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Generalized Search-Theoretic Models of Monetary Exchange

by Peter Rupert, Martin Schindler and Randall Wright

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We present new results on existence, the number of equilibria, and welfare for search-theoretic models of money that extend the literature in several ways. For example, we provide results for general bargaining parameters while previous papers consider only special cases. Also, we present two version of the model: in one agents holding money cannot produce, in the other they can. The former model has been used in essentially all the previous literature, although the latter seems more natural for some purposes, and avoids several undesirable implications. Since very little is known about this version, we analyze it in detail.

Generalized Search-Theoretic Models of Monetary Exchange*

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Abstract

We present new results on existence, the number of equilibria, and welfare for search-theoretic models of money that extend the literature in several ways. For example, we provide results for general bargaining parameters while previous papers consider only special cases. Also, we present two versions of the model: in one agents holding money cannot produce, in the other they can. The former model has been used in essentially all the previous literature, although the latter seems more natural for some purposes, and avoids several undesirable implications. Since very little is known about this version we analyze it in detail.

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1. Introduction

We present new results on existence, the number of equilibria, and welfare for search-theoretic models of monetary exchange that generalize the existing literature in several ways. The first and most straightforward extension is that we allow money to yield a positive rate of return. Although straightforward, this is interesting because, even in the simplest model with indivisible goods and money like Kiyotaki and Wright (1993), one now has to determine not only if agents are willing to trade goods for money but also if they are willing to trade money for goods, and this changes the set of equilibria qualitatively. Second, when we make goods divisible and let agents bargain, as in Shi (1995) or Trejos and Wright (1995), we present results for all values of the parameter θ representing bargaining power, where previous authors considered the symmetric case $\theta = 1/2$ or the extreme case $\theta = 1$. This generality is useful not only for its own sake, but also because it allows us to discuss the relationship between bargaining power and efficiency, as is done in labor market theory (Hosios [1990]; Mortensen and Pissarides [1994]).

The third and perhaps most interesting extension is that even though we focus attention on models where agents hold either zero or one unit of money, we present two very different versions of such models. In one version, we assume that after producing an agent needs to consume before producing again, which means that in equilibrium agents holding money cannot produce and so cannot acquire more than one unit of cash. In the other, we assume that agents holding money can still produce, and simply impose a unit bound on money holdings. This is interesting for the following reason. Although the former model where agents with money cannot produce is the one used in essentially all of the previous literature, it has several undesirable implications. The alternative model does not share some of these undesirable implications, but almost nothing is known about its properties;

hence, it seems worth investigating it in detail, and comparing the results to the standard model

An example of an undesirable implication of the standard model where agents holding money cannot produce is that if two agents with money meet and there is a double coincidence of wants, they do not trade. A related implication is that as we increase the fraction of agents holding money we necessarily decrease the productive capacity of the economy, which makes it very difficult to interpret the effects of changes in the money supply on endogenous variables such as welfare in the standard model. The alternative, where agents with money can produce, does not have these implications. Indeed, for many issues it is arguably the more reasonable model. Yet, as we said above, very little is known about this version. This paper provides a fairly complete set of results for both models, so that economists who use this type of monetary theory will understand what the alternative assumptions are doing, and so that they can choose the version that works best for their applications.[1]

The rest of the paper proceeds as follows. The next section considers the case of indivisible goods and provides results on existence, the number of equilibria, and welfare for the two versions of the model. The section after that considers the case

[1]There is by now an extensive literature in search-based monetary economics (over 100 papers are listed in Rupert et al. [2000]). All of the simple models in this literature adopt what we are calling the standard assumption, that agents with money cannot produce, with the exception of some unpublished work by Siandra (1993,1995). However, Siandra only considered a special version of the model – in particular, he only considered the model with indivisible goods – and also made some other assumptions that one might want to think about, as we discuss below. There are more complicated models where agents can hold more general inventories of money, including Green and Zhou (1998), Camera and Corbae (1999), Molico (1999), Taber and Wallace (1999), and Zhou (1999). This branch of the literature does allow agents with money to produce, but also typically assumes that there are no meetings that generate a double coincidence of wants, or makes other assumptions that render the main issue in this paper much less interesting. Moreover, it seems clear that models where agents hold only 1 or 0 units of money are still useful for many purposes (e.g., they deliver analytic results), and so it seems worthwhile studying these simpler models in this context, to see what the alternative assumptions imply.

of divisible goods and bargaining, and provides results for arbitrary bargaining power parameters for the two versions of the model. In the final section we conclude.

2. The Indivisible Goods Model

There is a $[0, 1]$ continuum of agents who live forever and discount at rate r. There are a variety of nonstorable and for now indivisible consumption goods. Each agent i produces just one type of good, at unit cost $c \geq 0$. We model preferences as follows: Given two agents i and j, let iWj indicate "i wants to consume the good that j produces" in the sense that i derives utility $u > c$ from consuming what j produces if iWj and i derives utility 0 from consuming what j produces otherwise. For no agent i is it the case that iWi, so trade is necessary for consumption. For a pair of agents i and j selected at random, $pr(iWj) = x$ and $pr(jWi|iWj) = y$. Thus, xy is the probability of a double coincidence of wants.[2]

There is an exogenous quantity $M \in [0, 1]$ of money. Storing a unit of money yields a utility flow γ per unit time. If $\gamma > 0$ money has a dividend, if $\gamma < 0$ it has a storage cost, and if $\gamma = 0$ it is pure fiat money. Initially one unit of money is given to each of M agents chosen at random. To simplify the presentation, we begin by assuming money cannot be disposed of, and relax this later. Agents meet bilaterally according to a Poisson process with parameter α. Upon meeting, agents trade iff mutually agreeable; e.g., if two agents without money meet, a direct barter trade occurs iff there is a double coincidence of wants.

[2] This specification nests several models in the literature. For example, in Kiyotaki and Wright (1989) or Aiyagari and Wallace (1991), there are N goods and N types of agents, and each type n produces good n and consumes good $n + 1$ (mod N). This yields $x = 1/N$, and either $y = 0$ if $N > 2$ or $y = 1$ if $N = 2$. In Kiyotaki and Wright (1993), the events $\{iWj\}$ and $\{jWi\}$ are independent, so $y = x$, and the double coincidence probability is x^2.

We analyze two different versions of the model: in the first, following Kiyotaki and Wright (1991,1993), agents holding money *cannot* produce; in the second, following Siandra (1993,1995), they *can*. Call the first $Model - K$ and the second version $Model - S$. The $Model - K$ assumption can be interpreted as saying that after producing an agent needs to consume before producing again, which implies that in equilibrium he will never acquire more than 1 unit of money. However, as discussed in the Introduction, it has some other implications that are not especially desirable. For instance, when two agents with money meet they cannot trade even if there is a double coincidence. Also, the assumption ties the number of productive agents, $1 - M$, to the money supply in an unnatural way. The $Model - S$ assumption avoids these problems, but does not rule out the accumulation of multiple units of money. Hence, we simply assume in $Model - S$ that agents can store at most 1 unit of money.

In $Model - K$, if you have money your only option is to try to trade it for consumption since by assumption you cannot barter. In $Model - S$, however, when there is a double coincidence you could offer to barter or pay with cash. We will show below that the unique pure strategy equilibrium is for agents to barter whenever they can. What remains to be determined is, what happens when i with money meets j without money and there is a single coincidence of wants (iWj but not jWi). Thus, we have two endogenous variables: π_0 is the probability agents give up goods to get money, and π_1 is the probability agents give up money to get goods. If $\pi = \pi_0\pi_1 > 0$ we say that money circulates.

2.1. $Model - K$

Beginning with $Model - K$, let V_j be the value function of an agent with $j \in \{0, 1\}$ units of money (these do not depend on agent type or time because we consider only symmetric steady state equilibria). If time proceeds in discrete periods of length h, the Bellman equations can be written

$$V_1 = \frac{1}{1+rh}\{\alpha hx\pi(1-M)(u+V_0) + [1-\alpha xh(1-M)\pi]V_1 + \gamma h + o(h)\} \tag{2.1}$$

$$V_0 = \frac{1}{1+rh}\{\alpha hxy(1-M)(u-c+V_0) + \alpha hxM\pi(V_1-c) + [1-\alpha hxy(1-M) - \alpha hxM\pi]V_0 + o(h)\}, \tag{2.2}$$

where $o(h)$ captures the payoff in the event of two or more Poisson arrivals in a period and, hence, $o(h)/h \to 0$ as $h \to 0$. For example, (2.2) sets the return to having $j = 0$ units of money to the sum of three terms. The first is the probability you meet someone without money, $\alpha h(1-M)$, times the probability of a double coincidence, xy, times the barter payoff, $u - c + V_0$. The second is the probability you meet someone with money, αhM, times the probability he wants your good (independent of whether you want his, since he cannot produce), x, times the payoff $\pi(V_1 - c)$. The third term is the probability of no trade times V_0.

Rearranging (2.1) and (2.2), letting $h \to 0$, and normalizing $\alpha x = 1$, we arrive at the continuous time equations

$$rV_1 = \pi(1-M)(u+V_0-V_1) + \gamma \tag{2.3}$$

$$rV_0 = y(1-M)(u-c) + \pi M(V_1 - V_0 - c). \tag{2.4}$$

Define the net gain from a monetary exchange for agents with and without money by $\Delta_1 = u + V_0 - V_1$ and $\Delta_0 = V_1 - V_0 - c$. Simplification yields

$$\Delta_1 = \frac{[M\pi + (1-M)y](u-c) + ru - \gamma}{r+\pi} \tag{2.5}$$

$$\Delta_0 = \frac{(1-M)(\pi - y)(u-c) - rc + \gamma}{r+\pi}. \tag{2.6}$$

Then the strategies (π_0, π_1) constitute an equilibrium if they satisfy the best response conditions

$$\pi_j \begin{cases} = 1 \\ = [0,1] \\ = 0 \end{cases} \quad \text{as} \quad \Delta_j \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} \quad \text{for } j = 0, 1. \tag{2.7}$$

This is basically the model in Kiyotaki and Wright (1993), except in that paper there is no mention of π_1 because $\gamma = 0$, and $\gamma = 0$ (more generally $\gamma \leq 0$) implies $\pi_1 = 1$ is a dominant strategy by virtue of (2.5). With $\gamma = 0$, the only interesting decision is whether to accept money, and there are exactly three possible types of equilibria corresponding to whether π_0 is 0, 1, or between 0 and 1. If we allow $\gamma > 0$, however, we need to determine π_1 endogenously, and there are potentially nine types of equilibria corresponding to whether each π_j is 0, 1, or between 0 and 1; however, only five types of equilibria actually can exist.

Proposition 1. *In Model-K, there are five potential types of equilibria and they exist in the following regions of parameter space:*

$$\begin{array}{lll} \pi_0 = 1 \text{ and } \pi_1 = 0 & \text{is an equilibrium iff} & r \leq \bar{r}_2 \\ \pi_0 = 0 \text{ and } \pi_1 = 1 & \text{is an equilibrium iff} & r \geq \bar{r}_3 \\ \pi_0 = 1 \text{ and } \pi_1 \in (0,1) & \text{is an equilibrium iff} & \bar{r}_1 < r < \bar{r}_2 \\ \pi_0 \in (0,1) \text{ and } \pi_1 = 1 & \text{is an equilibrium iff} & \bar{r}_3 < r < \bar{r}_4 \\ \pi_0 = 1 \text{ and } \pi_1 = 1 & \text{is an equilibrium iff} & \bar{r}_1 \leq r \leq \bar{r}_4 \end{array}$$

where the critical values of r are given by

$$\begin{aligned} \bar{r}_1 &= \frac{\gamma - [M + (1-M)y](u-c)}{u} \\ \bar{r}_2 &= \frac{\gamma - (1-M)y(u-c)}{u} \\ \bar{r}_3 &= \frac{\gamma - (1-M)y(u-c)}{c} \\ \bar{r}_4 &= \frac{\gamma + (1-M)(1-y)(u-c)}{c}. \end{aligned}$$

We have $\bar{r}_1 < \bar{r}_2 < \bar{r}_3 < \bar{r}_4$. These are the only (steady state, symmetric) equilibria.

Proof: For pure strategy equilibria, insert π_0 and π_1 into (2.5) and (2.6) and determine the region of parameter space in which the inequalities in (2.7) hold. Consider $\pi_0 = \pi_1 = 1$. For this to be an equilibrium we require $\Delta_0 \geq 0$ and $\Delta_1 \geq 0$. Inserting $\pi_0 = \pi_1 = 1$ into (2.5) and (2.6), one finds $\Delta_0 \geq 0$ and $\Delta_1 \geq 0$ iff $r \in [\bar{r}_1, \bar{r}_4]$, as stated in the Proposition. The other pure strategy cases are similar. For mixed strategies, solve $\Delta_j = 0$ for π_j and then determine the region of parameter space in which $\pi_j \in (0,1)$. Consider $\pi_0 \in (0,1)$ and $\pi_1 = 1$. For this to be an equilibrium we require $\Delta_0 = 0$ and $\Delta_1 \geq 0$. Now $\Delta_0 = 0$ implies

$$\pi_0 = y + \frac{rc - \gamma}{(1-M)(u-c)}.$$

It is easy to see that $\pi_0 > 0$ iff $r > \bar{r}_3$ and $\pi_0 < 1$ iff $r < \bar{r}_4$, and the condition $\Delta_1 \geq 0$ is redundant. Hence, this equilibrium exists iff $r \in (\bar{r}_3, \bar{r}_4)$, as stated. The other mixed strategy cases are similar. In this way we get the complete set of equilibria. Routine algebra yields $\bar{r}_1 < \bar{r}_2 < \bar{r}_3 < \bar{r}_4$. This completes the proof. ■

2.2. *Model* $-$ *S*

Now suppose agents with money *can* produce. The first issue that needs to be resolved is, what happens in a double coincidence when you have money and the other person does not – do you barter or pay with cash?[3] In this situation, with probability β the agent with money and with probability $1-\beta$ the agent without money is chosen to propose either barter, cash, or no trade. The other responds either by accepting, which executes the proposal, or rejecting, which implies they part company. See Figure 1. Note that rejecting a barter trade is

[3]This is the only ambiguous case, since in every other meeting there is only one feasible transaction (e.g., if you encounter a double coincidence and have no money, barter is the only option). Note that in *Model* $-$ *K* the issue does not come up, since agents with money cannot barter. Also note that Siandra simply assumes "that if a barter and a monetary transaction are possible upon a meeting, barter will always take precedence over monetary exchange ... the main justification is just tractability." (Siandra [1995, p.5]).

strictly dominated by accepting, and proposing no trade is strictly dominated by proposing barter. Hence, all we need to determine is (ψ_0, ψ_1), where ψ_j is the probability the agent with j units of money proposes barter and $1 - \psi_j$ is the probability he proposes cash.

Lemma 1. *In a double-coincidence meeting between an agent with and an agent without money, generically the unique subgame-perfect equilibrium in pure strategies is* $\psi_0 = \psi_1 = 1$.

Proof: See the Appendix. ■

Having resolved the ambiguity between barter and cash when both are available, the Bellman equations can be written

$$\begin{aligned} V_1 &= \frac{1}{1+rh}\{\alpha hxy(u-c+V_1) + \alpha hx(1-y)(1-M)\pi(u+V_0) \\ &\quad +[1-\alpha hxy - \alpha hx(1-y)(1-M)\pi]V_1 + \gamma h + o(h)\} \end{aligned} \quad (2.8)$$

$$\begin{aligned} V_0 &= \frac{1}{1+rh}\{\alpha hxy(u-c+V_0) + \alpha hx(1-y)M\pi(V_1-c) \\ &\quad +[1-\alpha hxy - \alpha hx(1-y)M\pi]V_0 + o(h)\}, \end{aligned} \quad (2.9)$$

where we temporarily reintroduce αx to facilitate comparison to (2.1) and (2.2). In particular, observe that in $Model-S$ the only time you use money is when you meet someone who produces what you want but does not want what you produce. Rearranging, letting $h \to 0$, and normalizing $\alpha x = 1$ as before, we have

$$rV_1 = y(u-c) + (1-y)\pi(1-M)(u+V_0-V_1) + \gamma \quad (2.10)$$

$$rV_0 = y(u-c) + (1-y)\pi M(V_1-V_0-c). \quad (2.11)$$

Although the value functions are different across the two models, we compute Δ_j and define equilibrium exactly as in $Model-K$.

Proposition 2. *In Model-S there are five potential types of equilibria and they exist in the following regions of parameter space:*

$$\begin{array}{lll}
\pi_0 = 1 \text{ and } \pi_1 = 0 & \text{is an equilibrium iff} & r \leq \hat{r}_2 \\
\pi_0 = 0 \text{ and } \pi_1 = 1 & \text{is an equilibrium iff} & r \geq \hat{r}_3 \\
\pi_0 = 1 \text{ and } \pi_1 \in (0,1) & \text{is an equilibrium iff} & \hat{r}_1 < r < \hat{r}_2 \\
\pi_0 \in (0,1) \text{ and } \pi_1 = 1 & \text{is an equilibrium iff} & \hat{r}_3 < r < \hat{r}_4 \\
\pi_0 = 1 \text{ and } \pi_1 = 1 & \text{is an equilibrium iff} & \hat{r}_1 \leq r \leq \hat{r}_4
\end{array}$$

where the critical values of r are given by

$$\begin{aligned}
\hat{r}_1 &= \frac{\gamma - M(1-y)(u-c)}{u} \\
\hat{r}_2 &= \gamma/u \\
\hat{r}_3 &= \gamma/c \\
\hat{r}_4 &= \frac{\gamma + (1-M)(1-y)(u-c)}{c}.
\end{aligned}$$

We have $\hat{r}_1 < \hat{r}_2 < \hat{r}_3 < \hat{r}_4$. These are the only (steady state, symmetric) equilibria, given that agents play pure strategies in the game determining whether to use barter or cash in a double coincidence meeting.

Proof: The argument mimics Proposition 1 exactly. ■

2.3. Discussion

The regions of (γ, r) space where the different equilibria exist in $Model - K$ and $Model - S$ are shown in Figures 2a and 2b. The same five types of equilibria exist in both models, and the regions where they exist are similar but quantitatively different (unless $y = 0$, since the models are identical when there is no barter). If γ is very low the only equilibrium is where no one accepts money, and if γ is very high the only equilibrium is where no one spends it; hence, money circulates iff its intrinsic properties are not too bad or too good. Also, in both models there is a region where the unique equilibrium is $\pi = 1$, but in other regions there are multiple equilibria: in one region we must have $\pi_1 = 1$ but π_0 can be either 0, 1, or between 0 and 1; and in another region we must have $\pi_0 = 1$ while π_1 can be 0, 1, or between 0 and 1.[4]

In terms of how the two models differ, it is actually more difficult to get money to circulate in $Model - S$ (the region in which $\pi > 0$ is possible is smaller). This is because agents with money are more willing to spend it in $Model - K$, since by doing so they can then barter. If $\gamma = c = 0$, the differences between the two models are especially stark: in $Model - K$ there are always three equilibria, $\pi = 0$, $\pi \in (0, 1)$, and $\pi = 1$; and in $Model - S$ there are two, $\pi = 0$ and $\pi = 1$. The intuition is as follows. In $Model - S$, there is no cost associated with acquiring money when $\gamma = c = 0$, and so as long as there is a strictly positive probability of money being accepted you should always accept it. This is not true in $Model - K$, because even when $\gamma = c = 0$ there is the opportunity cost of giving up your barter option.

To continue the comparison of the two models, define welfare by $W = MV_1 + (1 - M)V_0$. Algebra yields

$$rW_S = M\gamma + [y + M(1 - M)(1 - y)\pi](u - c)$$
$$rW_K = M\gamma + (1 - M)[(1 - M)y + M\pi](u - c)$$

in model $j = S, K$. Given any π, we have $W_S \geq W_K$ with strict inequality as long as $y > 0$ and $M > 0$, because in $Model - K$ money crowds out barter. Given $\pi = 1$, maximizing W with respect to M yields:

$$M_S = \begin{cases} 1 & \text{if } \gamma \geq (1 - y)(u - c) \\ 0 & \text{if } \gamma \leq -(1 - y)(u - c) \\ \frac{(1-y)(u-c)+\gamma}{2(1-y)(u-c)} & \text{otherwise.} \end{cases}$$

[4] It is well known that there is a strategic complementarity in the decision to accept money, π_0. It is less well understood, however, that the same is true of π_1: for some parameters, you are more willing to spend money if you believe other agents will do the same. Notice this only occurs when $\gamma > 0$, which is why it does not arise in the standard model.

$$M_K = \begin{cases} 1 & \text{if } \gamma \geq u - c \\ 0 & \text{if } \gamma \leq -(1-2y)(u-c) \\ \frac{(1-2y)(u-c)+\gamma}{2(1-y)(u-c)} & \text{otherwise.} \end{cases}$$

Hence, $M_S \geq M_K$, again because money crowds out barter in $Model - K$.

The preceding calculation ignores the fact that $\pi = 1$ is not an equilibrium for all parameters. If $\gamma = 0$ then it is easy to see $\pi = 1$ is an equilibrium iff $M \leq \overline{M} = 1 - \frac{rc}{(1-y)(u-c)}$. Then maximizing W subject to $M \leq \overline{M}$ implies:

$$M_S^* = \min\left(\tfrac{1}{2}, \overline{M}\right)$$
$$M_K^* = \begin{cases} 0 & \text{if } y \geq 1/2 \\ \min\left(\frac{1-2y}{2-2y}, \overline{M}\right) & \text{if } y < 1/2 \end{cases}$$

In Figure 3a, the labels W_1^j, W_0^j, and $W_{\pi_0}^j$ denote welfare in model $j = S, K$ in an equilibria with $\pi_0 = 1$, $\pi_0 = 0$, and $\pi_0 \in (0,1)$ (recall $\pi_1 = 1$ is a dominant strategy when $\gamma = 0$). The curves are only drawn for values of M such that the equilibria exist. Figure 3b shows welfare as a function of y, given we set $M = M_j^*$ (which depends on y). These figures illustrate various properties, including: W is always higher in $Model - S$; W increases with π; and $M_j^* = 0$ for big y.[5]

We close this section by mentioning two details. First, the above results are predicated on the assumption that agents cannot freely dispose of money.

Lemma 2. *No free disposal is not binding, except if* $\pi_0 = 0$ *and* $\gamma < 0$.

Proof: See Appendix.

Second, the results for $Model - S$ are predicated on the condition that when there is a double coincidence and one agent has money but the other does not,

[5] Assuming the constraint $M \leq \bar{M}$ is not binding (e.g., r is small), we have $M_S = 1/2$ for all parameter values, and $M_K \leq 1/2$ with strict inequality as long as $y > 0$. To understand this, note that the role of money is to allow trade when iWj but not vice-versa, and i has money but j does not. Maximizing the probability of this event implies giving money to 1/2 the agents. This explains $M_S = 1/2$. In $Model - K$, however, we also have to take into account the fact that money crowds out barter.

they will barter. Lemma 1 shows this is the unique subgame perfect equilibrium in pure strategies. However, if we allow mixed strategies, there actually can be other equilibria. These other equilibria are not robust to various perturbations,[6] but we record them here as follows.

Lemma 3. *In Model-S there are two other equilibria where barter is not the outcome in a double-coincidence meeting between an agent with money and an agent without money. These equilibria both have* $\pi = 1$*, and they exist in the following regions of parameter space:*

$$\begin{array}{lll} \psi_0 \in (0,1) \text{ and } \psi_1 = 1 & \text{is an equilibrium iff} & \hat{r}_0 < r < \hat{r}_1 \\ \psi_0 = 1 \text{ and } \psi_1 \in (0,1) & \text{is an equilibrium iff} & \hat{r}_4 < r < \hat{r}_5 \end{array}$$

where $\hat{r}_1$ *and* $\hat{r}_4$ *are defined in Proposition 2 and*

$$\begin{aligned} \hat{r}_0 &= \frac{\gamma - [(1-y)M + y(1-M)(1-\beta)](u-c)}{u} \\ \hat{r}_5 &= \frac{\gamma + [(1-y)(1-M) + yM\beta](u-c)}{c}. \end{aligned}$$

We have $\hat{r}_0 < \hat{r}_1$ *and* $\hat{r}_5 > \hat{r}_4$*. These are the only (symmetric steady state) equilibria other than those given in Proposition 2.*

Proof: See the Appendix.

3. The Divisible Goods Model

Assume now that goods are perfectly divisible, and when q units are exchanged, the consumer derives utility $u(q)$ and the producer incurs disutility $-c(q)$, where

[6] For example, in a double coincidence meeting where an agent proposed a cash trade instead of a barter trade, if the other agent could respond with a counteroffer (rather than simply accepting or rejecting), these equilibria would not exist. Allowing for "trembles" also eliminates these equilibria. That is, for any positive probability that a cash offer is rejected (note that in any of the mixed strategy equilibria described in the lemma, the agent is indifferent between acccepting and rejecting), proposing barter becomes strictly dominant.

$u'(q) > 0$ and $c'(q) > 0$. Also, $c''(q) \geq 0$ and $u''(q) \leq 0$ with at least one of the inequalities strict. Finally, $u(0) = c(0) = 0$, $u'(0) > c'(0) = 0$, and there exists $\hat{q} > 0$ such that $u(\hat{q}) = c(\hat{q})$. Otherwise, everything is the same as in the previous section, except to simplify the presentation we set $\gamma = 0$ and focus only on equilibria with $\pi = 1$ (equilibria with $0 < \pi < 1$ do not exist with divisible goods). Also, we will first present results for $Model - K$ and $Model - S$ when $y > 0$, and then give results for $y = 0$, in which case the two models are equivalent.

Assume q solves the generalized Nash bargaining problem,

$$q = \arg\max[u(q) + V_0 - T_1]^{\theta}[V_1 - c(q) - T_0]^{1-\theta} \tag{3.1}$$

subject to the constraints $q \geq 0$, $\Delta_1 = u(q)+V_0-V_1 \geq 0$ and $\Delta_0 = V_1-V_0-c(q) \geq 0$. In (3.1), T_j is the threat point of the agent with j units of money and θ represents bargaining power. We allow θ to take on any value in $[0,1]$ and consider two cases for T_j that have been analyzed in the literature: $T_j = V_j$ and $T_j = 0$. Note that in Shi (1995) and Trejos and Wright (1995) only the special case $\theta = 1/2$ is considered.[7]

3.1. $Model - K$

Given Q, the Bellman equation are

$$\begin{aligned} rV_1 &= (1-M)[u(Q) + V_0 - V_1] & (3.2) \\ rV_0 &= (1-M)yB + M[V_1 - V_0 - c(Q)], & (3.3) \end{aligned}$$

where B denotes the payoff to barter. We assume that in a barter trade each agent produces q^* for the other agent, so that $B = u(q^*) - c(q^*)$, where q^* satisfies

[7]As is well known, the Nash solution corresponds to the equilibrium of an explicit strategic bargaining model, where θ and T_j depend on details of the game; see Osborne and Rubinstein (1990) or Coles and Wright (1998).

$u'(q^*) = c'(q^*)$.[8] We are interested in determining the q that is traded for money. Taking as given the Q that other agents are trading, any two agents bargain over the q they will exchange for the money. In equilibrium, of course, $q = Q$.

Consider the case where the threat point is $T_j = V_j$. Then the first order condition for (3.1) is

$$\theta[V_1 - V_0 - c(q)]u'(q) - (1-\theta)[u(q) + V_0 - V_1]c'(q) = 0. \tag{3.4}$$

Solving (3.2) and (3.3) for V_j, setting $q = Q$, and inserting the results into (3.4), we have $e(q) = 0$, where

$$\begin{aligned} e(q) &= \theta[(1-M)u(q) - (r+1-M)c(q)]u'(q) \\ &\quad -(1-\theta)[(r+M)u(q) - Mc(q)]c'(q) \\ &\quad -(1-M)yB[\theta u'(q) + (1-\theta)c'(q)]. \end{aligned} \tag{3.5}$$

A solution $q > 0$ to $e(q) = 0$ would constitute a monetary equilibrium if there were no other constraints. The constraint $\Delta_1 \geq 0$ can be shown to be redundant if $\Delta_0 \geq 0$ holds, and $\Delta_0 \geq 0$ iff q is below a threshold given by the q that solves $c(q) = V_1 - V_0$.

So, an unconstrained monetary equilibrium is given by a value of $q > 0$ such that $e(q) = 0$ and $\Delta_0 \geq 0$. Another type of monetary equilibrium occurs when a constraint binds. Suppose $\Delta_0 = 0$ at some $\tilde{q}$; then the constraint binds at $q = \tilde{q}$ and is violated for any $q > \tilde{q}$. Thus, if $e(\tilde{q}) > 0$ then $\tilde{q}$ is a constrained solution to the Nash bargaining problem. It is easy to see that $\Delta_0 = 0$ iff $q = \underline{q}$ or $q = \bar{q}$, where $\underline{q}$ and $\bar{q}$ are the zeros of the strictly concave function

$$f(q) = (1-M)u(q) - (r+1-M)c(q) - (1-M)yB.$$

[8] It is easy to motivate this assumption, since this outcome is the unique equilibrium of a natural bargaining game; however, it actually does not matter, since all that we require for the results is the very weak assumption that B does not exceed $\frac{u(q^*)-c(q^*)}{y}$.

See Figure 4a. Note that f shifts down with an increase in r, and for small r the two zeros $\underline{q}$ and $\bar{q}$ exist, while for big r there are no solutions to $f(q) = 0$. Clearly, $q \in (\underline{q}, \bar{q})$ implies $\Delta_0 > 0$, and so any $q \in (\underline{q}, \bar{q})$ such that $e(q) = 0$ is an unconstrained equilibrium. Also, if $e(\underline{q}) > 0$ then $\underline{q}$ is a constrained equilibrium, and if $e(\bar{q}) > 0$ then $\bar{q}$ is a constrained equilibrium.

Results on existence and the number of equilibria follow. The key functions used in the constructions and the resulting set of equilibrium values for q as a function of θ are depicted in Figures 4 and 5 for the case of $T_j = V_j$ and $T_j = 0$, respectively.

Proposition 3. *Consider Model-K with $T_j = V_j$ and $y > 0$. There exists $\bar{r} > 0$ such that no monetary equilibria exist if $r > \bar{r}$, while if $r < \bar{r}$ there exists $\bar{\theta} \in (0, 1)$ (that depends on r) such that the following is true: for $\theta < \bar{\theta}$ no monetary equilibria exist; and for $\theta > \bar{\theta}$ there exists a generically even number of monetary equilibria. All monetary equilibria are unconstrained.*

Proof: Consider the limiting case of $r = 0$. By inspection of f, there exists $\underline{q} > 0$ and $\bar{q} > \underline{q}$ such that $f(\underline{q}) = f(\bar{q}) = 0$, as in Figure 4a. It is easy to check that $e(\underline{q}) \leq 0$ and $e(\bar{q}) \leq 0$, with equality iff $\theta = 1$. This implies that no constrained equilibria exist (when $\theta = 1$ we actually have an equilibrium where the constraint is satisfied at equality, but since the first order condition also holds with equality we say that the solution is unconstrained). Moreover, if any unconstrained monetary equilibria exist, generically there will be an even number since e will have an even number of zeros in $(\underline{q}, \bar{q})$. Setting $\theta = 0$ implies $e < 0$ for all q, so no monetary equilibria exist. Recall that $\theta = 1$ implies the existence of monetary equilibria. It can be shown that $\frac{\partial e}{\partial \theta}|_{e=0} > 0$ (see the Appendix), so that decreasing θ shifts e down in Figure 4a. This implies the existence of a unique $\bar{\theta}$ such that monetary equilibria exist iff $\theta > \bar{\theta}$. See Figure 4b. All of the above statements are for $r = 0$; by continuity, the results are the same for small positive r. It is easy to see that e is monotonically decreasing in r (for $q > 0$), and that $e < 0$ for all $q \in (\underline{q}, \overline{q})$ when r is sufficiently big. Hence, there is a unique $\bar{r}$ such that no monetary equilibria exist for any θ when $r > \bar{r}$. ■

Things are similar for the case $T_j = 0$, except for two things: now we can say that (generically) there exist 2 monetary equilibria when any exist, as opposed to simply an even number; and now one or both of the monetary equilibria is constrained, while in the case of $T_j = V_j$ they are all unconstrained (see Figures 4b and 5b).

Proposition 4. *Consider Model-K with $T_j = 0$ and $y > 0$. There exists $\bar{r} > 0$ such that no monetary equilibria exist if $r > \bar{r}$, while if $r < \bar{r}$ there exists $\bar{\theta} \in (0, 1)$ (that depends on r) such that the following is true: for $\theta < \bar{\theta}$ no monetary equilibria exist; and for $\theta > \bar{\theta}$ there are exactly two monetary equilibria. Both equilibria are constrained if θ is large, while one equilibrium is constrained and the other unconstrained if θ is not so large.*

Proof: With $T_j = 0$, the analogue of (3.5) is

$$\begin{aligned} e(q) &= \theta\{(1-M)(r+M)u(q) - [r(1+r) + M(1-M)]c(q)\}u'(q) \\ &\quad -(1-\theta)\{[r(1+r) + M(1-M)]u(q) - M(r+1-M)c(q)\}c'(q) \\ &\quad +(1-M)yB[\theta(1-M)u'(q) - (1-\theta)(r+1-M)c'(q)]. \end{aligned}$$

See Figure 5a. As before, $r = 0$ implies that $\underline{q} > 0$ and $\overline{q} > \underline{q}$ exist. If $\theta = 0$, $e(q) < 0$ for any $q > 0$ and no monetary equilibria exist. If $\theta = 1$, $e(q) > 0$ for all $q \in (0, \overline{q}]$, and both $\underline{q}$ and $\overline{q}$ are constrained equilibria. One can check $\frac{\partial e}{\partial \theta}|_{e=0} > 0$, so there exists a critical value $\bar{\theta} > 0$ such that monetary equilibria exist iff $\theta \geq \bar{\theta}$. Also, for $\theta \geq \bar{\theta}$ but not too large, $e(\overline{q}) < 0$, and so one monetary equilibrium is unconstrained at $q \in (\underline{q}, \overline{q})$ and one equilibrium is constrained at $\underline{q}$, while for very large θ, $e(\overline{q}) > 0$ (for arbitrary r), so both monetary equilibria are constrained. This can be seen by noting that

$$\begin{aligned} e(\overline{q})|_{\theta=1} &= (M+r)\left[(1-M)u(\overline{q}) - (r+1-M)c(\overline{q})\right]u'(\overline{q}) \\ &\quad +(1-M)^2 yBu'(\overline{q}) \\ &= (1+r)(1-M)yBu'(\overline{q}), \end{aligned}$$

which is positive for any r, combined with the fact that e is continuous and increasing in θ.

These results are qualitatively the same for small positive r. It is shown in the appendix that, as long as $e(\underline{q}) > 0$, we have $\frac{\partial e}{\partial r}|_{e=0} < 0$. Note that this is all we need as monetary equilibria exist iff $e(\underline{q}) > 0$. Setting $\theta = 1$, there exists a unique $\bar{r} > 0$ such that no monetary equilibria exist when $r > \bar{r}$ for any θ. For any $r < \bar{r}$, by the above reasoning, there exists $\bar{\theta} < 1$ such that monetary equilibria exist for $\theta > \bar{\theta}$. To see that there are exactly two monetary equilibria (whenever any exist), it is sufficient to show that any $q \in (0, \overline{q})$ such that $e(q) = 0$ is unique. This is verified by noting that at $e = 0$, we have

$$(1-\theta)\frac{c'}{u'} = \frac{\theta}{D}\left\{(1-M)(r+M)u - [r(1+r) + M(1-M)]c + (1-M)^2 yB\right\}$$

where $D = [r(1+r) + M(1-M)]u - M(r+1-M)c + (r+1-M)(1-M)yB$, and that the left side is increasing and the right side decreasing in q. ■

3.2. *Model* $-$ *S*

In $Model - S$, the Bellman equations are[9]

$$rV_1 = yB + (1-M)(1-y)[u(q) + V_0 - V_1]$$

[9]Given $B = u(q^*) - c(q^*) \geq u(q) - c(q)$ for all q, a version of Lemma 1 holds here: when someone with money meets someone without money and there is a double coincidence of wants, they always barter.

$$rV_0 \quad = \quad yB + M(1-y)[V_1 - V_0 - c(q)],$$

and the function $f(q)$ becomes

$$f(q) = (1-M)(1-y)u(q) - [r + (1-M)(1-y)]c(q).$$

Otherwise, the method is the same as in $Model - K$. However the results are quite different: in $Model - S$ there exists a unique monetary equilibrium for all $\theta > 0$. Graphically, this can be understood by noting that in this model, the analogues to the functions e and f shown in Figures 4a and 5a now go through the origin.[10]

Proposition 5. *Consider Model-S with* $T_j = V_j$ *and* $y > 0$. *For any* $r > 0$, *the following is true: if either* $\theta = 0$ *or* $y = 1$, *no monetary equilibrium exists; otherwise, there exists a unique monetary equilibrium and it is unconstrained.*

Proof: In this case the analogue of (3.5) is

$$e(q) = \theta f(q)u'(q) - (1-\theta)\{[r + M(1-y)]u(q) - M(1-y)c(q)\}c'(q).$$

For $y = 1$, we have $e(0) = 0$ and $e(q) < 0$ for $q > 0$, and so no monetary equilibria exist for any θ. Now suppose $y < 1$. It is easy to see that $f(q) = 0$ at $\underline{q} = 0$ and $\overline{q} > 0$. It can also be verified that $e(0) = 0$ and $e(\overline{q}) \leq 0$ for all θ, where the inequality is strict for $\theta < 1$, implying that there are no constrained equilibria. If $\theta = 0$, the only solution to $e = 0$ is $q = 0$. For $\theta \in (0, 1]$, $e'(0) > 0$, so that there exists a $q \in (0, \overline{q}]$ such that $e(q) = 0$. The monetary equilibrium is unique because at $e = 0$, we have

$$(1-\theta)\frac{c'}{u'} = \theta \frac{f(q)}{[r + M(1-y)]u - M(1-y)}$$

[10]Note however that multiple monetary equilibria can be recovered in $Model - S$ if we reintroduce γ, since $e(0)$ and $f(0)$ need not be 0 if $\gamma \neq 0$.

where the left hand side is increasing and the right hand side decreasing in q. Finally, when $\theta = 1$, the (unconstrained) unique monetary equilibrium is at $q = \overline{q}$. See Figure 6. ■

Proposition 6. *Consider Model-S with $T_j = 0$ and $y > 0$. For any $r > 0$, the following is true: if either $\theta = 0$ or $y = 1$, no monetary equilibrium exists; otherwise, there exists a unique monetary equilibrium. It is unconstrained for $\theta < \hat{\theta}$, while it is constrained for $\theta > \hat{\theta}$, where.$0 < \hat{\theta} < 1$.*

Proof: In this case the analogue of (3.5) is

$$\begin{aligned} e(q) &= \theta[r + M(1-y)]f(q)u' \\ &\quad -(1-\theta)[r(1+r-y)u + M(1-y)f(q)]c' \\ &\quad +(1+r-y)yB[\theta u' - (1-\theta)c']. \end{aligned}$$

The constraint is as before, and so $f(q) = 0$ at $\underline{q} = 0$ and $\overline{q} > 0$. For $\theta = 0$, we have $e(0) = 0$ and $e'(0) < 0$, and so no monetary equilibrium exists, while at $y = 1$, again, no monetary equilibria exist. Now assume $y < 1$. For $\theta \in (0, 1]$, we have $e(0) > 0$. Note that for $\theta = 0$, $e(\overline{q}) < 0$ and for $\theta = 1$, $e(\overline{q}) > 0$. By inspection e is monotonically increasing in θ. Thus, there exists $\hat{\theta} \in (0, 1)$ such that the monetary equilibrium is unconstrained for $\theta < \hat{\theta}$ and constrained (at $\overline{q}$) for $\theta > \hat{\theta}$. Uniqueness is shown in the usual way. See Figure 7. ■

3.3. The Case $y = 0$

Here we present the results for $y = 0$, which makes $Model - S$ and $Model - K$ equivalent. It turns out that the results in this common case are most similar to the $Model - S$ results for $y > 0$: for all $\theta > 0$ there always exists a monetary equilibrium and it is unique. Note that with $y = 0$, the equilibrium is always unconstrained. (This is not inconsistent with Proposition 6, because $\hat{\theta}$ defined in that proposition goes to 1 as y goes to zero.)

Proposition 7. *For either $T_j = V_j$ or $T_j = 0$, if $y = 0$ then for any $r > 0$, the following is true: if $\theta = 0$ no monetary equilibria exist; and if $\theta > 0$ there exists a unique monetary equilibrium and it is unconstrained.*

Proof: For $y = 0$, we have $\underline{q} = 0$ and $\overline{q} > 0$ by inspection of $f(q)$. For $\theta = 0$, the only solution to $e = 0$ is $q = 0$. For $\theta \in (0, 1)$ it can be verified that $e(0) = 0$ and $e(\overline{q}) < 0$, implying that there are no constrained equilibria. Moreover, it is easy to see that $e'(0) > 0$, which implies that there exists $q \in (\underline{q}, \overline{q})$ such that $e(q) = 0$. Finally, when $\theta = 1$, the unique non-zero solution to $e = 0$ is $q = \overline{q}$, and so this is the (unconstrained) unique equilibrium. Function e and equilibria are qualitatively the same as in $Model - S$ with $T_j = V_j$ and $y > 0$ (see Figure 6). ■

3.4. Discussion

The main elements that distinguish our analysis of the divisible goods case from the previous literature are: (i) we allow any $\theta \in [0, 1]$; and (ii) we consider $Model-S$ as well as $Model-K$. We now discuss the implications of these generalizations. First, note that many of the main results from the previous literature go through: for example, for any θ, in $Model - K$ with $y > 0$ the existence of monetary equilibria depends on agents being sufficiently patient. Also, as in earlier analyses, the equilibrium quantity q is generically not socially efficient. However, given our generalized model, the interaction between efficiency and bargaining power θ can be explored in more detail.

It is easy to see that the efficient (welfare maximizing) q is q^*, the quantity that solves $u'(q^*) = c'(q^*)$, and that q^* generically does not satisfy the equilibrium conditions. For example, as remarked in Trejos and Wright (1995), in $Model - K$

with $T_j = 0$ and $\theta = 1/2$, the equilibrium implies $q < q^*$ for all $r > 0$, although $q \to q^*$ in the limit as $r \to 0$. Once we allow for general bargaining power, however, any quantity $q \in [\underline{q}, \overline{q}]$ is an equilibrium for some θ (in every version of the model). At least if r is not too big, we have $q^* \in [\underline{q}, \bar{q}]$, and so there exists some θ^* such that the equilibrium value of money coincides with the efficient q^*.[11]

Proposition 8. *In either model, for $r < r^* = (1-M)(1-y)B/c(q^*)$ there exists $\theta^* \in (0,1)$ such that the equilibrium quantity is efficient: $q = q^*$.*

Proof. In $Model - K$, we have

$$f = (1-M)(u-c) - rc - (1-M)yB.$$

From the proofs of the previous propositions we know that q^* can be obtained by choosing θ appropriately iff $\bar{q} \geq q^*$. We only need to determine when this is the case. For $r = 0$, it is easy to see $\bar{q} > q^*$. It is also easy to see that $\bar{q} \to 0$ as $r \to \infty$. Because f is monotonically decreasing in r, there exists $r^* > 0$ such that $\bar{q} > q^*$ for $r < r^*$. The critical r^* is found by setting $f(q^*) = 0$ and solving for r. The proof for $Model - S$ is analogous. ■

The value of θ^* that generates $q = q^*$ can be derived by setting $e(q^*) = 0$ and solving for θ. The resulting expressions are not particularly instructive, in general, but simplify nicely when $r \to 0$. On the one hand, when $T_j = 0$ we have $\theta^* = \frac{1}{2}$ in either $Model - K$ or $Model - S$. Hence, at least in the limit as frictions vanish, symmetric bargaining power generates the efficient q in either model as long as $T_j = 0$. On the other hand, when $T_j = V_j$ we have $\theta^* = M + (1-M)y$ in $Model - K$ and $\theta^* = M$ in $Model - S$. Notice that θ^* is greater in $Model - K$ for an exogenously fixed M. However, if we set M to maximize W, conditional

[11]This relationship between bargaining power and efficiency is reminiscent of a similar relationship in matching models of the labor market, as mentioned in the Introduction.

on $q = q^*$, we have $M = \frac{1}{2}$ in $Model - S$ and $M = \frac{1-2y}{2-2y}$ in $Model - K$. Inserting these into θ^* again yields $\theta^* = \frac{1}{2}$ in both cases. Summarizing:

Proposition 9. *For either $Model - S$ or $Model - K$, in the limit as $r \to 0$, we have the following: when $T_j = 0$ the efficient bargaining weight is $\theta = 1/2$ for any M; when $T_j = V_j$ the efficient bargaining weight varies between 0 and 1 depending on the exogenous value of M, but if M is set optimally then again $\theta = 1/2$.*

Intuitively, in the case $T_j = V_j$ the reason that θ^* is greater in $Model - K$ than in $Model - S$ for an exogenously fixed M can be understood as follows. First, as illustrated in Figure 8a, the equilibrium q is lower in $Model - K$, not only for $r = 0$ but for any r, again because accepting money is more costly when it crowds out barter.[12] Hence, θ has to be larger in $Model - K$ to generate any given q, including q^*. Note, however, that it is not true in general that the equilibrium q is lower in $Model - K$ when $T_j = 0$, as shown by example in Figure 8b, where for low M we see that q is higher in $Model - K$ (the figure is drawn with $u = q$, $c = q^2$, $y = 0.1$, $r = 0.5$, and $\theta = 0.5$). It is also not true that welfare is necessarily lower in $Model - K$ here, contrary to the result in the indivisible goods model; see Figure 8a, where $W_S < W_K$ for low M. The intuition is as follows. For exogenously given values of r, M, θ etc., suppose q is much greater than q^* in $Model - S$. Then welfare can be higher in $Model - K$, even though money crowds out barter in this model, because it generates a lower q.

[12]Note that q is lower in $Model - K$ iff $e^K = 0$ implies $e^S > 0$, where e^j is the equilibrium condition in model $j = K, S$. By inspection,

$$e^K = e^S + \theta(1-M)y(u-c)u' - (1-\theta)yM(u-c)c' - (1-M)yB[\theta u' + (1-\theta)c']$$

and so $e^K = 0$ implies

$$e^S = [(1-M)yB + My(u-c)](1-\theta)c' + \theta(1-M)yBu' - \theta(1-M)y(u-c)u'$$

which is positive since $B \geq u(q) - c(q)$ for all q.

4. Conclusion

We have extended search-theoretic models of money in several ways, including: introducing a flow return to holding money; generalizing the bargaining solution; and developing the model in which agents with money can produce and comparing it to the conventional model where they cannot. Results on existence, the number of equilibria, and welfare were derived for a variety of different versions of the model. We hope that these results will be useful to others who want to extend or apply search-based monetary theory in the future.

5. Appendix

Proof of Lemma 1: Suppose that the seller gets to propose, and he proposes a cash transaction. There are three possibilities. First, if $\Delta_1 < 0$ then the proposal will be rejected, so the seller would have been strictly better off proposing barter (Figure 1). Second, if $\Delta_1 > 0$ then the proposal will be accepted, but in this case $\Delta_0 < u - c$ (because $\Delta_0 + \Delta_1 = u - c$), and so again the seller would have been strictly better off proposing barter. So a seller would never propose a cash trade over barter except possibly if $\Delta_1 = 0$. A symmetric argument implies that a buyer would never propose a cash trade except possibly if $\Delta_0 = 0$. So we have two cases to consider: (i) $\Delta_0 = 0$, which implies $\Delta_1 = u - c$, which further implies $\psi_0 = 1$ (since $\Delta_1 > 0$ implies the seller strictly prefers barter), which is the only case in which we can have $\psi_1 < 1$; and (ii) $\Delta_1 = 0$, which implies $\Delta_0 = u - c$ and $\psi_1 = 1$, which is the only case in which we can have $\psi_0 < 1$.

Consider case (ii), where $\Delta_1 = 0$, $\Delta_0 = u - c$, $\psi_0 < 1$ and $\psi_1 = 1$ in equilibrium. Note $\Delta_0 = u - c$ implies $\pi_0 = 1$. Suppose $\pi_1 < 1$; then the agent without money gets $\pi_1 \Delta_0 = \pi_1(u - c)$ from proposing a cash transaction, which is strictly less than what he gets proposing barter. So $\psi_0 < 1$ requires $\pi_0 \pi_1 = 1$. Then the value functions generalizing (2.10) and (2.11) are

$$\begin{aligned} rV_1 &= yM(u-c) + (1-y)(1-M)\Delta_1 + y(1-M)[\beta\psi_1 + (1-\beta)\psi_0](u-c) \\ &\quad + y(1-M)[1-\beta\psi_1 - (1-\beta)\psi_0](1-\psi_0)\Delta_1 + \gamma \\ rV_0 &= y(1-M)(u-c) + (1-y)M\Delta_0 + yM[\beta\psi_1 + (1-\beta)\psi_0](u-c) \\ &\quad + yM[1-\beta\psi_1 - (1-\beta)\psi_0](1-\psi_0)\Delta_0. \end{aligned}$$

Since we are in case (ii), we have $\Delta_1 = 0$, $\Delta_0 = u - c$ and $\psi_1 = 1$. Hence, subtracting V_1 and V_0 and simplifying, we have

$$\psi_0 = \frac{ru - \gamma + [(1-y)M + y(1-M)(1-\beta)](u-c)}{y(1-M)(1-\beta)(u-c)}. \tag{5.1}$$

This equality is violated for generic parameter values when $\psi_0 = 0$. Hence there is no equilibrium where sellers propose cash with probability 1. A symmetric argument for case (i) implies there is no equilibrium where buyers propose cash with probability 1. This means the unique pure strategy equilibrium is for agents to propose barter with probability 1: $\psi_0 = \psi_1 = 1$. ■

Proof of Lemma 2: We need to be clear about what agents do after they dispose of their money. In $Model - K$, they cannot trade, since they are not able to produce. Hence agents will dispose of money and drop out of the trading process iff $V_1 < 0$. Since it is easy to see that $V_0 \geq 0$ in any equilibrium and $V_1 \geq V_0$ in any equilibrium with $\pi_0 > 0$, the only case where disposal could potentially occur is $\pi_0 = 0$, which implies $V_1 = \gamma/r$. Hence, agents dispose of money iff $\pi_0 = 0$ and $\gamma < 0$. In $Model - S$, if agents dispose of money they can continue to barter. Hence, in this model they dispose of money iff $V_1 < V_0$. Again this occurs iff $\pi_0 = 0$ and $\gamma < 0$. ■

Proof of Lemma 3: The proof of Lemma 1 shows that there is potentially an equilibrium with $\pi = 1$, $\Delta_1 = 0$, $\Delta_0 = u - c$, $\psi_0 \in (0, 1)$ and $\psi_1 = 1$ as long as (5.1) holds. The only thing to check is $\psi_0 \in (0,1)$. It is straightforward to see that this holds iff $r \in (\hat{r}_0, \hat{r}_1)$, where $\hat{r}_0$ is defined in Lemma 3 and $\hat{r}_1$ is defined in Proposition 2. One can show $\hat{r}_0 < \hat{r}_1$. Hence, the equilibrium with $\pi = \psi_1 = 1$ and $\psi_0 \in (0,1)$ exists iff $r \in (\hat{r}_0, \hat{r}_1)$. The symmetric argument shows that there is potentially an equilibrium with $\pi = 1$, $\Delta_1 = u - c$, $\Delta_0 = 0$, $\psi_0 = 1$ and $\psi_1 \in (0,1)$ as long as

$$\psi_1 = \frac{\gamma - rc + [(1-y)(1-M) + yM\beta](u-c)}{yM\beta(u-c)}.$$

The only thing to check is $\psi_1 \in (0,1)$, which holds iff $r \in (\hat{r}_4, \hat{r}_5)$, where $\hat{r}_4$ is defined in Proposition 2 and $\hat{r}_5$ is defined in Lemma 3. Hence, the equilibrium with $\pi = \psi_0 = 1$ and $\psi_1 \in (0,1)$ exists iff $r \in (\hat{r}_0, \hat{r}_1)$. These are the only possibilities. ■

Claims in Proofs of Propositions 3 and 4: In Proposition 3, we claim $e_\theta|_{e=0} > 0$. Differentiating (3.5) yields

$$e_\theta = [(1-M)u - (r+1-M)c]u' - (1-M)yB[u' - c'] + [(r+M)u - Mc]c'.$$

Using $e = 0$, we have

$$e_\theta|_{e=0} = [ru + M(u-c) + (1-M)yB]c' > 0.$$

In Proposition 4 we make the same claim. In this case,

$$e_\theta|_{e=0} = [ru + M(u-c) + (1-M)yB](r+1-M)c' > 0.$$

We also claim $e_r|_{e=0} < 0$ if $e(\underline{q}) > 0$. Differentiation yields

$$e_r = \theta[(1-M)u - (1+2r)c]u' - (1-M)yB(1-\theta)c' - (1-\theta)[(1+2r)u - Mc]c',$$

which can be rewritten

$$\begin{aligned} re_r &= e - \theta\{(1-M)Mu - [M(1-M) - r^2]c\}u' \\ &\quad +(1-\theta)\{[M(1-M) - r^2]u - M(1-M)c\}c' \\ &\quad -(1-M)yB[\theta(1-M)u' - (1-\theta)(1-M)c']. \end{aligned}$$

Setting $e = 0$ and rearranging, we have

$$re_r|_{e=0} = -r^2[\theta cu' + (1-\theta)uc'] - [M(u-c) + (1-M)yB](1-M)[\theta u' - (1-\theta)c'].$$

As the first term is strictly negative, a sufficient condition for $e_r|_{e=0} < 0$ is $\theta u' - (1-\theta)c' > 0$. It can be shown easily that $e(\underline{q}) > 0$ implies $\theta u' - (1-\theta)c' > 0$, verifying the claim. ■

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Figure 1: Game Tree

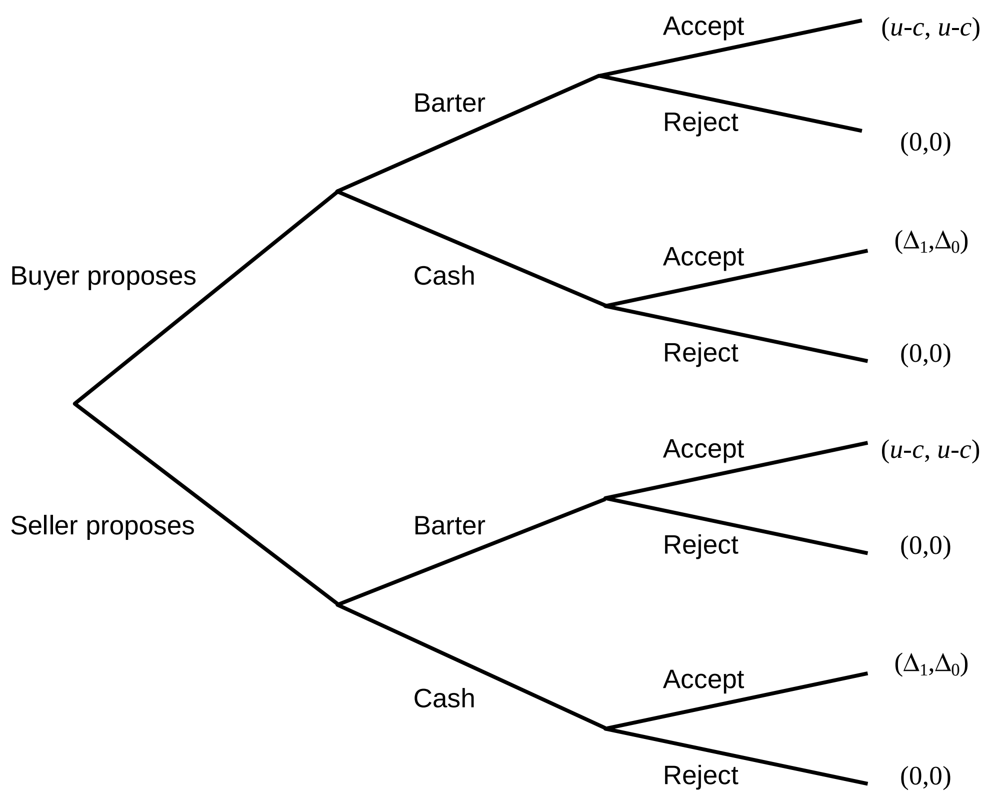

Figure 2a: ***Model-K***

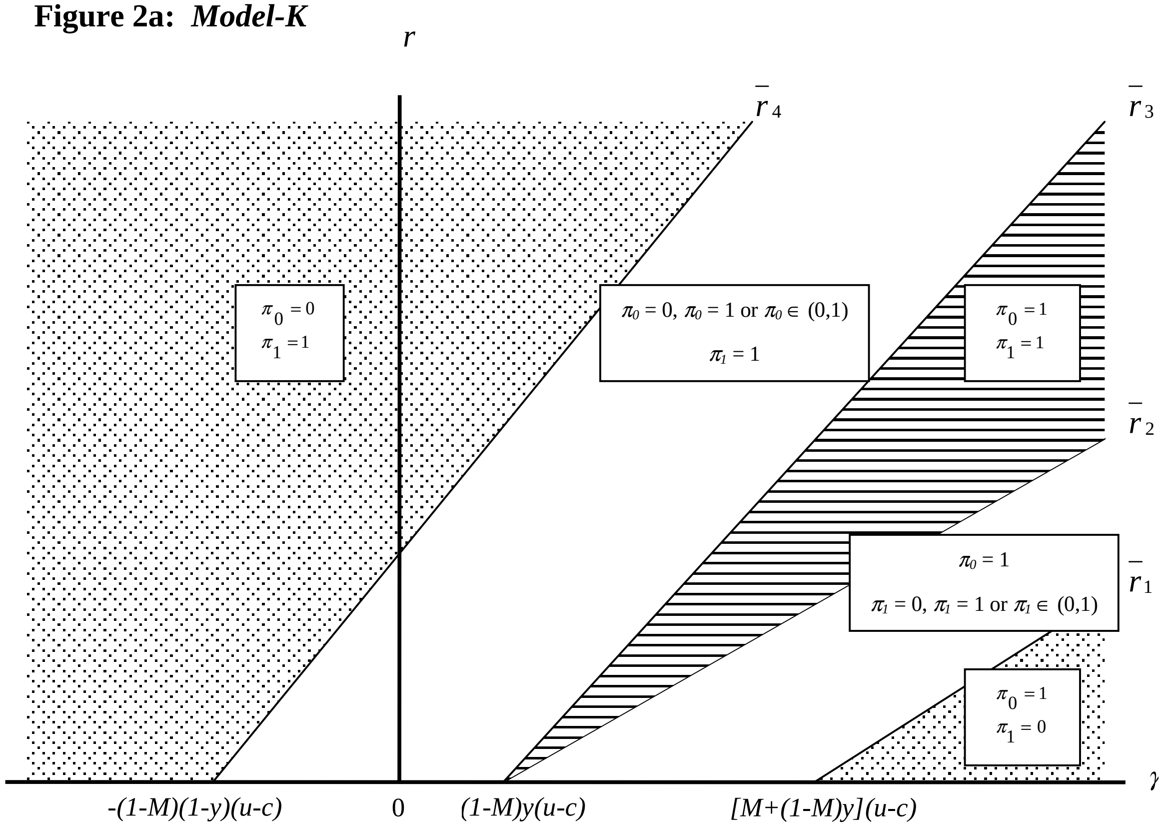

Figure 2b: ***Model-S***

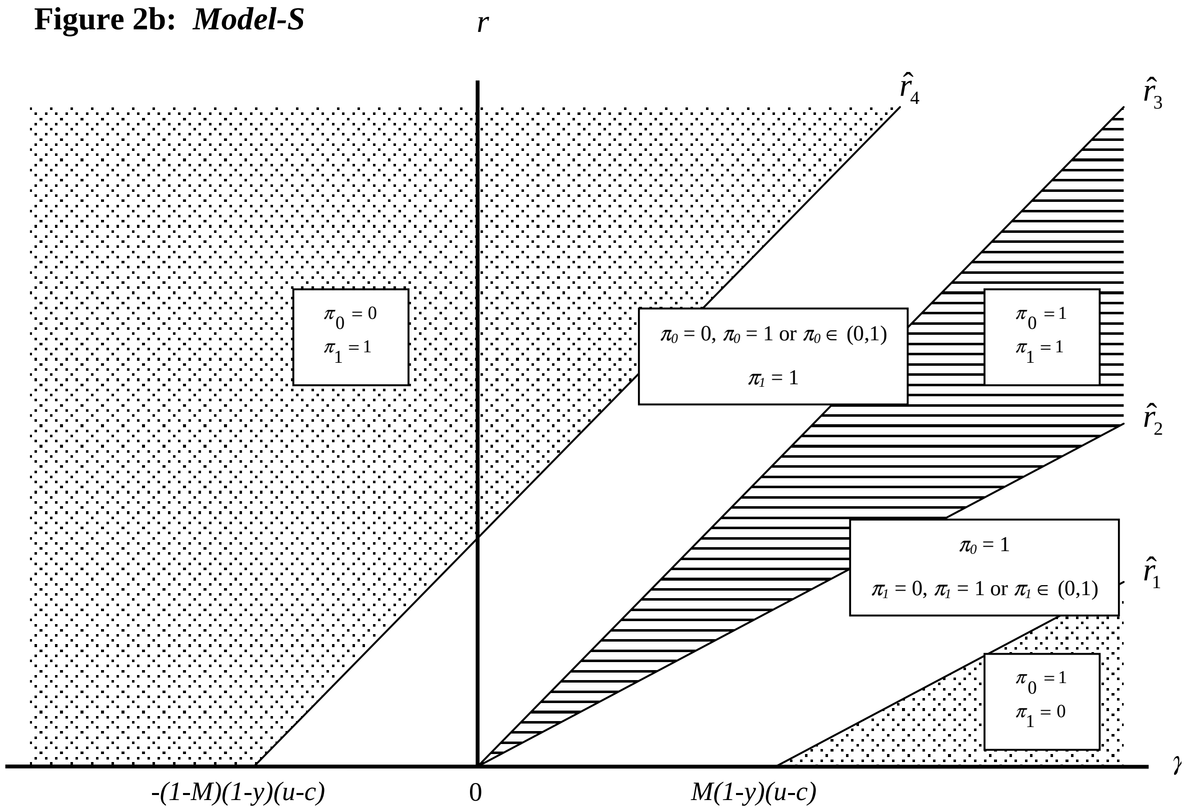

Figure 3a: Welfare as a function of *M*

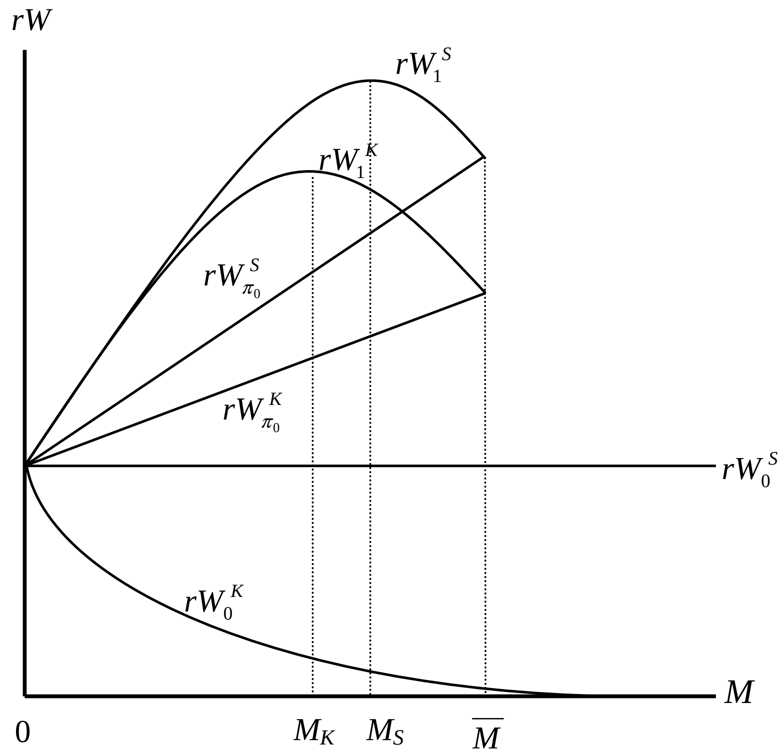

Figure 3b: Welfare as a function of *y* (optimal *M*)

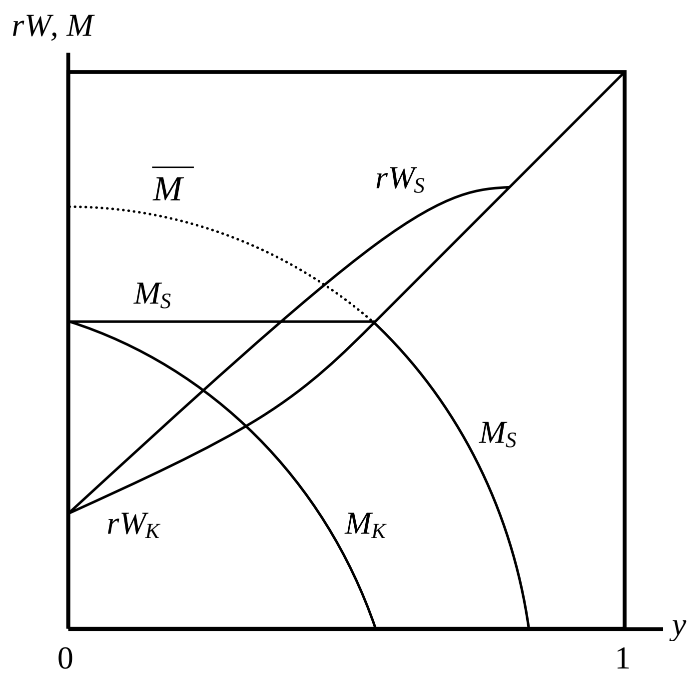

Figure 4a: Functions *e* and *f* in *Model-K* with $y > 0$, $T_j = V_j$

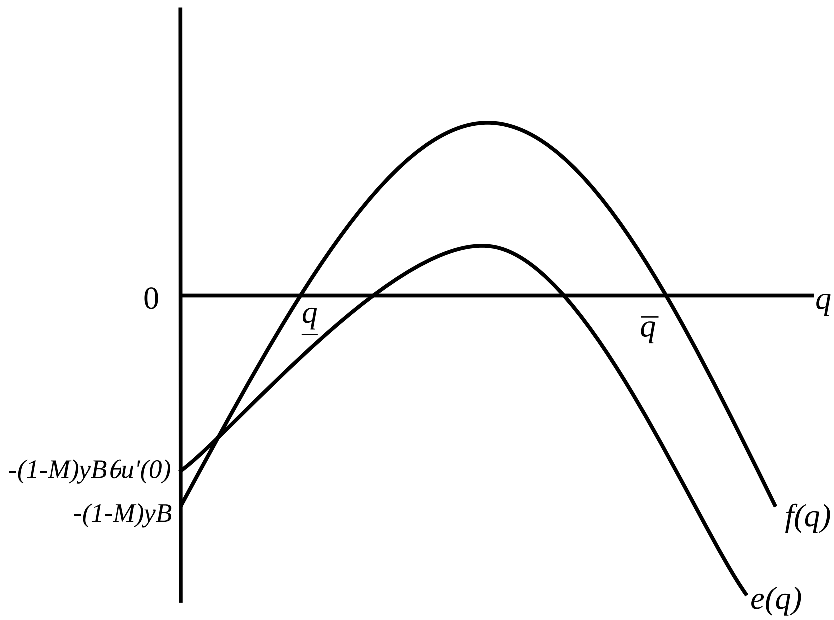

Figure 4b: Equilibria as a function of in *Model-K* with $y > 0$, $T_j = V_j$

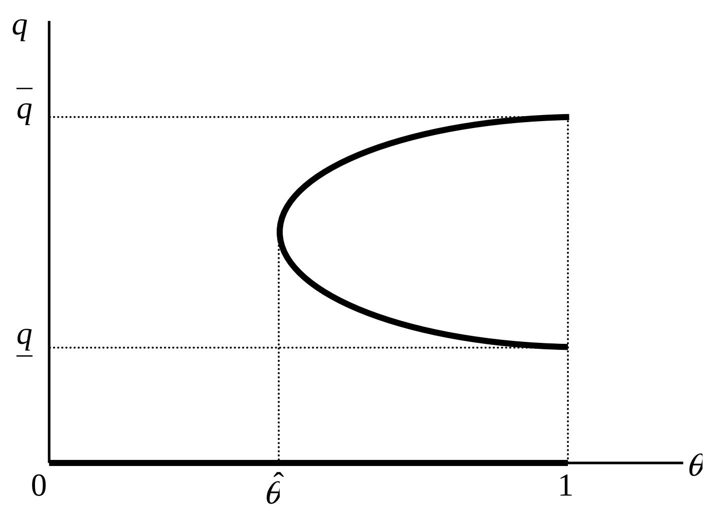

Figure 5a: Functions *e* and *f* in *Model-K* with $y > 0$, $T_j = 0$

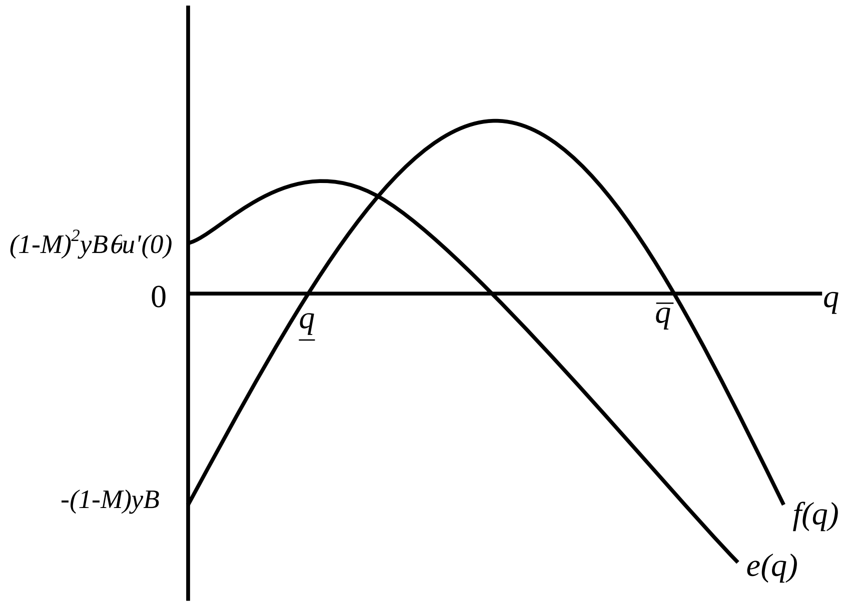

Figure 5b: Equilibria as a function of in *Model-K* with $y > 0$, $T_j = 0$

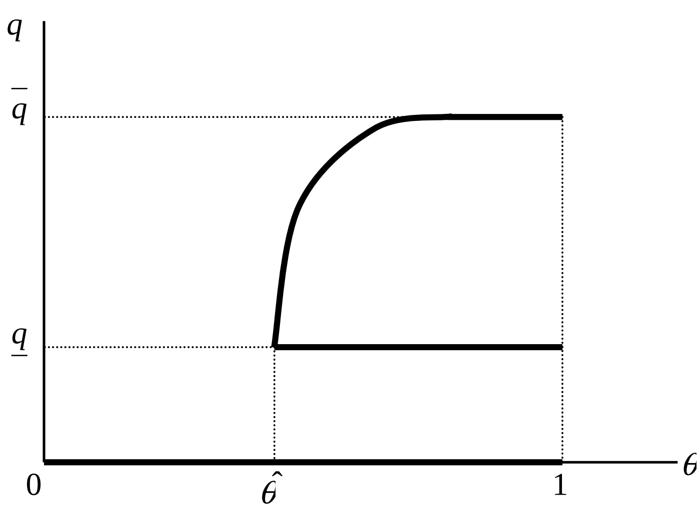

Figure 6a: Functions *e* and *f* in *Model-S* with $y > 0$, $T_j = V_j$

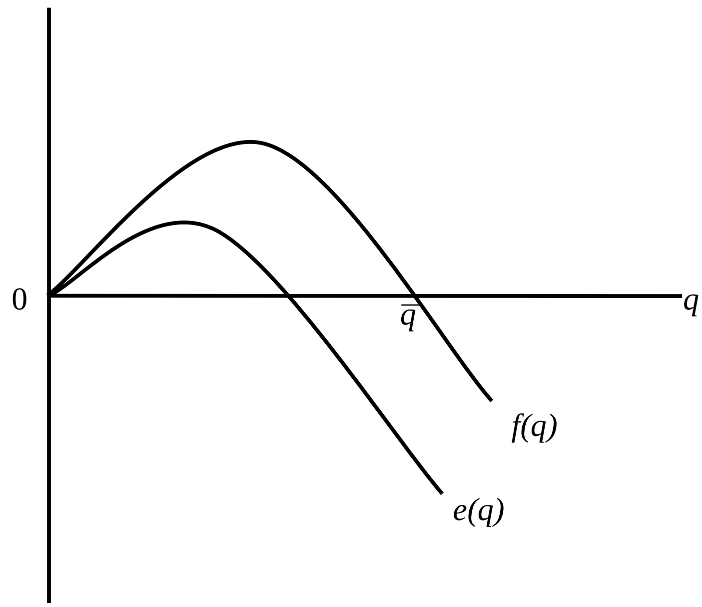

Figure 6b: Equilibria as a function of in *Model-S* with $y > 0$, $T_j = V_j$

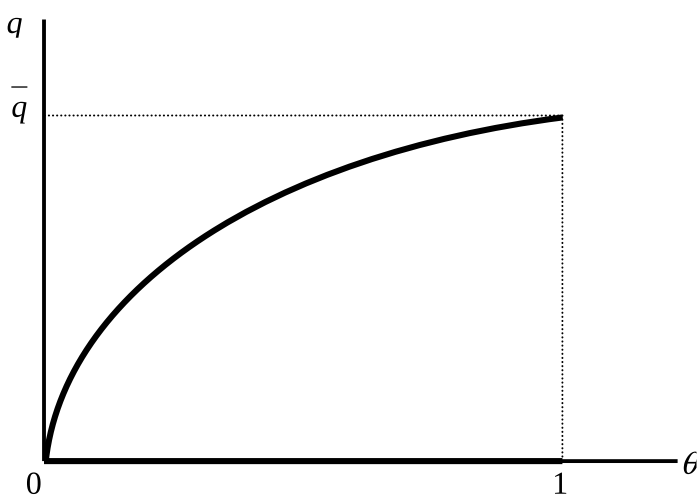

Figure 7a: Functions *e* and *f* in *Model-S* with $y > 0$, $T_j = 0$

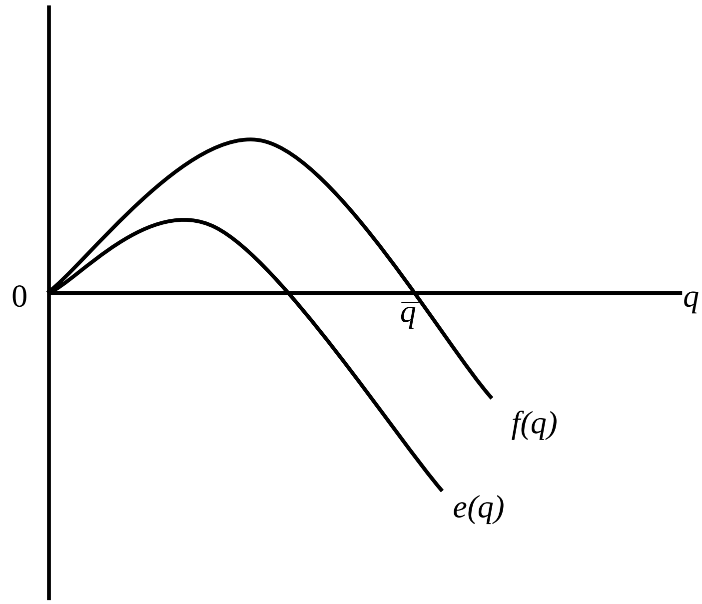

Figure 7b: Equilibria as a function of in *Model-S* with $y > 0$, $T_j = 0$

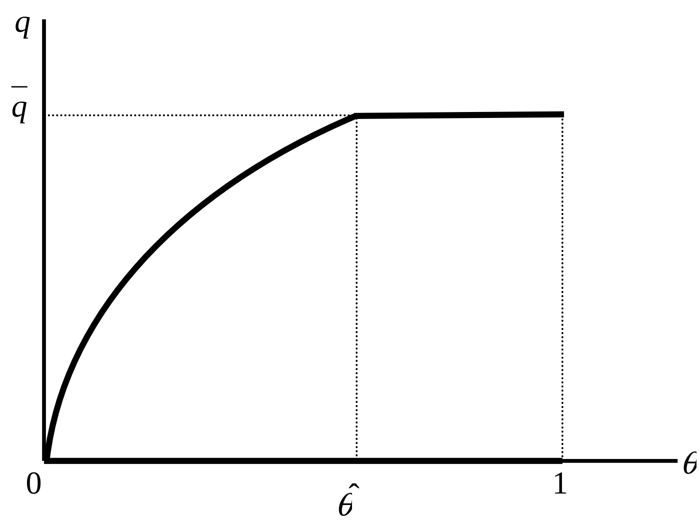

Figure 8a: Welfare as a function of *M* in *Model-S* with *y* > 0, $T_j = V_j$

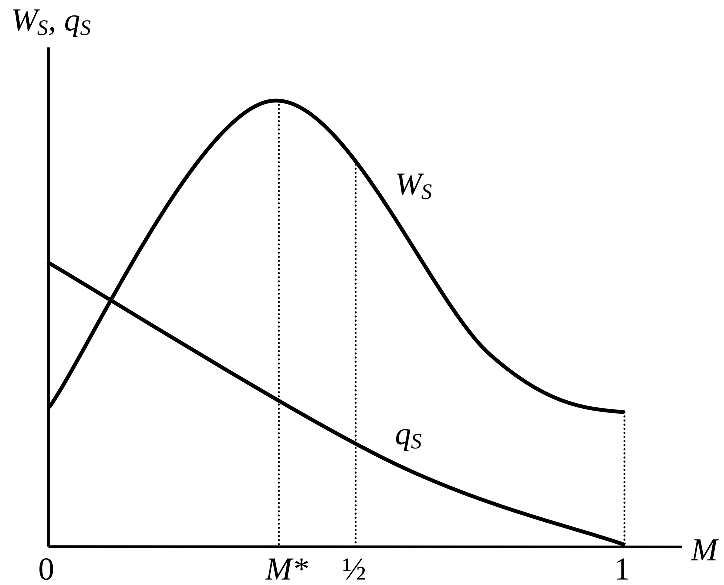

Figure 8b: Welfare as a function of *M* in *Model-S* with *y* > 0, $T_j = V_j$

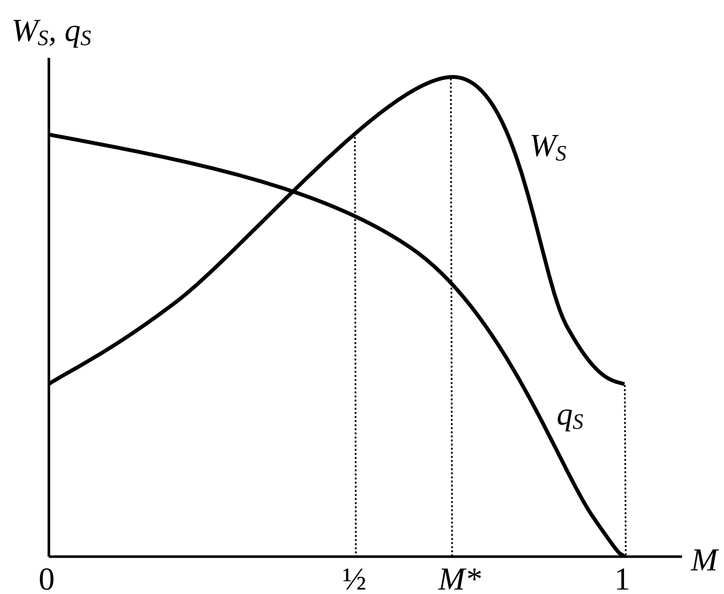

Figure 8c: Welfare as a function of *M* in *Model-S* with *y* > 0, T_j = 0

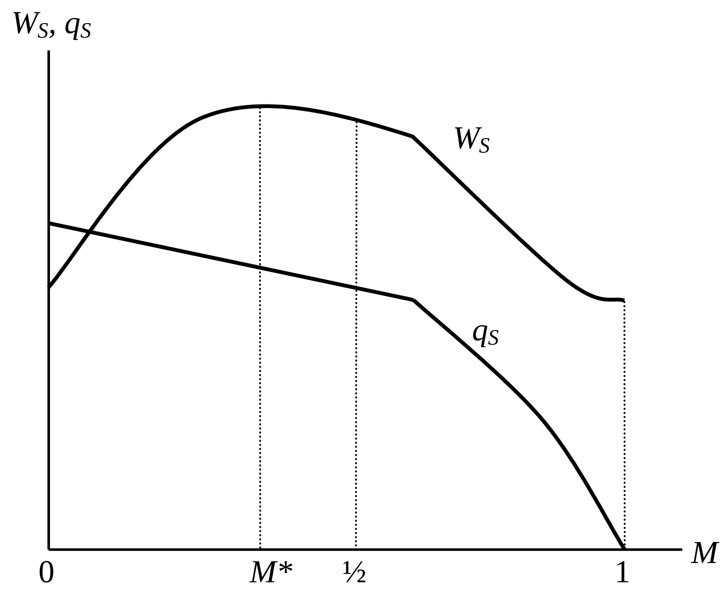

Figure 8d: Welfare as a function of *M* in *Model-S* with *y* > 0, T_j = 0

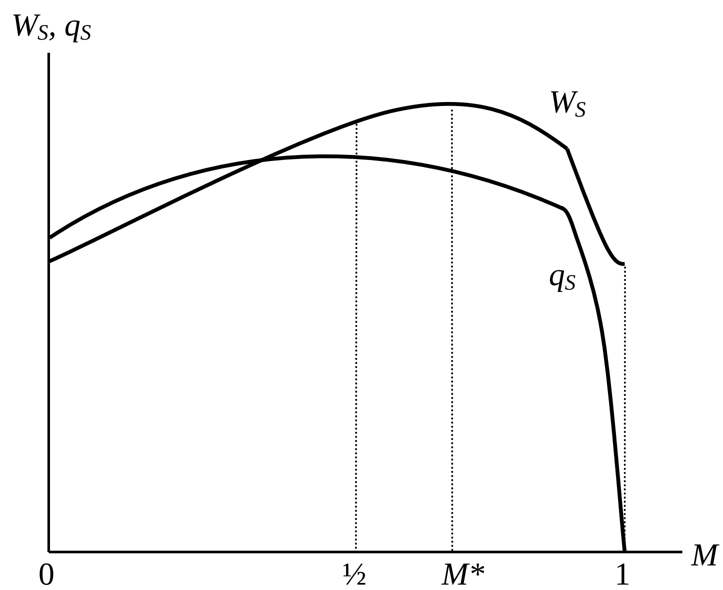

Figure 9a: Welfare as a function of *M* in *Model-S*, *Model-K* with $T_j = V_j$

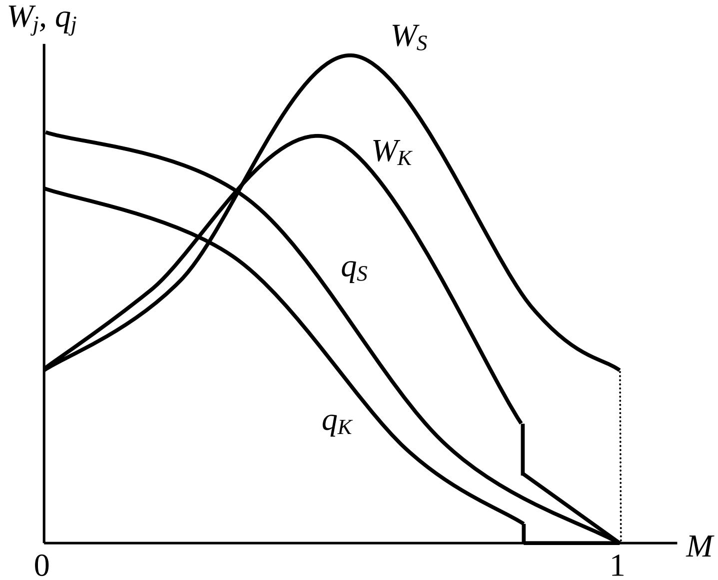

Figure 9b: Welfare as a function of *M* in *Model-S*, *Model-K* with T_j = 0
(θ = .5, r = .5, y = .1)