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OPTIMAL BANK PORTFOLIO CHOICE UNDER FIXED-RATE DEPOSIT INSURANCE

by Anlong Li

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Abstract

This paper analyzes the optimal investment decisions of insured banks under fixed-rate deposit insurance. In the presence of charter value, trade-offs exist between preserving the charter and exploiting deposit insurance. Allowing banks to dynamically revise their asset portfolios has a significant impact on both the investment decisions and the fair cost of deposit insurance. The optimal bank portfolio problem can be solved analytically for constant charter value. The corresponding deposit insurance is shown to be a put option that matures sooner than the audit date. An efficient numerical procedure is also developed to handle more general situations.

1. Introduction

The current system of fixed-rate deposit insurance in the United States gives insured banks the incentive to take on riskier investments than they otherwise would. To relate the cost of deposit insurance to a bank's investment risk, Merton (1977) shows that deposit insurance grants a put option to the insured bank. Under this model, banks tend to take on extremely risky projects to exploit the put option. As a result, fixed-rate deposit insurance is apt to be underpriced for high-risk-taking banks and overpriced for low-risk-taking banks. Implementation of option models for valuing deposit insurance can be found in Marcus and Shaked (1984) and Ronn and Verma (1986).

In reality, not all banks take extreme risks. Being in business is a privilege and is reflected in a firm's charter value or growth option. Extreme risk-taking may lead a bank into insolvency, forcing it out of business by regulators. The charter value comes from many sources, such as monopoly rents in issuing deposits, economies of scale, superior information in the financial markets, and reputation.

Taking into account the charter value, Marcus (1984) shows that banks either minimize or maximize their risk exposure as a result of the trade-offs between the put option value and the charter value. Under a different setting, Buser, Chen, and Kane (1981) show that the trade-offs reestablish an interior solution to the capital structure decision. They also argue that capital requirements and other regulations serve as additional implicit constraints to discourage extreme risk-taking.

Almost all models of deposit insurance assume that banks' asset risk is exogenously given. With the exception of the discussion in Ritchken et al. (1991), the flexibility for banks to dynamically adjust their investment decisions has been mostly ignored. However, their model allows only a finite number of portfolio revisions between audits.

In this paper, I establish a continuous-trading model to identify how an equity-maximizing bank dynamically responds to flat-rate deposit insurance schemes and how this affects the actuarially fair value of deposit insurance. Since investment decisions are carried out by optimizing the investment portfolio, I model the problem as the optimal control of a diffusion process. Upon obtaining the optimal portfolio, the actuarially fair cost of insurance can be easily calculated.

In this model, I use the traditional dynamic programming approach (Fleming and Rishel [1975]). The disadvantage of this approach is that it often reduces the problem to an intractable partial differential equation (PDE) where analytical solutions are rare. Merton's (1971) application to the optimal consumption problem is among the few cases in which analytical solutions are obtained. Fortunately, in this problem the resulting PDE can be explicitly solved provided that the charter value is constant. Even though I assume lognormal price to warrant an analytical solution, general price distributions can be easily built into the model.

The dynamic programming procedure can also be carried out numerically by lattice approximation. This is especially attractive when more realistic assumptions are made. As the bank changes its portfolio risk over time, the most common binomial model is no longer path-independent, and the problem size grows exponentially with the number of partitions. This difficulty is resolved by using a trinomial lattice. The lattice is set up in such a way that the decision variable is incorporated into the transition probabilities rather than into the step size.

This paper is organized as follows: Section 2 formulates the model and summarizes the results under no portfolio revision. Section 3 solves the optimal portfolio problem under continuous portfolio revision. The value of deposit insurance is derived based on the optimal portfolio decisions. Section 4 presents the trinomial approximation of controlled diffusion process. Section 5 extends the model to more general situations, and section 6 concludes the paper. The proof of the main results can be found in the appendix.

2. The Static Model - No Portfolio Revision

Investment Opportunities: Assume that financial markets are complete. The bank can invest in both riskless bonds (earning rate r) and a portfolio of risky securities that follows a geometric Wiener process

$$dS/S = \mu dt + \sigma dW. \tag{1}$$

Capital and Liability: The bank's initial asset $X(0)$ consists of capital $K(0)$ and deposit base $D(0)$. For simplicity, I assume no net external cash inflows into the deposit base, no capital injections, and no dividend payments during the time interval $[0,T]$. Because all deposits are insured, I assume that deposits earn the riskless rate r. Let $L(t)$ be the liability at time t; then

$$L(t) = L(0)e^{rt}. \tag{2}$$

Investment Decisions: Management decides at time zero to put a fraction q of its assets in risky securities and the remaining in riskless bonds. Without portfolio revision, q is fixed before the audit.
The market value of the assets at time t is

$$X(t) = qX(0)e^{\mu t-\sigma^2 t/2+\sigma\sqrt{t}\xi} + (1-q)X(0)e^{rt}, \tag{3}$$

where ξ is the standard normal random variable with density and distribution function n() and N(), respectively.

Auditing and Closure Rules: The regulator conducts an audit at time T. If the bank is solvent, i.e., the market value of its assets exceeds its liabilities, it claims the residual $X(T) - L(T)$ and keeps its charter. If the bank is insolvent, the regulator takes over and equityholders receive nothing. Let $G(T)$ represent the charter value of a solvent bank at time T. $G(T)$ is assumed to be a constant fraction of total liabilities. Define

$$G(t) = fL(t), \qquad 0 < f < 1. \tag{4}$$

Let $V(t;q)$ be the equity value at time t under policy q. Then

$$V(T,q) = \begin{cases} X(T) - L(T) + G(T) & \text{if } X(T) > L(T) \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The equity value at time 0 can be obtained by using standard option pricing techniques,

$$V(0,q) = \begin{cases} qX(0)N(d_1)-[L(0)-G(0)-(1-q)X(0)]N(d_2) & \text{if } (1-q)X(0)<L(0) \\ X(0)-L(0)+G(0) & \text{otherwise,} \end{cases} \tag{6}$$

where

$$d_1 = \frac{ln\{qX(0)/[L(0)-(1-q)X(0)\} + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}.$$

On behalf of the shareholders, management will maximize the equity value by choosing the optimal fraction q^* such that

$$V(0,q^*) = \max_q \{ V(0,q) \}. \tag{7}$$

This optimization problem can be solved analytically. Solvent and insolvent banks are treated separately. Even though an initially insolvent bank would be an unusual case, it is included to complete the analysis. I summarize these results in theorems 1 and 2.

Theorem 1. For an insolvent bank without portfolio revisions, $q^* = 1$ is optimal. Consequently, the value of the deposit insurance[1] is

$$I(0) = -X(0)N(-d_1^1) + L(0)N(-d_2^1), \tag{8}$$

where

$$d_1^1 = \frac{ln[X(0)/L(0)] + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2^1 = d_1 - \sigma\sqrt{T}.$$

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[1] The value of deposit insurance always refers to the actuarially fair cost of deposit insurance.

Theorem 2. For a solvent bank without portfolio revision, the optimal policy is

$$q^* = \begin{cases} 1 & \text{if } f < 1-\mathrm{H}(m) \\ 0 & \text{if } f \geq 1-\mathrm{H}(m), \end{cases} \tag{9}$$

where

$$m = [X(0) - L(0)]/L(0)$$

$$\mathrm{H}(m) = (1+m)\mathrm{N}(-d_1^1)/\mathrm{N}(-d_2^1).$$

Consequently, the value of deposit insurance is

$$\mathrm{I}(0) = \begin{cases} -X(0)\mathrm{N}(-d_1^1) + L(0)\mathrm{N}(-d_2^1) & \text{if } f < 1-\mathrm{H}(m) \\ 0 & \text{if } f \geq 1-\mathrm{H}(m). \end{cases} \tag{10}$$

Theorems 1 and 2 show that without revision opportunities between audits, banks always take extreme positions. Regardless of the charter value, an insolvent bank always takes the riskiest position. With a small charter value, a solvent bank may be better off by taking the riskiest position so as to maximize the value of the deposit insurance. Only solvent banks with a sufficiently large capital-deposit ratio m or a relatively high charter value will invest in riskless bonds.[2]

The value of insurance for an insolvent bank, or for a solvent bank with $f < 1 - \mathrm{H}(m)$, is the same as in Merton (1977) where the charter value is zero. When $f \geq 1 - \mathrm{H}(m)$, risk-taking is discouraged and the insurance has no intrinsic value.[3]

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[2] This can be shown from the fact that $\mathrm{H}(m)$ is an increasing function of m with $\mathrm{H}(-1) = 0$ and $\mathrm{H}(\infty) = 1$.

[3] To be precise, when $f = 1 - \mathrm{H}(m)$, a bank is indifferent between $q = 0$ (preserving the charter) and $q = 1$ (exploiting the insurance). However, the bank's actual decision on q does affect the value of insurance. This discontinuity in the insurance value is one of the drawbacks of

3. Continuous Portfolio Revision

In this section, I assume that banks can revise their investment portfolios continuously over time at no cost. Let $X(t)$ be the market value of the assets and $q = q(t,X(t))$ be the fraction of risky assets in the portfolio at time $t \in [0,T]$. Then $X(t)$ follows a diffusion process

$$dX(t) = [q\mu + (1-q)r]X(t)dt + q\sigma X(t)dW(t), \qquad X(0) = X_0, \tag{11}$$

where $W(t)$ is a standard Brownian motion. The liability and charter value are given by equations (2) and (4), respectively. For valuation purposes, one can substitute μ with r in equation (11). Let $J(t,X(t))$ be the maximum equity value of the bank at time t. Then

$$J(t,X(t)) = \max_q E_t[J(T,X_T)e^{-r(T-t)}]. \tag{12}$$

It has the boundary condition

$$J(T,X(T)) = \begin{cases} X(T) - L(T) + G(T) & \text{if } X(T) \geq L(T) \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

We are interested in the maximum equity value $J(0,X(0))$ for any given $X(0) = X_0$ at time zero and the corresponding optimal policy $q^*(t)$ for all $t \in [0,T]$. This problem is solved by using dynamic programming. The results are presented in the following theorem.

Theorem 3. Let τ be the solution of the following equation[4]

$$\frac{G(T)}{L(T)}\left[N(\sigma\sqrt{T-\tau}/2) + \frac{2n(\sigma\sqrt{T-\tau}/2)}{\sigma\sqrt{T-\tau}}\right] = 1. \tag{14}$$

Suppose the asset value at time t is $X(t)$. Under the assumptions of section 2 and continuous portfolio revision, the optimal decision $q^*(t)$ and the corresponding equity value $J(t,X(t))$ are as follows.

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static models.

[4] If the solution is negative, simply let $\tau = 0$.

(1) If $t \in [\tau,T)$ and $X(t) \geq L(t)$, then $q^*(t) = 0$, and

$$J(t,X(t)) = X(t) - [L(t)-G(t)]. \tag{15}$$

(2) If $t \in [\tau,T)$ and $X(t) < L(t)$, then $q^*(t) = 1$, and

$$J(t,X(t)) = \frac{G(t)}{L(t)} [X(t)N(\gamma_1) + L(t)N(\gamma_2)] \tag{16}$$

where

$$\gamma_1 = \frac{ln[X(t)/L(t)] + \sigma^2(T-t)/2}{\sigma\sqrt{T-t}}$$

$$\gamma_2 = \gamma_1 - \sigma\sqrt{T-t}.$$

(3) If $t \in [0,\tau)$, then $q^*(t) = 1$, and

$$J(t,X(t)) = \frac{G(t)}{L(t)} [X(t)N(\gamma_1,-\gamma_3,\rho) + L(t)N(\gamma_2,-\gamma_4,\rho)] + X(t)N(\gamma_3) - [L(t)-G(t)]N(\gamma_4) \tag{17}$$

where

$$\gamma_3 = \frac{ln[X(t)/L(t)] + \sigma^2(\tau-t)/2}{\sigma\sqrt{\tau-t}}$$

$$\gamma_4 = \gamma_3 - \sigma\sqrt{\tau-t}$$

$$\rho = \sqrt{\tau-t}/\sqrt{T-t}$$

and $N(x,y,\rho)$ is the standard cumulative bivariate normal distribution with correlation coefficient ρ.

In summary, the optimal policy is[5]

$$q^* = \begin{cases} 0 & \text{if } t \in [\tau,T) \text{ and } X(t) \geq L(t) \\ 1 & \text{if } t \in [0,\tau) \text{ or } X(t) < L(t). \end{cases}$$

Theorem 3 clearly illustrates the trade-offs between preserving the

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[5] Actually, when $t \in [\tau,T)$ and $X(t) > L(t)$, any q is optimal as long as q is set at 0 when $X(t)$ hits the solvency curve $L(t)$.

charter and exploiting the deposit insurance. The deposit insurance is essentially a put option on the bank's assets that matures at the time of the audit. The longer the time before an audit, the higher the value of the deposit insurance. Prior to time τ, the deposit insurance is more valuable than the fixed charter value, and shareholders exploit the deposit insurance by choosing $q = 1$. After time τ, since the audit is near, the deposit insurance is less valuable than the charter, and shareholders will do their best to ensure that the market value of the bank's assets remains above the solvency curve $L(t)$ in order to preserve its charter. Figure 1 shows this optimal policy where the riskless rate is set to zero.

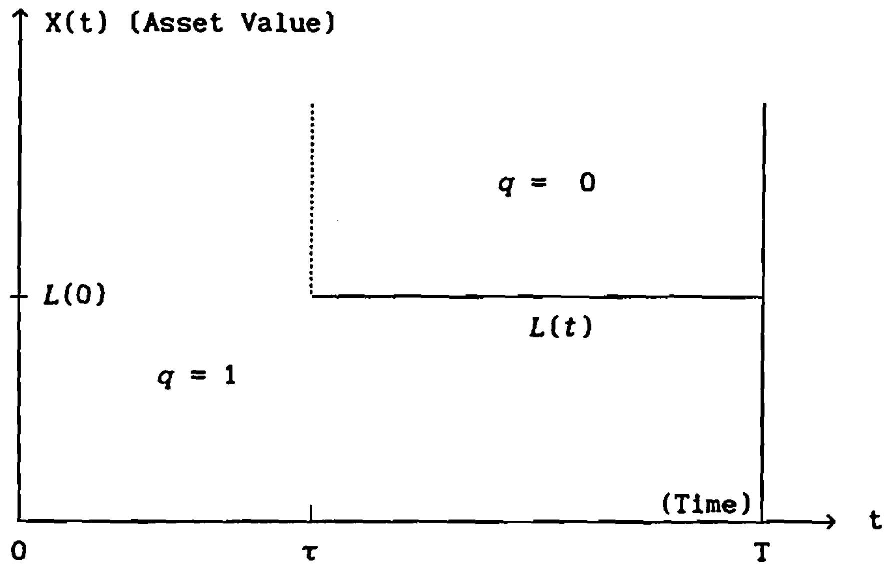


Figure 1. Optimal Portfolio Policies

The critical time τ is uniquely determined by equation (14) for any $0 \leq f \leq 1$. To see this, rewrite equation (14) with $\beta = \sigma\sqrt{T-\tau}/2$:

$$N(\beta) + \frac{e^{-\beta^2/2}}{\sqrt{2\pi}\beta} = 1/f. \tag{18}$$

Since the left-hand side of (18) decreases from $+\infty$ to 1 as β goes from 0 to $+\infty$, a positive β is uniquely determined. We can also show that τ is increasing in σ and decreasing in f. However, it depends on neither the the riskless interest rate r nor the banks' capital-deposit ratio m.

As an example, consider an audit period of one year. Suppose the volatility of the risky assets is σ = 10 percent annually, and the charter value is f = 10 percent of the deposit base. Solving equation (14) yields τ = 0.293. If f drops to 5 percent, τ will increase to 0.834. If there is no charter at all, τ equals T, the audit date.

To obtain the value of the deposit insurance $I(0)$, note that the equity value comes from three sources: namely, the initial capital $K(0)$, the deposit insurance $I(0)$, and the charter value $G(0)$. That is,

$$J(0,X(0)) = K(0) + I(0) + G(0)P\{X(T)\geq L(T)\}, \tag{19}$$

where $P\{X(T)\geq L(T)\}$ is the probability that the bank passes the audit. Following the same argument as in the proof of theorem 3, we have

$$P\{X(T)\geq L(T)\} = \begin{cases} 1 & \text{if } \tau=0,\ X(0)\geq L(0) \\ \frac{X(0)}{L(0)}\,N(\gamma_1) + N(\gamma_2) & \text{if } \tau=0,\ X(0)<L(0) \\ N(\gamma_4) + \frac{X(0)}{L(0)}\,N(\gamma_1,-\gamma_3,\rho) + N(\gamma_2,-\gamma_4,\rho) & \text{if } \tau>0, \end{cases}$$

where the γ's are evaluated at time t = 0. Substituting this into equation (19), we have the actuarially fair value of deposit insurance for a bank with continuous revision opportunities

$$I(0) = \begin{cases} 0 & \text{if } \tau=0,\ X(0)\geq L(0) \\ L(0) - X(0) & \text{if } \tau=0,\ X(0)\leq L(0) \\ -X(0)N(-\gamma_3) + L(0)N(-\gamma_4) & \text{if } \tau>0, \end{cases} \tag{20}$$

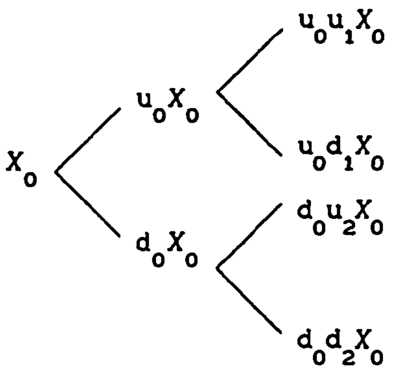
where γ_3 and γ_4 are evaluated at time t = 0.

This insurance value can be viewed as a put option on the bank's assets with maturity τ instead of T. This clearly explains the impact of the charter value and the continuous portfolio revision on the value of deposit insurance. Since $\tau < T$ as long as $f > 0$, the deposit insurance is less valuable in the presence of charter value. Compared to the static model, the insurance value in equation (20) is continuous in terms of charter value and capital-asset ratio. Even for very highly capitalized banks, as long as $\tau > 0$, the insurance has a positive value.

4. Trinomial Approximation

For general terminal payoff functions other than the one in equation (5), analytical solutions may not always exist, and numerical procedures must be used to solve the optimal portfolio problem. Without portfolio revisions, a simple binomial model can be used to approximate the bank's asset value. However, when the portfolio is revised, the resulting lattice becomes path-dependent.

To see this, partition the audit period $[0,T]$ into n subintervals of equal length $h = T/n$. The asset portfolio may be revised at discrete decision points $t_i = ih$, $i = 0,1,\dots,n-1$. Let $q(t_i, X(t_i))$ be the revised fraction of risky investments at time t_i if the market value of the bank's assets is $X(t_i)$. Let q be initially set to q_0. The portfolio is revised at time t_1 by changing q_0 to q_1 at the up state and q_2 at the down state, respectively. The two-period binomial lattice looks like



where

$$u_i = (1-q_i)e^{rh} + q_i e^{\sigma\sqrt{h}}$$
$$d_i = (1-q_i)e^{rh} + q_i e^{-\sigma\sqrt{h}}$$

for $i = 0,1,2$. Obviously, if $u_0d_1 \neq d_0u_2$, the lattice is path-dependent.

To overcome this difficulty, a path-independent lattice is first set up as if there is no portfolio revision. Then, when the portfolio is revised to a new q value at a revision point, one changes only the transition probabilities such that the drift and variance terms match locally. This suggests adding one more degree of freedom to the lattice. Consider the following trinomial lattice when the asset value at time t_i is X_i:

$$X_i \begin{array}{l} \overset{p_1}{\nearrow} \; X_i \exp\{rh+\sigma\sqrt{h}\} \\ \overset{p_0}{\longrightarrow} \; X_i \exp\{rh\} \\ \overset{p_{-1}}{\searrow} \; X_i \exp\{rh-\sigma\sqrt{h}\}. \end{array} \tag{21a}$$

The transition probabilities are set to

$$p_1 = \frac{q^2}{2}\left(1 - \frac{\sigma\sqrt{h}}{2}\right) \tag{21b}$$

$$p_0 = 1-q^2 \tag{21c}$$

$$p_{-1} = \frac{q^2}{2}\left(1 + \frac{\sigma\sqrt{h}}{2}\right). \tag{21d}$$

Obviously, $\sum_j p_j = 1$. The first and second local moments are

$$\begin{aligned} \mu_h(X_i, t_i) &= \sum_j p_j[\exp\{rh+j\sigma\sqrt{h}\}-1]X_i/h \\ &= rX_i + O(\sqrt{h}), \end{aligned} \tag{22}$$

$$\begin{aligned} \sigma_h^2(X_i, t_i) &= \sum_j p_j[X_i\exp\{rh+j\sigma\sqrt{h}\}-X_i]^2/h \\ &= (q\sigma X_i)^2 + O(\sqrt{h}). \end{aligned} \tag{23}$$

As $h \longrightarrow 0$, these moments converge to the true mean and variance of the diffusion process $X(t)$ in equation (11). This ensures that the trinomial process converges to the process $X(t)$ in distribution.

To find the optimal policy q^*, a dynamic programming procedure can be applied to the trinomial lattice. At the very end-nodes, payoff values are given. Working backward, at any node X_i, an optimal policy $q_i^*(h)$ and equity value can be easily obtained. Under certain smoothness conditions on the payoff function, as $h \longrightarrow 0$, $q_i^*(h)$ will converge to the optimal policy q^*. The optimal policy of theorem 3 can be easily confirmed using this procedure.

5. A Second Look at the Charter Value

In sections 2 and 3, we adopted Marcus's (1984) specification of the charter value. The bank either retains or loses the full charter value depending on whether or not it is solvent at the audit date. This corresponds to the terminal payoff curve OBCD in figure 2. However, despite its simplicity, this specification is far from realistic.

For example, regulators may, for economic or political reasons, choose to inject additional funds into a slightly insolvent bank rather than simply to close it. Thus, the payoff curve OBCD in figure 2 should stretch farther to the left. As for the equityholders, if the market value of the bank's assets is below the liability value just before the audit, it would be to the bank's advantage to inject additional funds in order to preserve the charter. It may do so as long as the charter value exceeds the liability minus asset value. This suggests the payoff curve OAD of a call option with strike price $L(T) - G(T)$. In this case, the charter can be viewed as part of the bank's tangible assets.

However, when a bank is close to insolvency, it may face financial distress or bankruptcy costs, which would decrease the charter value. Usually the charter value depends not only on the size of the deposit base, but also on the soundness of the bank (such as the capital-deposit ratio). When this ratio drops below a certain level, a regulatory tax is likely to be charged (Buser, Chen, and Kane [1981]). Therefore, a more reasonable payoff function would be somewhat like the OEFD curve in figure 2. For a highly capitalized bank, the charter value is proportional to the deposit base (the F-D segment). As the bank lowers its capital, the charter-deposit ratio decreases (the E-F segment). If the capital is too low, the charter value is zero (the O-E segment).

After the payoff curve is specified, we can use the trinomial approximation of section 4 to calculate the present value of bank equity and the actuarially fair price of deposit insurance. For demonstration purposes, suppose the payoff curve has the following form:

$$V(T) = \begin{cases} \{1-\exp[-\delta(\frac{K(T)}{fL(T)})^{\alpha}]\}K(T) & \text{if } K(T) \geq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{24}$$

where $K(T) = X(T) - (1-f)L(T)$, $\alpha, \delta \geq 0$.

This payoff function contains many interesting special cases. When $f = 0$, it reduces to the case of Merton (1977). When $\delta = +\infty$ and $\alpha < +\infty$, it reduces the OAD curve in figure 3 where an insolvent bank can inject additional funds at no extra cost in order to retain its charter. When $\alpha = +\infty$ and $\delta < +\infty$, it reduces to that of Marcus (1984), which corresponds to the OBCD payoff curve in figure 2.

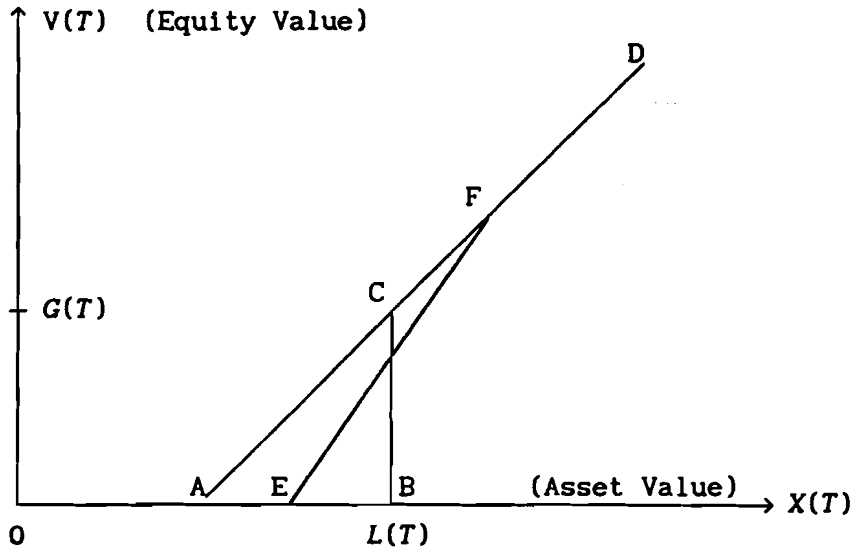


Figure 2. Alternative Payoff Functions

Figure 3 shows the payoff function (24) for α = 1, 2, 4 and ∞, while $\delta = 1$. The corresponding optimal policies are shown in figure 4, where the other parameters are $T = 1$, r = 0, $X_0 = L_0 = 100$, $\sigma = 0.1$, and $f = 0.05$. All of the optimal policies are similar to the one in theorem 3. Banks initially choose $q = 1$. After a critical time τ, there is a critical curve $K(t)$. If asset value $X(t)$ is above $K(t)$, $q = 0$ is optimal; otherwise $q = 1$ is optimal. In contrast to theorem 3, the critical curve $K(t)$ is no longer a straight line. It is interesting to note that the larger the value of α, the larger the critical time τ, because the charter value erodes faster as α increases.

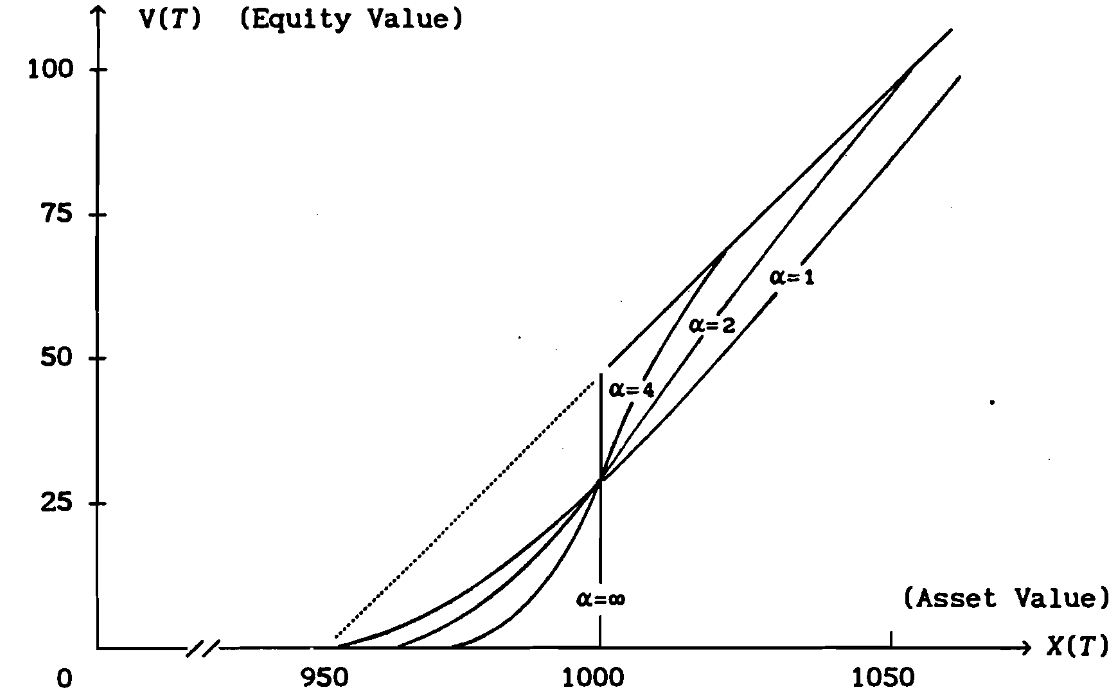


Figure 3. Some Specific Payoff Functions

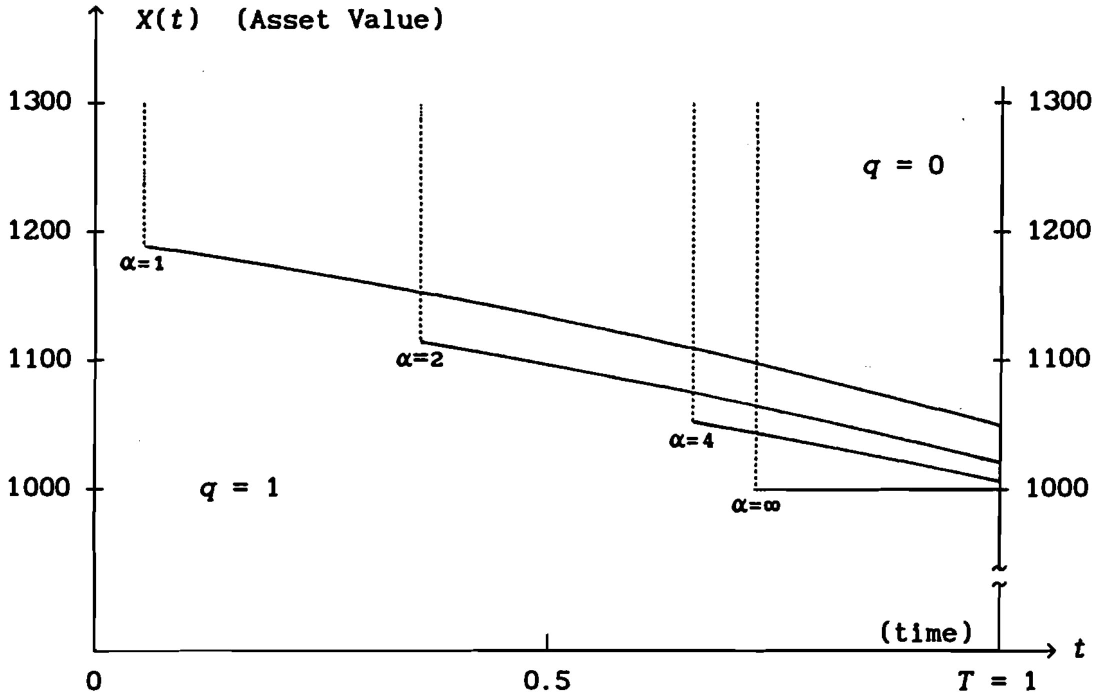


Figure 4. Optimal Policies Under the Payoff Functions in Figure 3

6. Conclusion

This paper develops a stochastic control model to analyze the investment decisions of a bank whose deposits are fully insured under a fixed-rate insurance premium. I show how banks dynamically adjust their investment portfolios in response to market information and how this flexibility affects both investment decisions and the value of deposit insurance. The optimal portfolio problem is solved analytically assuming lognormal asset price and constant charter value. For general payoff patterns, an efficient numerical procedure is presented.

Under continuous portfolio revision I show that, before some critical time τ, the bank always takes the riskiest position regardless of its solvency situation. The bank may act cautiously only between time τ and the audit date T. The value of deposit insurance remains a put option, but with maturity τ instead of T. This critical time τ depends on the charter value, on the volatility of the risky assets, and on the time between audits. This gives the regulators some guidelines, at least in theory, on the timing of audits.

The major limitation of this model is the empirical difficulty in specifying the charter value. This is further complicated by other factors such as transaction costs, asymmetric information, reputation, and economic conditions.

Appendix

Proof of Theorem 1. Since $X(0) < L(0)$, from equation (6) we have

$$\partial V/\partial q = X(0)[\mathrm{N}(d_1)-\mathrm{N}(d_2)] + \frac{X(0)G(0)[L(0)-X(0)]}{[L(0)-(1-q)X(0)]^2}\frac{1}{\sigma\sqrt{T}} > 0.$$

The equity value V is increasing in q; $q^* = 1$ is optimal. Q.E.D.

Proof of Theorem 2. For a solvent bank, when $q \le q_{\min} \equiv 1 - L(0)/X(0)$, the riskless bonds alone will be enough to pay off the obligation at time T, and the bank will pass the audit with certainty. In this case, $V(0,q) = X(0) - [L(0) - G(0)]$.

When $q > q_{\min}$, i.e., $L(0)-(1-q)X(0) > 0$, we have

$$\frac{\partial^2 V}{\partial q^2} = X(0)\mathrm{n}(d_1)\left(\frac{L(0)-X(0)}{L(0)-(1-q)X(0)}\right)^2\frac{1}{q\sigma\sqrt{T}} + \frac{2X^2(0)G(0)[L(0)-X(0)]}{\sigma\sqrt{T}[L(0)-(1-q)X(0)]^3} \ge 0.$$

Hence, the equity value $V(0,q)$ is flat on interval $[0, q_{\min}]$ and convex on interval $[q_{\min}, 1]$. The optimal policy q^* is either 1 or any value in $[0,q_{\min}]$. Therefore, from equations (6) and (7)

$$\begin{aligned} V(0,q^*) &= \max\{ V(0,0), V(0,1) \} \\ &= \max\{ X(0)-[L(0)-G(0)],\ X(0)\mathrm{N}(d_1^1)-[L(0)-G(0)]\mathrm{N}(d_2^1) \}. \end{aligned}$$

This leads to equation (10). Q.E.D.

To prove theorem 3, a few lemma are necessary. Lemma 1 is an adaptation of Fleming and Rishel (1975, p. 124, theorem V.5.1). Lemma 2 is a classic result (Bhattacharya and Waymire [1990, p. 32]). In the rest of the proof, I use the shorthand notations J and f for $J(t,X(t))$ and $f(s;t,X(t))$, respectively, as long as no confusion arises.

Lemma 1. (Sufficient optimality condition for discounted stochastic dynamic programming) Let $X(t)$ be a diffusion process on $[0,T]$

$$dX(t) = \mu(X)dt + \sigma(X)dW(t), \qquad X(0) = X_0, \tag{A.1}$$

where μ and σ satisfy the linear growth and the Lipschitz conditions. Let $M(t,X)$ and $J(T,X)$ be continuous and satisfy the polynomial growth condition. Let $J(t,X)$ be the solution of the dynamic programming equation

$$rJ = \max \{J_t + \mu(X)J_x + \tfrac{1}{2}(\sigma(X))^2 J_{xx} + M(t,X)\} \tag{A.2}$$

with boundary value $J(T,X(T))$. If $J(t,X)$ is twice differentiable for $t \in [0,T)$ and continuous for $t \in [0,T]$, then

$$J(t,X(t)) \geq E_q[\int_0^T M(s,X(s))ds + J(T,X(T))] \tag{A.3}$$

for any admissible policy q.

Lemma 2. Let $X(t)$ be a Brownian motion with drift μ. Let T_z be the first time the process reaches level z conditioned on $X(0) = x$. Then the probability density and distribution functions of T_z are

$$f(t;x,z) = \frac{(z-x)}{\sqrt{2\pi}\sigma t^{3/2}} \exp[-\frac{(z-x-\mu t)^2}{2\sigma^2 t}], \qquad t > 0, \tag{A.4}$$

$$F(t;x,z) = \int_0^t f(s;x,z)ds = N(\frac{x-z+\mu t}{\sigma\sqrt{t}}) - e^{2(z-x)\mu/\sigma^2} N(\frac{x-z-\mu t}{\sigma\sqrt{t}}). \tag{A.5}$$

Lemma 3. The functional $J(t,X(t))$ and the policy q^* defined in theorem 3 is optimal if

(1) when J_{xx} is continuous at $(t,X(t))$, the maximizing q is

$$q^* = \begin{cases} 1 & \text{if } J_{xx} \geq 0 \\ 0 & \text{if } J_{xx} \geq 0 \end{cases} \tag{A.6}$$

and

$$rJ = J_t + rX(t)J_x + \tfrac{1}{2}(\sigma X(t))^2 J_{xx} \qquad \text{if } q^* = 1 \tag{A.7}$$

$$rJ = J_t + rX(t)J_x \qquad \text{if } q^* = 0 \tag{A.8}$$

(2) when J has a jump at $(t,X(t))$,

$$J_x^+ < J_x^- \tag{A.9a}$$

$$q^* = 0 \tag{A.9b}$$

where

$$J_x^{\pm} = J_x(t,X(t)\pm 0).$$

Proof: Part (1) follows immediately from lemma 1 with $\mu(X) = rX$. To show part (2), note that $J(t,X(t))$ is twice differentiable except when $X(t) = L(t)$ and $t \in (\tau,T]$ where $J_x^- > J_x^+ = 1$ and $J_{xx}^- = J_{xx}^+ = 0$; when $t \in (\tau,T]$, $J(t,X(t))$ is convex for $X(t) \le L(t)$ and linear for $X(t) \ge L(t)$ (see the proof of theorem 3). To apply lemma 1, add a smoothing term P^{ε} to J such that $J^{\varepsilon}(t,X(t)) = J(t,X(t)) + P^{\varepsilon}(t,X(t))$ is twice differentiable, convex for $X(t) \le L(t)$, and concave for $X(t) \ge L(t)$ for $t \in (\tau,T)$ and for any small number $\varepsilon > 0$. For example, one such P^{ε} is

$$P^{\varepsilon} = \begin{cases} -\varepsilon\pi\dfrac{\Delta J^L}{2} & \text{if } X(t)>L(t)+\varepsilon\pi \text{ and } t\in(\tau,T) \\ -[X(t)-L(t)+sin(\dfrac{X(t)-L(t)}{\varepsilon})\varepsilon]\dfrac{\Delta J^L}{2} & \text{if } 0\le X(t)-L(t)\le\varepsilon\pi \text{ and } t\in(\tau,T) \\ 0 & \text{otherwise,} \end{cases}$$

where $\Delta J^L = J_x^+(L(t)) - J_x^-(L(t))$. Define

$$\phi(P^{\varepsilon}) = - rP^{\varepsilon} + P_t^{\varepsilon} + rX(t)P_x^{\varepsilon}.$$

Then for any admissible policy q,

$$\begin{aligned} &-rJ^{\varepsilon} + J_t^{\varepsilon} + rX(t)J_x^{\varepsilon} + \tfrac{1}{2}(qX(t)\sigma)^2 J_{xx}^{\varepsilon} - \phi(P^{\varepsilon}) \\ &\quad\le - rJ^{\varepsilon} + J_t^{\varepsilon} + rX(t)J_x^{\varepsilon} + \tfrac{1}{2}(q^*X(t)\sigma)^2 J_{xx}^{\varepsilon} - \phi(P^{\varepsilon}) \\ &\quad= 0, \end{aligned}$$

where q^* is the policy in theorem 3. Therefore, $J^{\varepsilon}(t,X(t)) = J(t,X(t)) + P^{\varepsilon}(t,X(t))$ is the solution of the dynamic programming equation

$$rJ^{\varepsilon} = \max_q\, [J_t^{\varepsilon} + rX(t)J_x^{\varepsilon} + \tfrac{1}{2}(qX(t)\sigma)^2 J_{xx}^{\varepsilon} - \phi(P^{\varepsilon})]$$

for $t \in (\tau, T-\delta)$ and any small number $\delta > 0$. Applying lemma 1, we have

$$
\begin{aligned}
J(t,X(t)) &= J^{\varepsilon}(t,X(t)) - P^{\varepsilon}(t,X(t)) \\
&\ge \mathbb{E}_q[-\int_t^{T-\delta} \phi(P^{\varepsilon})dt + J^{\varepsilon}(T-\delta,X(T-\delta))] - P^{\varepsilon}(t,X(t)) \\
&= \mathbb{E}_q J(T-\delta,X(T-\delta)) - \mathbb{E}_q[\int_t^{T-\delta} \phi(P^{\varepsilon})dt] + \mathbb{E}_q[P^{\varepsilon}(T-\delta,X(T-\delta))] \\
&\qquad - P^{\varepsilon}(t,X(t)).
\end{aligned}
$$

Let $\varepsilon,\delta \longrightarrow 0$. The last three terms on the right-hand side all go to zero. Then $J(t,X(t)) \ge \mathbb{E}_q[J(T,X(T))]$ for any q. This implies $J(t,X(t))$ and q^* are optimal. Q.E.D.

Proof of Theorem 3. We need to show that the given functional $J(t,X(t))$ and the corresponding policy q^* satisfy the conditions in lemma 3.

Case 1. Let $t \in (\tau,T]$ and $X(t) \ge L(t)$. When $X(t) > L(t)$, $q^*(t) = 0$ and $J(t,X(t))$ in equation (15) together satisfy the conditions (A.6) and (A.8) in lemma 3. When $X(t) = L(t)$, as we will show later, J_x is not continuous in $X(t)$. However, from lemma 3, $q^* = 0$ is optimal if $J_X^+ < J_X^-$. Since $J_X^+ = 1$, we need only to show that $J_X^- > 1$ at $X(t) = L(t)$.

First note that $J(t,X(t))$ is continuous at $X(t) = L(t)$. In fact, as $X(t)\uparrow L(t)$, $\gamma_1 \longrightarrow -\sigma\sqrt{T-t}/2$ and $\gamma_2 \longrightarrow \sigma\sqrt{T-t}/2$ in equation (16). Further manipulation yields $J(t,X(t)) \longrightarrow G(t) = J(t,X(t))$. Now differentiate $J(t,X(t))$ in equation (16), and let $X(t)\uparrow L(t)$. Then

$$
\begin{aligned}
J_X(t,X(t)) &= \frac{G(t)}{L(t)}\left[\mathrm{N}(\gamma_1) + X(t)\mathrm{n}(\gamma_1)\frac{\partial\gamma_1}{\partial X} + L(t)\mathrm{n}(\gamma_2)\frac{\partial\gamma_2}{\partial X}\right] \\
&\longrightarrow \frac{G(t)}{L(t)}\left[\mathrm{N}(\sigma\sqrt{T-t}/2) + \frac{2\mathrm{n}(\sigma\sqrt{T-t}/2)}{\sigma\sqrt{T-t}}\right]. \qquad \text{(A.10)}
\end{aligned}
$$

Since $\partial J_x^-/\partial t = -\dfrac{G(t)}{L(t)}\dfrac{\mathrm{n}(\sigma\sqrt{T-t}/2)}{\sigma(T-t)^{3/2}} < 0$, J_x^- is strictly increasing in t. Noting that $J_x^- = 1$ at $t = \tau$, we have $J_x^- > 1$ for all $t \in (\tau,T]$.

Case 2. Let $t \in (\tau,T]$, $X(t) < L(t)$. Differentiating equation (16), and noting that $X(t)\mathrm{n}(\gamma_1) = L(t)\mathrm{n}(\gamma_2)$, we have

$$J_{XX} = \frac{\partial^2}{\partial X^2}[X(t)N(\gamma_1) + L(t)N(\gamma_2)]\frac{G(t)}{X(t)}$$

$$= \frac{2f(T;t,X(t))G(t)}{\sigma^2X^2(t)} \geq 0. \qquad \text{(A.11)}$$

J_{XX} is obviously continuous. To show that $q^*(t) = 1$ is optimal, we need only to check that condition (A.7) in lemma 3 is satisfied. Toward this goal, let $Y(t) = \ln[X(t)] - rt$; then

$$dY(t) = -(\sigma^2/2)dt + \sigma dW(t), \qquad Y(0) = \ln[X(0)].$$

The first passage times are the same for the geometric Wiener process $X(t)$ to reach $L(s)$ given $X(t)$ at time t and for the Brownian motion $Y(t)$ to reach $\ln[L(s)] - rs$ given $Y(t) = \ln X(t) - rt$ at time t. From lemma 2, the density function of this first passage time is

$$f(s;t,X(t)) = \frac{\ln[L(t)/X(t)]}{\sqrt{2\pi}\sigma(s-t)^{3/2}} \exp\{- \frac{[\ln\frac{L(t)}{X(t)} + \frac{1}{2}\sigma^2(s-t)]^2}{2\sigma^2(s-t)}\} \qquad (s>t).$$

It is easy to show that $J(t,X(t)) = \int_t^T G(t)f(s;t,X(t))ds$. Since the density function f satisfies the backward Kolmogorov equation

$$f_t + rX(t)\, f_X + \tfrac{1}{2}(X(t)\sigma)^2 f_{XX} = 0, \qquad \text{(A.12)}$$

condition (A.7) can be easily checked:

$$-rJ + J_t + rX(t)J_X + \tfrac{1}{2}(X(t)\sigma)^2J_{XX}$$

$$= -rG(t)\int_t^T fds + [rG(t)\int_t^T fds + G(t)\int_t^T f_t ds]$$

$$+ rX(t)G(t)\int_t^T f_X ds + \tfrac{1}{2}(X(t)\sigma)^2G(t)\int_t^T f_{XX}ds$$

$$= G(t)\int_t^T \{f_t + rX(t)f_X + \tfrac{1}{2}(X(t)\sigma)^2f_{XX}\}\, ds$$

$$= 0.$$

Case 3. Let $t \in [0,\tau]$. We first show that $J(t,X(t))$ in equation (17) is the risk-neutral value of a contingent claim with terminal value $J(\tau,X(\tau))$ at time τ. To see this, let

$$J(t,X(t)) = e^{-r(\tau-t)}\int_{-\infty}^{\infty} J(\tau,X(\tau))\, \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz, \qquad \text{(A.13)}$$

where $X(\tau) = X(t)e^{(r-\sigma^2/2)(\tau-t)+\sigma(\tau-t)^{1/2}z}$. Substituting equations (15) and (16) into (A.13),

$$J(t,X(t)) = e^{-r(\tau-t)}\int_{-\gamma_4}^{\infty} \{X(\tau)-[L(\tau)-G(\tau)]\}\, \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz$$

$$+\ e^{-r(\tau-t)}\int_{-\infty}^{-\gamma_4} \frac{G(\tau)}{L(\tau)}\, [X(\tau)N(\gamma_1) + L(\tau)N(\gamma_2)]\, \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz,$$

where γ_1 and γ_2 are evaluated at τ rather than at t. Carrying out the integrations above gives equation (17). From (A.13) we have

$$\tfrac{1}{2}(X(t)\sigma)^2 J_{XX}(t,X(t)) = e^{-r(\tau-t)}\int_{-\infty}^{\infty} \tfrac{1}{2}[X(\tau)\sigma]^2 J_{XX}(\tau,X(\tau))\, \frac{e^{-z^2/2}}{\sqrt{2\pi}}\, dz.$$

Since $J_{XX}(\tau,X(\tau)) \geq 0$ from cases 1 and 2, $J_{XX}(t,X(t)) \geq 0$.

Now we need only to check condition (A.7) in order to show $q^*(t) = 1$ is optimal. Let $p = p(\tau,y;t,X(t))$ be the density function of the lognormal price $X(\tau)$ conditioned on $X(t)$. Rewrite equation (17) as

$$J(t,X(t)) = e^{-r(\tau-t)}\int_{0}^{\infty} J(\tau,y)p(\tau,y;t,X(t))dy. \qquad \text{(A.14)}$$

Then equation (A.7) can be established by the fact that $p(\tau,y;t,X(t))$ satisfies the backward Kolmogorov equation (A.10). Q.E.D.

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