are the agents' probabilities and y is the intermediary's production plan. The program is written differently than Program 1 in the text. The economy-wide resource constraints have been rewritten in terms of market clearing and the constraints on the intermediary's production. This substitution brings y explicitly into the Pareto program.

All variables are vectors in this program so there may be more than one constraint of each type and more than one good. However, there are a finite number of variables and constraints so the program is a linear program. An important feature of this program is that the market clearing constraint have been rewritten into two sets of inequality constraints (the first two sets of constraints in program). This alternative method for representing equality constraints will be useful when the mapping is described. The last set of constraints in the program $g_i x_i \le b_i$ define the consumption sets X_i (equation (13) in the text). By the same logic used in describing the market clearing constraints, the notation $g_i x_i \le b_i$ is general enough to incorporate any equality constraints in the consumption sets.

Theorem 9 *Assuming that a feasible solution to the Pareto program exists, then for any set of weights $\lambda \in S^{I-1}$, a solution to the Pareto program exists.*

Proof: The constraint set is closed. It is also bounded because the probability measure constraints in $g_i x_i \le b_i$ bound each x_i and feasible y are then bounded by the resource constraint in conjuction with the bounds on $\{x_i\}$. Therefore, $\{x_i\}$ and y are chosen from compact sets. The objective function is continuous. Therefore, a maximum exists. **Q.E.D.**

We will denote goods by the index j. Consequently, the jth coefficient of the vector x_i will be denoted $x_{i,j}$. For matrices like f or g_i we let f_j and $g_{i,j}$ denote the jth column of that matrix.

Theorem 10 *A solution to the Pareto program is a saddle point of*

$$L(\{x_i\}, y, p^l, p^g, \mu, \gamma) = \sum_i \lambda_i \alpha_i u_i x_i - (p^l - p^g)\left[\sum_i \alpha_i(x_i - \xi_i) - y\right] - \mu f y - \sum_i \gamma_i (g_i x_i - b_i), \quad (58)$$

where $x \geq 0$ and y are maximizing variables and $(p^l \geq 0, p^g \geq 0, \mu \geq 0, \gamma \geq 0)$ are minimizing variables. The necessary and sufficient conditions are

$$\begin{aligned}
\forall i,j,\;\; \lambda_i \alpha_i u_{i,j} - (p^l_j - p^g_j)\alpha_i - \gamma_i g_{i,j} &= 0, \;\; (\leq 0 \text{ if } x_{i,j} = 0) && (59)\\
\forall j,\;\; (p^l_j - p^g_j) - \mu f_j &= 0, && (60)\\
\sum_i \alpha_i (x_i - \xi_i) - y &= 0, && (61)\\
\mu f y = 0, \qquad & f y \leq 0, && (62)\\
\forall i,\;\; \gamma_i (g_i x_i - b_i) = 0, \qquad & g_i x_i \leq b_i. && (63)
\end{aligned}$$

Proof: The Pareto program is a linear program. Use Theorems 6 and 7. **Q.E.D.**

At this point, it is worth proving a preliminary result.

Lemma 1 *For $((p^l - p^g), y)$ to be part of a saddle-point of (58), $(p^l - p^g)y = 0$.*

Proof: Multiplying each (60) by y_j and summing over j gives $(p^l - p^g)y = \mu f y$. By (62), $(p^l - p^g)y = 0$. **Q.E.D.**

The other lemma we need is that with non-satiation there will not be a solution with $p^l - p^g = 0$. Otherwise, if these two vectors are equal, then prices will be zero.

Lemma 2 *With non-satiation, there does not exist a saddle point $(x, y, p^l, p^g, \mu, \gamma)$ to the Lagrangian such that $p^l - p^g = 0$.*

Proof: A saddle point solves

$$\max_{\{x\}\geq 0, y} \sum_i \alpha_i \lambda_i u_i x_i - (p^l - p^g)\left[\sum_i \alpha_i (x_i - \xi_i) - y\right] - \mu f y - \sum_i \gamma_i (g_i x_i - b_i),$$

By non-satiation there exists for some i, $x'_i \in X_i$ such that $u_i x'_i > u_i x_i$. Furthermore, $\gamma_i(g_i x_i - b_i) \geq \gamma_i(g_i x'_i - b_i), \forall x'_i \in X_i$. (The latter hold trivially if $\gamma_i = 0$. And if $\gamma_i > 0$ then $g_i x_i = b_i$ so if $g_i x'_i \leq b_i$ then $\gamma_i g_i x' \leq \gamma_i b_i = \gamma_i g_i x$.) Since $\mu f y = 0$ by (62), if $p^l = p^g$ then $\alpha_i \lambda_i u_i x'_i - \gamma_i g_i x'_i > \alpha_i \lambda_i u_i x_i - \gamma_i g_i x_i$, which, when summed over i, contradicts the assumption that (x, p^l, p^g) is a saddle point. **Q.E.D.**

B.2 Second Welfare Theorem

We first express the compensated equilibrium conditions in terms of the necessary and sufficient conditions.

Lemma 3 *Conditions 1 and 2 in the definition of a compensated equilibrium can be rewritten respectively in the following form:*

1. $\forall i$, x_i^* *is part of a saddle point of*

$$L(x_i, \beta_i, \nu_i) = px_i + \beta_i(u_i x_i - u_i x_i^*) - \nu_i(g_i x_i - b_i),$$

where $x_i \geq 0$ *is a minimizing variable, and* $\beta_i \geq 0$, $\nu_i \geq 0$ *are maximizing variables. The necessary and sufficient conditions for it are:*

$$\forall j, \;\; \beta_i u_{i,j} - p_j - \nu_i g_{i,j} = 0, (\leq 0 \text{ if } x_{i,j} = 0) \tag{64}$$

$$\beta_i(u_i x_i - u_i x_i^*) = 0, \;\; u_i x_i - u_i x_i^* \geq 0, \tag{65}$$

$$\nu_i(g_i x_i - b_i) = 0, \;\; g_i x_i - b_i \leq 0. \tag{66}$$

2. y *is a saddle point of*

$$L(y, \tilde{\mu}) = py - \tilde{\mu} f y,$$

where y *is a maximizing variable and* $\tilde{\mu} \geq 0$ *is a minimizing variable. The necessary and sufficient conditions for it are:*

$$\forall j, \;\; p_j - \tilde{\mu} f_j = 0 \tag{67}$$

$$\tilde{\mu} f y = 0, \;\; f y \leq 0. \tag{68}$$

Proof: Both problems are linear programs. If they have finite solutions then Kuhn-Tucker conditions hold. **Q.E.D.**

Theorem 2 (Second Welfare Theorem) *Any solution to the Pareto program for some Pareto weights* $\lambda \geq 0$ *can be supported as a compensated equilibrium.*

Proof: Given a solution to the Pareto program, necessary and sufficient condition (61) implies that condition 3 of a compensated equilibrium, market clearing, holds. Let $p = (p^l - p^g)$, $\beta_i = \lambda_i$, $\nu_i = \gamma_i / \alpha_i$, and $\tilde{\mu} = \mu$. Comparison of necessary and sufficient conditions (60) and (62) of Program B with the necessary and sufficient conditions (67) and (68) of the Intermediary's problem shows that the Intermediary's problem is satisfied. Comparison of (59) and (63) from Program B with (64) and (66) shows that these two necessary and sufficient conditions from Consumer's problems hold. The remaining condition from the Consumer's problem (65) holds trivially since $x_i = x_i^*$. **Q.E.D.**

The conditions for a competitive equilibrium are nearly identical. The only change is the necessary and sufficient conditions for the Consumers' problems. These conditions are $\forall i$, x_i is part of a saddle point of

$$L(x_i, \delta_i, \omega_i) = u_i x_i - \delta_i p(x_i - \xi_i) - \omega_i (g_i x_i - b_i),$$

where $x_i \geq 0$ is a maximizing variable and $\delta_i \geq 0$, $\omega_i \geq 0$ are minimizing variables. The necessary and sufficient conditions for it are:

$$\forall j, \quad u_{i,j} - \delta_i p_j - \omega_i g_{i,j} = 0, \quad (\leq 0 \text{ if } x_i = 0) \tag{69}$$

$$\delta_i p(x_i - \xi_i) = 0, \quad p(x_i - \xi_i) \leq 0, \tag{70}$$

$$\omega_i (g_i x_i - b_i) = 0, \quad g_i x_i - b_i \leq 0. \tag{71}$$

Theorem 3 *Any solution to the Pareto program for some Pareto weights $\lambda > 0$ can be supported as a competitive equilibrium.*

Proof: As in the proof of Theorem 2, set $p = (p^l - p^g)$, and $\tilde{\mu} = \mu$. In addition, set $\forall i$, $\xi_i = x_i^*$, $\delta_i = 1/\lambda_i$ and $\omega_i = \gamma_i/(\lambda_i \alpha_i)$. Market clearing and the Intermediary's problem are satisfied by the same arguments used in the proof to Theorem 2. Consumer's problem condition (70) holds trivially because $\xi_i = x_i^*$. Substituting into (59) gives us that condition (69) holds. Finally, substituting into (63) and dividing by the scalar $(\lambda_i \alpha_i)$ demonstrates that condition (71) holds. **Q.E.D.**

We now state a version of Arrow's remark.

Theorem 4 *Take a solution to the Pareto program for $\lambda \geq 0$. If, at the corresponding compensated equilibrium, there exists for all i, a cheaper point satisfying $x_i \in X_i$, then $\lambda > 0$.*

Proof: A solution to the Pareto program must be a saddle point of the Lagrangian. The saddle point condition requires for all i,

$$\lambda_i \alpha_i u_i x_i - \alpha_i (p^l - p^g) x_i - \gamma_i (g_i x_i - b_i) \geq \lambda_i \alpha_i u_i x_i' - \alpha_i (p^l - p^g) x_i' - \gamma_i (g_i x_i' - b_i), \ \forall x_i'.$$

Assume a cheaper point exists. Now substitute $p = p^l - p^g$ and consider the $\lambda_i = 0$ case. Then,

$$px_i^{'} \geq px_i + \gamma_i(g_i x_i - b_i)/\alpha_i - \gamma_i(g_i x_i^{'} - b_i)/\alpha_i,\ \forall x_i^{'}.$$

We know, however, that $\gamma_i(g_i x_i - b_i) = 0$ and that $\gamma_i(g_i x_i^{'} - b_i) \leq 0$. Therefore, $px_i^{'} \geq px_i$, $\forall x_i^{'}$ satisfying $g(x_i^{'} - b_i) \leq 0$, which contradicts the cheaper good assumption. **Q.E.D.**

B.3 Existence Theorem

A Negishi-style proof works by showing that at a fixed point of Negishi's mapping, consumers' budget constraints are satisfied by a solution to the Pareto program. To show that such weights exist, we use the following mapping of $(\lambda, x, y, p^l, p^g) \rightarrow (\lambda, x, y, p^l, p^g)$. The mapping consists of the following two parts

$$\lambda \rightarrow (x', y', p^{l'}, p^{g'})$$
$$(\lambda, x, p^l, p^g) \rightarrow \lambda$$

Each variable is restricted to lie in a compact, convex set. Remembering that I is the number of types, we restrict $\lambda \in S^{I-1}$ (the unit simplex), $x \in X$, $y \in \Gamma$, $(p^l, p^g) \in S^{2n-1}$, where n is the number of goods, where X is the cross-product over I of the X_i, and $\Gamma = \{y \in Y | y \text{ is feasible}\}$. Market clearing and the compactness of the X_i guarantee that Γ is a compact set. The set Γ is also convex.[22]

First, for any $\lambda \in S^{I-1}$ there is a solution to the Pareto program $(x^*, y^*, p^{l*}, p^{g*}, \mu^*, \gamma^*)$. Then we normalize (p^{l*}, p^{g*}) to be in the unit simplex with $p_i^{j'} = p_i^{j*}/\sum_{i,j} p_i^{j*}$. This gives us $(x', y', p^{l'}, p^{g'})$. This mapping is non-empty, compact-valued, convex-valued, and upper hemi-continuous.

The second part of the mapping calculates the new Pareto weights as a function of the transfers needed to support the Pareto optimum. Recall that the sets S^{2n-1} and X are compact. Therefore, there exists a positive number A such that

$$\sum_i |(p^l - p^g)(\xi_i - x_i)| \leq A,$$

[22] There are two minor differences between this mapping and Negishi's. The production plan is not explicitly mentioned in the second part of the mapping because with our constant-returns-to-scale technology there are no profits that may affect agent's income. The other difference is the splitting of prices into negative (p^l) and positive (p^g) components.

for any $(x, p^l, p^g) \in X \times S^{2n-1}$. Then, for any $\lambda \in S^{I-1}$, let

$$\hat{\lambda}_i = \max\{0, \lambda_i + \frac{(p^l - p^g)(\xi_i - x_i)}{A}\}, \text{ and } \lambda_i' = \frac{\hat{\lambda}_i}{\sum_i \hat{\lambda}_i}. \tag{72}$$

This mapping is a continuous function so it is trivially a non-empty, convex valued, and upper hemi-continuous correspondence.

To summarize, we have defined a mapping from $S^{I-1} \times X \times \Gamma \times S^{2n-1} \to S^{I-1} \times X \times \Gamma \times S^{2n-1}$. Each of these sets is convex and compact and cross-products of convex and compact sets are convex and compact. Each part of the mapping is non-empty, convex-valued, compact-valued, and upper hemi-continuous. Cross-products of correspondences preserve these properties. Therefore, we have proven the following theorem.

Theorem 11 *A fixed point* $(\lambda, x, y, p^l, p^g)$ *of the mapping exists.*

Proof: Apply the conditions of Kakutani's fixed point theorem.

The fixed point is a solution to the Pareto program so by Theorem 2 it is a compensated equilibrium. The second part of the mapping, subject to one condition, will provide us with a proof that the compensated equilibrium is a competitive equilibrium.

The next step is to show that the fixed point is a competitive equilibrium. We know that the intermediary's problem and market clearing are satisfied. It remains to show that the Consumer's problem is satisfied. We will not be able to do this in general. However, we will be able to show that in all but the most extreme cases the fixed point will also be a competitive equilibrium.

Theorem 5 (Existence) *For any given distribution of endowments, if the Pareto weights at a fixed point of the mapping are non-zero, then a competitive equilibrium exists.*

Proof: By Theorem 11 a fixed point to the mapping exists. The first part of the mapping is a solution to the Pareto program so by Theorem 2 the solution can be supported as a compensated equilibrium. Now assume that $\lambda > 0$. Set $p = p^l - p^g$. From the second part of the mapping (72), at a fixed point, $p(\xi_i - x_i)$ must be the same sign for all i. From the necessary and sufficient conditions to the Pareto program, we know that $p(\sum_i \alpha_i(x_i - \xi_i) - y) = 0$. From Lemma 1, we know that $py = 0$. Therefore, $p(\xi_i - x_i) = 0$, $\forall i$.

This satisfies one part of the necessary and sufficient conditions of the Consumer's problem. The remaining conditions simply require setting $\delta_i = 1/\lambda_i$ and $\omega_i = \gamma_i/(\lambda_i \alpha_i)$. Since $\lambda_i > 0$ we can make this substitution. **Q.E.D.**

The term $\delta_i = 1/\lambda_i$ is consumer i's marginal utility of income.

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