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# EXCLUSION IN ALL-PAY AUCTIONS

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## Abstract

In a recent paper, Baye, Kovenock, and de Vries (1993) derive a general formula for the seller's expected revenue in an all-pay auction, where the buyers' valuations are common knowledge. They use this formula to show that excluding the buyer with the highest valuation sometimes increases the seller's expected revenue. Lobbying may be modeled as an all-pay auction, and the exclusion result may therefore explain the practice of reducing fields of contestants vying for a political prize. One implication is that a politician may, perversely, exclude the individuals who place the highest value on a political prize.

The exclusion result depends critically on the assumption that the politician must award the prize to the highest bidder. We describe an alternate procedure that gives the politician more expected revenue. Although a politician adopting our procedure sometimes increases her expected revenue by excluding potential bidders, she should never exclude the buyer with the highest valuation.

In a recent paper, Baye, Kovenock, and de Vries (1993, henceforth BKV) derive a general formula for the seller's expected revenue in an all-pay auction, where the buyers' valuations are common knowledge. (In an all-pay auction, each bidder forfeits his bid, no matter who wins the auction.) They use this formula to show that excluding the buyer with the highest valuation sometimes increases the seller's expected revenue. The intuition is that if the highest valuation is much higher than the others, then excluding that buyer may produce more intense rivalry among the remaining bidders, which works strongly to the seller's advantage in an all-pay auction.

BKV also observe that politicians typically do not refund the contributions of lobbyists who fail to win a political prize, perhaps because giving refunds would suggest that the prize was sold. Lobbying may therefore be modeled as an all-pay auction, and BKV's exclusion result may explain the practice of reducing fields of contestants vying for a political prize. One implication is that a politician may, perversely, exclude the buyers who place the highest value on the prize.

In this paper, we argue that the exclusion result depends critically on BKV's assumption that the politician must award the prize to the highest bidder. We describe an alternate procedure that gives the politician more expected revenue and, by sometimes awarding the prize to someone other than the highest bidder, further obscures the reality of selling prizes. Although a politician adopting our procedure sometimes increases her expected revenue by excluding potential bidders, she should never exclude the buyer with the highest valuation.

BKV study the following complete-information game. A seller must grant an indivisible prize to one of N>2 buyers. BKV allow buyers to value the prize equally, but we confine attention to the plausible case in which the valuations can be strictly ranked, $v_1>v_2>v_3>...>v_N\geq0$. (Only if $v_1=v_2$ would our conclusions change substantially; we discuss this case below.) In the first stage of the game, the seller selects from among the N buyers a pool of $M \leq N$ *bidders*. We denote the valuations of the bidders $x_1 > x_2 > x_3 > \ldots > x_M$. In the second stage, the bidders simultaneously choose nonnegative bids and pay the amount bid to the seller. Finally, in the third stage, the seller awards the prize to one of the bidders, according to a rule announced in advance. We follow BKV in assuming that the seller is indifferent to who receives the prize, implying that any rule is credible. The payoff to the winning bidder is his valuation less his bid, while the payoff to a losing bidder is the negative of his bid. The payoff to the seller is the sum of the bids. Like BKV, we assume that the bidders play Nash equilibrium strategies in the second stage.

BKV assume that the third-stage rule is to award the prize to the highest bidder (breaking ties randomly). We call this the *high-bid rule*. Since the valuations are publicly known, the high-bid rule can induce only mixed strategy equilibria. BKV show that, in equilibrium, the seller's expected revenue equals $x_2$ if $x_1 = x_2$ and is strictly less than $x_2$ if $x_1 > x_2$. Given our assumption that $v_1 > v_2 > v_3$, it follows that the seller's revenue is always less than $v_2$ if she uses the high-bid rule.

An obvious way for the seller to earn more revenue is to adopt what we call the *rent-extraction rule*: if bidder 1 (the bidder having valuation $x_1$) bids at least $x_1 - \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small constant, then he receives the prize; otherwise, the seller awards the prize randomly to another bidder. Suppose that buyer 1 (the buyer having valuation $v_1$) is allowed to bid, so that $x_1 = v_1$. Then the rent-extraction rule induces a bid of $v_1 - \varepsilon$ from buyer 1 and zero bids from all other bidders, yielding a certain revenue that exceeds the expected revenue generated by the high-bid rule.

In the context of lobbying, an objection to the rent-extraction rule is that only one buyer bids, and that buyer always receives the prize. The bid thus has the conspicuous appearance of a bribe, much as if the seller had awarded the prize through a standard English auction. The high-bid rule provides a solution to this problem, but we study a different solution, which we call the $\Delta$-*rule.* It requires two or more bidders: if bidder 1's bid beats every other bid by at least $\Delta$, where $\Delta \in (0, x_1 - x_2)$ is announced in advance, then bidder 1 receives the prize; otherwise, the highest bidder other than bidder 1 receives the prize (again breaking ties randomly). We show below that if buyer 1 is allowed to bid, then for an appropriate choice of $\Delta$, this rule always generates expected revenue exceeding $v_2$. Since $v_2$ exceeds the expected revenue generated by the high-bid rule, regardless of who is allowed to bid, the optimal $\Delta$-rule is always better for the seller. Moreover, although it may be optimal to exclude some buyers from the bidding, it is never optimal to exclude buyer 1.

The first step toward these results is the following preliminary result.

*Proposition 1.* Bidder 1 bids at least $\Delta$ in any equilibrium supported by the $\Delta$-rule.

*Proof:* For any bidder j, a bid $b_j > x_j$ yields a negative surplus, so it is strictly dominated by a bid of zero. Therefore, in any equilibrium, the maximum bid from bidders 2 through N cannot exceed $x_2$, implying that bidder 1 can earn a positive surplus by bidding $b_1 \in (x_2 + \Delta, x_1)$. Because a bid $b_1 < \Delta$ cannot win, the result follows. ‖

BKV show that, given unequal valuations, the high-bid rule supports a unique equilibrium. In this equilibrium, bidder 1 bids with uniform density $1/x_2$ on the interval $(0, x_2]$, while bidder 2 bids zero with probability $(x_1 - x_2)/x_1$ and bids with uniform density $1/x_1$ on $(0, x_2]$. All other bidders bid zero. We next show that, under the $\Delta$-rule, the equilibrium is the same, except that $\Delta$ is added to bidder 1's bid. Hence, each buyer's probability of receiving the prize is unchanged, and $\Delta$ is, in effect, an unconditional extraction of rent from buyer 1.

*Proposition 2.* The $\Delta$-rule induces a unique equilibrium. Bidder 1's bid is distributed uniformly on the interval $(\Delta,\Delta+x_2]$. Bidder 2's bid is zero with probability $(x_1-x_2)/x_1$ and is distributed uniformly, with density $1/x_1$, on $(0,x_2]$. All other bidders bid zero.

*Proof*: It is straightforward to confirm that the prescribed strategies constitute an equilibrium. (Bidder 1 receives an expected surplus of $x_1-x_2-\Delta$ and bidder 2 receives an expected surplus of zero, from all bids in their respective supports.) Given our Proposition 1, uniqueness of the equilibrium follows from Lemma 1 of Hillman and Riley (1989), following the argument of BKV. ‖

Since bidder 1's expected bid is $\Delta+(x_2/2)$ and bidder 2's expected bid is $(x_2/x_1)(x_2/2)$, the seller's expected revenue in the equilibrium induced by the $\Delta$-rule is the sum $(x_1+x_2)x_2/2x_1 + \Delta$. Increasing $\Delta$ thus raises the seller's expected revenue. In the limit, as $\Delta$ approaches $x_1-x_2$, the seller's expected revenue approaches

$$R^*(x_1,x_2) \equiv x_2 + (x_1-x_2)(2x_1-x_2)/2x_1.$$

We focus on this case henceforth, and we refer to the limiting $\Delta$-rule as the *$\Delta^*$-rule.*

Given the high-bid rule, both BKV and Hillman and Riley (1989) find that the seller's expected revenue is less than $v_2$. Here, in contrast, allowing buyers 1 and 2 to bid ensures $R^*>x_2=v_2$.[1] The next proposition shows, moreover, that it may be possible to do even better. Although it never pays to exclude buyer 1, it may pay to exclude buyer 2.

*Proposition 3.* If the seller adopts the $\Delta^*$-rule, then the seller maximizes expected revenue by excluding no buyers if $v_1-v_2\leq v_N$ and excluding buyers 2,3,...,N-1 if $v_1-v_2>v_N$.

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[1] This note considers the case of buyers with equal valuations, which is excluded in the main text. Suppose that $v_1>v_2$ but $v_i=v_{i+1}$ for some $i>1$. BKV's analysis includes this case, so their conclusion that the seller's expected revenue from the high-bid rule is less than $v_2$ still applies. Under the $\Delta^*$-rule, allowing only buyers 1 and 2 to bid would eliminate the equal valuations and $R^*(v_1,v_2)>v_2$ would still apply. Hence, the $\Delta^*$-rule still earns more expected revenue than the high-bid rule. If $v_1=v_2$, then it is optimal under the high-bid rule to let buyers 1 and 2 bid, in which case the expected revenue is $v_2$, an outcome that cannot be improved upon. We rule out equal valuations mainly to ensure that $x_2>x_3$, which implies that the equilibrium in the second stage is unique.

*Proof:* The seller's expected revenue $R^*$ depends only on $x_1$ and $x_2$. Fixing $x_2$, $\partial R^*/\partial x_1 = 1 - (1/2)(x_2/x_1)^2 > 0$. Therefore, it is never optimal to exclude buyer 1, because it would increase revenue to substitute buyer 1 for whichever buyer, among the included buyers, has the highest valuation. Hence, let $x_1=v_1$. $R^*(v_1,x_2)$, as a function of $x_2$, is convex and quadratic, so it is symmetric about its minimum, which occurs at $x_2=v_1/2$. By excluding buyers, the seller has the option of setting $x_2=v_2,v_3,\ldots,v_N$. Symmetry implies that the revenue-maximizing choice is $x_2=v_2$ or $x_2=v_N$, whichever is further from $v_1/2$. The result follows. ‖

Proposition 3 implies that the best possible circumstance for the seller is the presence of a buyer N who places no value on the prize. Then, the seller should accept bids from only buyers 1 and N, and her expected revenue is $R^*(v_1,0)=v_1$. The $\Delta^*$-rule thus replicates the outcome of the rent-extraction rule, but (in this case) it shares the undesirable feature that buyer 1 is the only buyer who makes a payment, and he always wins the prize, giving the appearance of a bribe. A more elaborate analysis might incorporate in the seller's utility function some preference for rules that induce more apparent rivalry. Even if the seller chose to exclude no buyers, however, the $\Delta^*$-rule would still outperform the high-bid rule.

On balance, the $\Delta^*$-rule has several advantages over the high-bid rule assumed by BKV. It earns more expected revenue, and it gives a better appearance by sometimes awarding the prize to someone other than the highest bidder. It is hardly more complicated than the high-bid rule, nor would it be difficult to communicate to buyers. Given the seller's indifference to who receives the prize, the rules are equally credible. Finally, although optimal implementation of the high-bid rule typically requires knowledge of all buyers' valuations (to determine which buyers to exclude), knowledge of just the top two valuations ensures a better outcome under the $\Delta^*$-rule.

The $\Delta^*$-rule also does not have the property of the high-bid rule (which BKV emphasize) that it may be optimal to exclude the buyer who places the greatest value on the prize.

Some of our conclusions extend to environments of incomplete information. For instance, if buyers are symmetric ex ante and valuations are statistically independent, then the revenue equivalence results of Myerson (1981) and Riley and Samuelson (1981) hold, and the seller finds it optimal to include as many buyers as possible. If the buyers are asymmetric ex ante, then it may pay to put some buyers at a competitive disadvantage, *but not to exclude them.* Any information that would be useful for reducing the field can instead be incorporated in the rule for awarding the prize. One may therefore wish to look elsewhere to explain the practice of excluding buyers.

A different reason to exclude buyers might be found in information and computation costs. It may reduce the seller's cost of decision-making to reduce the field through a coarse first screening before making a final decision. Reducing the field may also lower the buyers' aggregate information expenditures, thereby reducing the costs borne by the seller. (With free entry of buyers, unrecoverable entry costs that are not paid to the seller are ultimately borne by her. For example, see French and McCormick [1984].) A thorough examination of these possibilities is beyond the scope of this note.

# REFERENCES


Baye, Michael, Dan Kovenock, and Casper de Vries (1993), "Rigging the Lobbying Process: An Application of the All-Pay Auction," *American Economic Review*, 83, 289-294.

Baye, Michael, Dan Kovenock, and Casper de Vries (1990), "The All-Pay Auction with Complete Information," mimeo.

French, Kenneth and Robert McCormick (1984), "Sealed Bids, Sunk Costs, and the Process of Competition," *Journal of Business*, 57, 417-441.

Hillman, Arye and John Riley (1989), "Politically Contestable Rents and Transfers," *Economics and Politics*, 1, 17-39.

Myerson, Roger (1981), "Optimal Auction Design," *Mathematics of Operations Research*, 6, 58-73.

Riley, John and William Samuelson (1981), "Optimal Auctions," *American Economic Review*, 71, 381-392.