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CONSUMPTION AND FRACTIONAL DIFFERENCING: OLD AND NEW ANOMALIES

by Joseph G. Haubrich

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1. Introduction

Consumption depends on income, so testing theories of consumption involves testing theories of income. A prominent recent example is the work by Campbell and Deaton (1989), which uncovers a paradox. They model income as having a unit root instead of as a fluctuation around a trend, and so they find that consumption looks too smooth: the permanent-income hypothesis does not hold. Like some previous researchers, they find that a difference-stationary process fits the data better than a trend-stationary process.

The choice between a difference-stationary process and a trend-stationary process, however, ignores the intermediate class of fractionally differenced processes. Since fractional processes exhibit long-term dependence, they are often classified as having a unit root rather than as trend stationary. This makes permanent income seem rougher than it really is, while consumption, which responds to the true, fractional income, looks too smooth. Specifying consumption correctly removes the paradox.

This paper reviews the techniques of fractionally differenced stochastic processes, calculates the stochastic properties of consumption when income follows a fractional stochastic process, and shows how this may explain the excess-smoothness results.

2. Fractional Methods

Intuition suggests that differencing a time series roughens it, while summing a time series smooths it. A fractional difference between 0 and 1 can be described as a filter that roughens a series less than does a first difference: The series is rougher than a random walk but smoother than white noise. This is apparent from the infinite-order moving-average representation. Let X_t follow

$$(1 - L)^d X_t = \epsilon_t, \tag{1}$$

where ϵ_t is white noise, d is the degree of differencing, and L is the lag operator. If d = 0, X_t is white noise, and if d = 1, X_t is a random walk. However, as Granger and Joyeux (1980) and Hosking (1981) show, d need not be an integer. The binomial theorem provides the relation

$$(1 - L)^d = \sum_{k=0}^{\infty} (-1)^k \binom{d}{k} L^k, \tag{2}$$

with the binomial coefficient $\binom{d}{k}$ defined as

$$\binom{d}{k} \equiv \frac{d(d - 1)(d - 2)\cdots(d - k + 1)}{k!} \tag{3}$$

for real d and nonnegative integer k. Using this definition, the autoregressive (AR) form of X_t follows

$$A(L)Xt = \sum_{k=0}^{\infty} A_k L^k X_t = \sum_{k=0}^{\infty} A_k X_{t-k} = \epsilon_t, \tag{4}$$

with the AR coefficient expressed compactly in terms of the gamma function

$$A_k = (-1)^k \binom{d}{k} = \frac{\Gamma(k - d)}{\Gamma(-d)\Gamma(k + 1)}. \tag{5}$$

Manipulating equation (5) yields the corresponding moving average (MA) representation of X_t:

$$X_t = (1 - L)^{-d}\epsilon_t = B(L)\epsilon_t \quad B_k = \frac{\Gamma(k - d)}{\Gamma(-d)\Gamma(k + 1)}. \tag{6}$$

The time-series properties of X_t depend crucially on the difference parameter, d. For example, when d is less than one-half, X_t is stationary; when d is greater than minus one-half, X_t is invertible (Granger and Joyeux [1980], Hosking [1981]). Likewise, the autocorrelation properties of X_t depend on the parameter d. The MA coefficients, B_k, indicate the effect of a shock K periods ahead and the extent to which current levels depend on past values. Using Stirling's approximation, we know that

$$B_k \approx \frac{k^{d-1}}{\Gamma(d)}. \tag{7}$$

Comparing this with the decay of an AR(1) process highlights the central "long-memory" feature of fractional processes: They decay hyperbolically, at rate k^{1-d}, rather than at the exponential rate, ρ^k, of an AR(1). For example, compare in Figure 1 the autocorrelation function of the fractionally differenced series $(1-L)^{0.475}X_t = \epsilon_t$ with the AR(1)$X_t = 0.9X_{t-1} + \epsilon_t$. Although both have first-order autocorrelations of 0.90, the AR(1)'s autocorrelation function decays much more rapidly. Figure 2A plots the impulse-response functions of these two processes. At lag 1, the MA coefficients of the fractionally differenced series and the AR(1) are 0.475 and 0.900, respectively; at lag 10, they are 0.158 and 0.349, and at lag 100, they are 0.048 and 0.000027. The persistence of the fractionally differenced series is apparent at the longer lags. Alternatively, we may ask what value of an AR(1)'s autoregressive parameter will, for a given lag, yield the same impulse response as the fractionally differenced series (equation [1]). This value is simply the k-th root of B_k, and is plotted in Figure 2B for various lags when $d = 0.475$. For large values of k, this autoregressive parameter must be very close to unity.

These representations also show how standard econometric methods can fail to detect fractional processes. Although a high-order ARMA process can mimic the hyperbolic decay of a fractionally differenced series in finite samples, the large number of parameters required would give the estimation a poor rating from the usual Akaike or Schwartz criteria. An explicitly fractional process, however, captures that pattern with a single parameter, d. Granger and Joyeux (1980) and Geweke and Porter-Hudak (1983) provide empirical support by showing that fractional models often out-predict fitted ARMA models.

The lag polynomials A(L) and B(L) provide a metric for the persistence of X_t. Suppose X_t represents GNP, which falls unexpectedly this year. How much should this decline change a forecast of future GNP? To address this issue, define C_k as the coefficients of the lag polynomial, C(L), that satisfies the relation $(1 - L)X_t = C(L)\epsilon_t$, where the process X_t is given by equation (1). One measure used by Campbell and Mankiw (1987) is

$$\lim_{k\to\infty} B_k = \sum_{k=0}^{\infty} C_k = C(1). \qquad (8)$$

For large values of k, the value of B_t measures the response of X_{t+k} to an innovation at time t, a natural metric for persistence. From equation (7), it is immediate that for $0 < d < 1$, $C(1) = 0$, and that, asymptotically, there is no persistence in a fractionally differenced series, even though the autocorrelations die out very slowly. This holds true not only for $d = 1/2$ (the stationary case), but also for $1/2 < d < 1$, when the process is nonstationary.

From these calculations, it is apparent that the long-run dependence of fractional processes relates to the slow decay of the autocorrelations, not to any permanent effect. This distinction is important; for example, an IMA(1,1) can have small but positive persistence, but the coefficients will never mimic the slow decay of a fractional process.

3. Fractional Differencing and the Theory of Consumption

The excess-smoothness paradox can be stated more precisely as follows. Assuming the standard certainty equivalence framework (for example, quadratic utility; see Hall [1978], Flavin [1981], and Zeldes [1989]), we can find how the variance of consumption depends on the income process:

$$\mathrm{var}(\Delta C_t) = \left\{ \frac{r}{1+r} \frac{[1 + \Sigma(1+r)^{-k}\theta_k]}{[1 - \Sigma(1+r)^{-k}\phi_k]} \right\}^2 \sigma_\epsilon^2, \tag{9}$$

where

C_t = consumption,

r = the real interest rate,

θ_t = the MA coefficients of income Y_t,

ϕ_t = the AR coefficients of Y_t,

Δ = the difference operator $\Delta = (1 - L)$, and

σ_ϵ^2 = the variance of income shocks.

Hansen and Sargent (1981) show that this formula holds for both stationary and nonstationary processes. Since consumption is a random walk (more generally a martingale) in this framework, the variance of the change in consumption (equation [9]) also represents the variance of innovations to consumption. Under the traditional assumption that income follows a trend-stationary process (because the shocks die out), the variance of innovations to consumption, var(ΔC_t), should be less than the variance of innovations to income, i. This is what Friedman was trying to explain with the permanent-income hypothesis -- namely, that consumption looks smoother than income. If, however, income is first-difference stationary, as researchers since Nelson and Plosser (1982) have claimed, the revision in permanent income exceeds the revision in actual income. Consumption innovation should then exceed income innovation, σ_ϵ^2. Deaton (1987) finds that it does not.

A numerical example based on the data used in this paper illustrates excess smoothness. Suppose income is a random walk. In that case, the variance of the change in consumption should equal the variance of the change in income, as intuition or equation (9) suggests. In fact, the figure for consumption is 11.65, while that for income is 61.14.

The key point to note, both in predicting the variance of consumption and in determining the variance of income innovations, is that we must make some assumptions or estimates of the income process. By making a different and better assumption about income -- fractional differencing -- the paradox can be resolved.

Another advantage of assuming a fractional-differencing process for income is that it allows us to retain two assumptions jettisoned by others. First, the income process is univariate, and consumers have no information about it that is hidden from the econometrician. West (1988) shows that such hidden information can spuriously create excess smoothness, because true income surprises would then be less than measured income surprises. Various methods that correct for hidden information (Campbell and Deaton [1989], Flavin [1988]) still show excessive smoothness, however. Second, the permanent-income hypothesis is maintained throughout. Both Campbell and Deaton and Flavin show that departures from this can simultaneously produce both excess smoothness and excess sensitivity.

The remainder of this section attempts to answer two basic questions. First, does there exist a difference parameter, d, that resolves the paradox -- that is, if income follows such a process, consumption will no longer look too smooth? Second, does actual income follow such a process? In other words, will the fractional parameter that provides a solution fit the income data that we have?

Using data for the United States, I proceed in four basic steps.[1] Section 3.1 reports estimates of the variance of income and consumption changes using both Generalized Method of Moments (GMM) and classical chi-squared techniques to determine the estimates' precision. In section 3.2, using the permanent-income hypothesis, I find a range of d in the income process that will produce the variance of consumption found in the first step. In section 3.3, I employ a test for fractional differencing in the income series. Finally, in section 3.4, I use simulations to estimate the probability that fractional parameters reported in section 3.2 would produce the value found in section 3.3.

3.1 Distribution of the Sample Variance

I begin by estimating and comparing the variance of income changes and the variance of consumption changes. Calculating the distribution of the sample variance depends on assumptions about the underlying process. The classical approach assumes an i.i.d. sample from a normal distribution and then produces the familiar result that the scaled sample variance is distributed chi-squared with degrees of freedom one less than the sample size:

$$\frac{nS^2}{\sigma^2} \sim X^2(n - 1). \qquad (10)$$

This may be appropriate for consumption, which, according to theory, should follow a random walk. It has the advantage of being correct for finite samples.

The GMM approach allows for heteroskedasticity and autocorrelation. Designed to handle much more complicated estimation problems (Hansen [1982], Hansen and Singleton [1982]), it reduces to a fairly simple form when used to determine the distribution of the sample variance. (See Ng Lo [1988] for a rigorous and clear demonstration of this.) In fact, it reduces to estimating the covariance matrix. Therefore, I use the Newey-West (1987) covariance matrix. This provides a positive, definite heteroskedastic and autocorrelation-consistent covariance matrix. The disadvantage is that it provides an asymptotic result.

The Newey-West matrix also requires that a choice be made on the number of lags used to compute the matrix. The authors suggest using the fourth root of the sample size, but the convergence results for this small number depend on mixing conditions, which will generally be violated in the case of long-term dependence. In more general cases, they suggest employing the cube or square root, while Chatfield (1984, p. 141) recommends using twice the square root. With a sample size of 120 for the consumption series and 137 for the two income series, I use five lags. This follows Ng Lo (1988), who finds that this choice works well even in larger samples for a variety of series.

Table 1 shows the sample variances for per-capita consumption of nondurables and services, plus both per-capita income measures used (labor and disposable). It also reports the 95 percent confidence bounds obtained using both the classical and GMM approaches. Since the GMM bounds are broader (because income shows autocorrelation), they are used in the next part of this exercise.

3.2 Implied Variance

The variance of income and consumption depends on an unobservable (to the econometrician) variable: shocks to income. If income follows a fractional process with parameter d, we have from Hosking (1981) that

$$\mathrm{var}(\Delta y) = \frac{\Gamma(3 - 2d)}{\{\Gamma(2 - d)\}^2}\,\sigma_\epsilon^2 \tag{11}$$

Likewise, the variance-of-consumption formula (equation [9]) specializes in this case to

$$\mathrm{var}(\Delta C_t) = \left\{\frac{r}{1+r}[\Sigma(1+r)^k\theta_k]^2\right\}\,\sigma_\epsilon^2, \tag{12}$$

where C_t is consumption, θ_t are the MA coefficients of income Y_t, and Δ is the difference operator, $\Delta = (1 - L)$. The estimates for income and consumption variance give estimates of the shock variance, σ_ϵ^2.

Notice that the implied shock variance changes with different assumptions about the income process, that is, with changes in the differencing parameter, d. Inverting equations (11) and (12) yields the variance of income shocks as a function of d. Then, comparing the implied shock variances across income and consumption yields the d values that make the income process consistent with observed consumption behavior.

Implementing the above procedure requires choosing an interest rate. I use three different quarterly rates: r = 0.2 percent, which corresponds to the long-run average rate used in Mehra and Prescott (1985); r = 1 percent, a high interest rate; and r = 0.05 percent, a low interest rate. Using these numbers made a noticeable, if not dramatic, difference in the variance estimates.

Tables 2A and 2B report the results of this investigation and make clear the choice of bounds on d used: 0.79 and 0.95 for labor income, and 0.72 and 0.96 for disposable income.

3.3 Testing for Fractional Differencing

The next step ascertains whether the d values obtained above are consistent with the observed income process. This section tests for fractional differencing using the modified rescaled range (R/S) statistic developed by Lo ([forthcoming] and Haubrich and Lo [1989]). In section 3.4, I use simulations to determine the probability that the values obtained from the test could come from distributions with a d parameter in the range calculated above.

The modified R/S statistic tests whether a process X_t shows long-term dependence. (It is based on a statistic originally developed by Hurst [1951] and popularized by Mandelbrot [1972].) More formally, consider a process defined as

$$X_t = \mu+\epsilon_t, \tag{13}$$

where μ is an arbitrary but fixed constant. For the null hypothesis H, assume that the disturbances (ϵ_T) satisfy the conditions

(C1) $E(\epsilon_t) = 0$ for all t,

(C2) $\sup_t E\,[|\epsilon_t|^\beta] < \infty$ for some $\beta > 2$,

(C3) $\sigma^2 = \lim_{n\to\infty} E\left[\frac{1}{n}\left(\sum_{j=1}^{n}\epsilon_j\right)^2\right]$ exists and $\sigma^2 > 0$, and

(C4) (ϵ_t) is strong-mixing, with mixing coefficients α_k that satisfy $\sum_{k=1}^{\infty}\alpha_k^{1-\frac{2}{\beta}} < \infty$.

Conditions (C2) through (C4) allow dependence and heteroskedasticity, but prevent them from being too large. Thus, short-term dependent processes, such as finite-order ARMA models, are included in the null hypothesis, as are models with conditional heteroskedasticity. Unlike the statistic used by Mandelbrot, the modified R/S statistic used here is robust to short-term dependence. A more in-depth discussion of these conditions appears in Phillips (1987), Haubrich and Lo (1989), and Lo (forthcoming).

To construct the modified R/S statistic, take a sample X_1, X_2, ... X_n, with sample mean $\bar{X}_n$, choose q lags, and calculate:

$$Q_n \equiv \frac{1}{\hat{\sigma}_n(q)}\left[\max_{1\le k\le n}\sum_{j=1}^{k}(X_j - \bar{X}_n) - \min_{1\le k\le n}\sum_{j=1}^{k}(X_j - \bar{X}_n)\right],$$

where

$$\hat{\sigma}_n^2(q) = \frac{1}{n} \sum_{j=1}^{n} (X_j - \bar{X}_n)^2 + \frac{2}{n} \sum_{j=1}^{q} w_j(q) \left\{ \sum_{i=j+1}^{n} (X_i - \bar{X}_n)(X_{i-j} - \bar{X}_n) \right\} \tag{14}$$

$$= \hat{\sigma}_x^2 + 2\sum_{j=1}^{q} w_j(q)\hat{\gamma}j \qquad w_j(q) \equiv 1 - \frac{j}{q+1}q < n.$$

Intuitively, the numerator in equation (14) measures the memory in the process via the partial sums. White noise does not stay long above the mean: Positive values are soon offset by negative values. A random walk will remain above or below zero for a long time, and the partial sums (positive or negative) will grow quickly, making the range large. Fractional processes fall in between. Mandelbrot (1972) refers to this as the "Joseph Effect" -- seven fat and seven lean years. The denominator normalizes not only by the variance, but by a weighted average of autocovariances.[2] This innovation over Hurst's R/S statistic provides the robustness to short-term dependence.

The partial sums of white noise constitute a random walk, so $Q_n(q)$ grows without bound as n increases. A further normalization makes the statistic easier to work with and interpret:

$$V_n(q) = Q_n(q)/\sqrt{(n)}. \tag{15}$$

Haubrich and Lo derive the asymptotic distribution of V, calculating a mean and standard deviation of approximately 1.25 and 0.27. Tables 3A and

3B present fractiles of the distribution of V and confidence intervals about the mean. Figure 3 plots the distribution and density. Note that the distribution is skewed, with most of its mass between three-fourths and two.

Table 4 reports the results of the modified R/S statistic applied to first differences of labor income and disposable income. Note that none are significantly different from the mean at the 5 percent level.

3.4 Simulation Results

Although the modified R/S statistic provides a good test (in terms of size and power) for detecting long-term dependence, it does not directly provide the d parameter. To better assess the chances that a d parameter from the correct range will fit the data, I use simulation methodology.

Simulations employed here ran as follows. I used a Vax Fortran program (a modification of one written by Lo) to generate 10,000 series of length 135 (not quite matching the data-series length of 136, to compare this study to other papers). The series were generated to have fractional differencing parameter d for several d. I then computed the modified R/S statistic for each series and counted the number of times that this value fell below the value obtained from the income data above (Table 4). This gives the percentage of times the statistic would be that low if the income series actually had that d parameter. I emphasize low because in first-difference form the relevant d would be negative, which should show up as a low R/S statistic. Table 5 reports these results and also answers the question: If the process is really fractionally differenced with a particular d, what is the probability that we would see the $V_n(q)$ number found in the data, or even a lower number? Of course, subtracting these numbers from one gives the probability of obtaining a higher R/S statistic. The reader may draw different conclusions from Table 5, but I think that the results provide mild support for the belief that fractional processes can explain the excess-smoothness problem. It seems unlikely that the actual d for either income process is smaller than the lower bounds obtained above; we would expect to see much lower numbers than those in Table 4. That is, Table 5 tells us that the probability of seeing that number or a lower one is very high for such a process with a d of -0.21 or -0.28. On the other hand, the chance of d = -0.04 or -0.05 producing such a number is more reasonable.

Earlier in this section, we saw what range d could fall into and still resolve the Deaton paradox. Now we see, in a general way, how likely it is that d could be in that range. The chance remains that d is too close to zero to resolve the paradox by invoking fractional methods. I submit that Table 5 opens the very real possibility that d falls into the relevant range.

4. Conclusion

Judging the smoothness of consumption depends on the estimate of permanent income, which in turn depends on our estimate of income. Paradoxes under one specification -- excess smoothness when income is assumed to have a unit root -- do not arise when income is fractional.

The explanation that I propose leaves intact two similar problems in the consumption literature. First, panel studies have found excess sensitivity of precisely the opposite type Campbell and Deaton find in aggregate data. Consumption variance is too high given the estimates for income. Flavin finds a different type of excess sensitivity, namely, that consumption depends on past income; it is not a martingale (the expected future value equals today's value), as the permanent-income hypothesis predicts. Campbell and Deaton refer to this as the "nonorthogonality" problem.

Nonetheless, without dropping either the permanent-income hypothesis or the univariate representation of income, fractional processes resolve the Deaton paradox. Theoretically, a fractional-income process matches the observed variance of both income and consumption. Empirically, on the basis of a new statistic and simulations, the evidence supports income following such a process.

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Table 1

Sample Variances

	Variance		95% Confidence Bounds	
Consumption	11.65	GMM	6.37	16.93
		Classical	9.17	15.30
Labor income	65.35	GMM	36.43	94.28
		Classical	52.18	84.24
Disposable income	61.14	GMM	28.90	93.38
		Classical	48.82	78.81

Consumption = First difference of real per-capita consumption of nondurables and services, 1989:IQ-1989:IIQ (quarterly data, seasonally adjusted). Source: National Income and Product Accounts.

Population = U.S. total resident population, including armed forces. Source: National Income and Product Accounts.

Labor income = First difference of quarterly real per-capita labor income, 1952:IQ-1986:IQ. Sources: Auerbach and Hassett (1989) and National Income and Product Accounts.

Disposable income = As above. Source: Auerbach and Hassett (1989).

Table 2A

Implied Income Innovation Variances
Labor Income

d	Implied variance from consumption: Lower bound	Implied variance from consumption: Upper bound	Implied variance from income	
Interest rate = 0.05%				
0.750	284.43	755.94	33.77	87.40
0.825	91.02	241.91	34.49	90.56
0.830	84.36	224.22	35.06	90.75
0.950	13.62	36.20	36.29	93.92
0.955	12.62	33.55	36.32	93.99
0.975	9.32	24.76	39.39	94.19
Interest rate = 1%				
0.650	157.86	419.56	31.84	82.40
0.715	87.03	231.29	33.13	85.73
0.720	83.13	220.94	33.22	85.98
0.920	13.28	35.30	36.08	93.39
0.925	12.68	33.71	36.13	93.50
0.980	7.65	20.34		
Interest rate = 0.2%				
0.750	141.89	377.11	33.77	87.40
0.785	91.91	244.28	34.37	88.96
0.790	86.38	229.59	34.45	89.17
0.940	13.42	35.67	36.23	93.77
0.945	12.61	33.52	36.26	93.85
0.960	10.47	27.82	36.34	94.05

Source: See table 1.

Table 2B

Implied Income Innovation Variances
Labor Income

d	Implied variance from consumption: Lower bound	Implied variance from consumption: Upper bound	Implied variance from income	
Interest rate = 0.05%				
0.750	284.43	755.94	26.79	86.57
0.825	91.02	241.91	27.76	89.70
0.830	84.36	224.22	27.82	89.88
0.960	11.70	31.10	28.83	93.15
0.965	10.84	28.82	28.84	93.20
0.990	7.42	19.71	28.90	93.36
Interest rate = 1%				
0.650	157.86	419.56	25.26	81.61
0.710	91.11	242.14	26.20	84.67
0.715	87.03	231.29	26.28	84.91
0.945	10.56	28.06	28.77	92.95
0.950	10.08	26.80	38.79	92.02
0.960	9.20	24.45	28.83	93.15
Interest rate = 0.2%				
0.750	141.89	377.11	29.79	86.57
0.785	91.91	244.28	27.27	88.11
0.790	86.38	229.59	27.33	88.32
0.955	11.85	29.61	28.81	93.09
0.960	10.47	27.82	28.83	93.15

Source: See table 1.

Note, Tables 2A and 2B

Approximations: Closed-form solutions for the infinite sums used in these calculations do not exist. An upper bound on the finite sum of N terms and the infinite sum is $\frac{1}{r}(1/1+r)^N$. The approximation is in fact better. 10,000 terms were used for the interest rates $r = 0.01$ and $r = 0.002$, leading to errors of less than 1×10^{-8} and 1.05×10^{-6}. 20,000 terms used for $r = 0.0005$ give an error of less than 0.09.

Table 3A

Fractiles of the Distribution $F_V(v)$

$P(V < v)$	.005	.025	.050	.100	.200	.300	.400	.500
v	0.721	0.809	0.861	0.927	1.018	1.090	1.157	1.223

$P(V < v)$	.543	.600	.700	.800	.900	.950	.975	.995
v	$\sqrt{\frac{\pi}{2}}$	1.294	1.374	1.473	1.620	1.747	1.862	2.098

Source: Haubrich and Lo (1989).

Table 3B

Symmetric Confidence Intervals About the Mean

$P\sqrt{\frac{\pi}{2}} - \gamma < V < \sqrt{\frac{\pi}{2}} + \gamma$	γ
.001	0.748
.050	0.519
.100	0.432
.500	0.185

Source: Haubrich and Lo (1989).

Table 4

R/S Analysis of Income

Series	$V_n(0)$	$V_n(1)$	$V_n(2)$	$V_n(3)$	$V_n(5)$
Labor Income	1.310	1.193	1.140	1.062	1.018
Disposable Income	1.268	1.261	1.245	1.176	1.170

Note: Both series per capita.

Sources: See table 1.

Table 5

Probability of Observing R/S Statistic

	Probability of $\leq V_n(q)$			
LAG(1)	Labor Income		Disposable Income	
	d=-0.05	d=-0.21	d=-0.04	d=-0.28
0	0.86	0.99	0.80	1.00
1	0.71	0.97	0.77	0.99
2	0.60	0.90	0.73	0.99
3	0.43	0.74	0.61	0.95
5	0.29	0.40	0.55	0.87

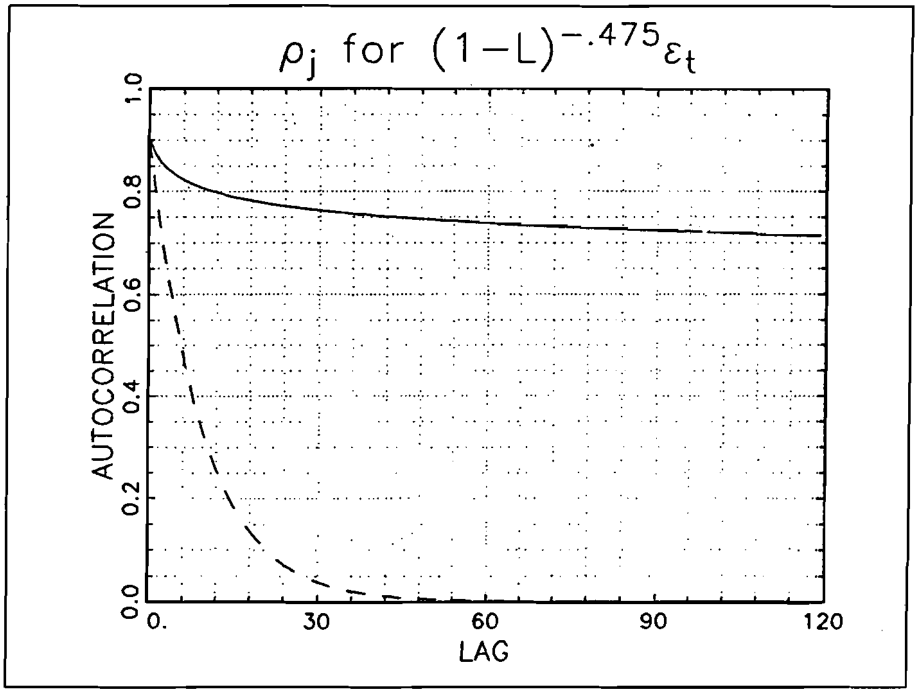
Source: Author's simulations.

Footnotes

1. For an estimate of income with a view to explaining consumption anomalies in the spirit of this section, see the interesting (independent) work of Diebold and Rudebusch (1989). Quah (1990) explains the paradox using permanent and temporary movements in income.

2. These weights define the Bartlett window. Newey and West (1987) enumerate the advantages of this specification.

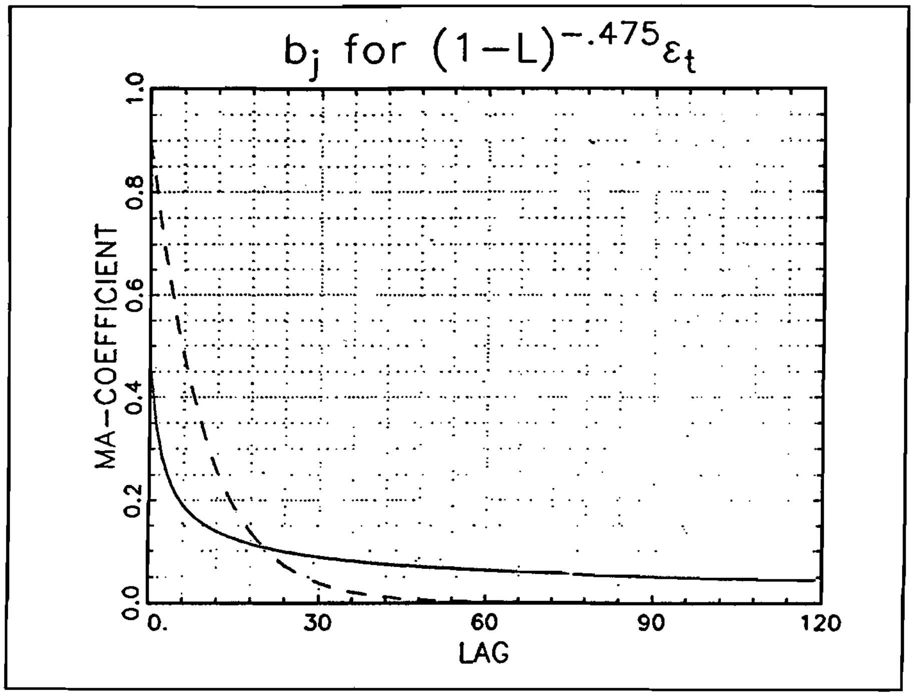
Figure 1



Autocorrelation functions of an AR(1) with coefficient 0.90 [dashed line] and a fractionally differenced series $X_t = (1-L)^{-d}\epsilon_t$ with differencing parameter $d = 0.475$ [solid line]. Although both processes have a first-order autocorrelation of 0.90, the fractionally differenced process decays much more slowly.

Source: Haubrich and Lo (1989).

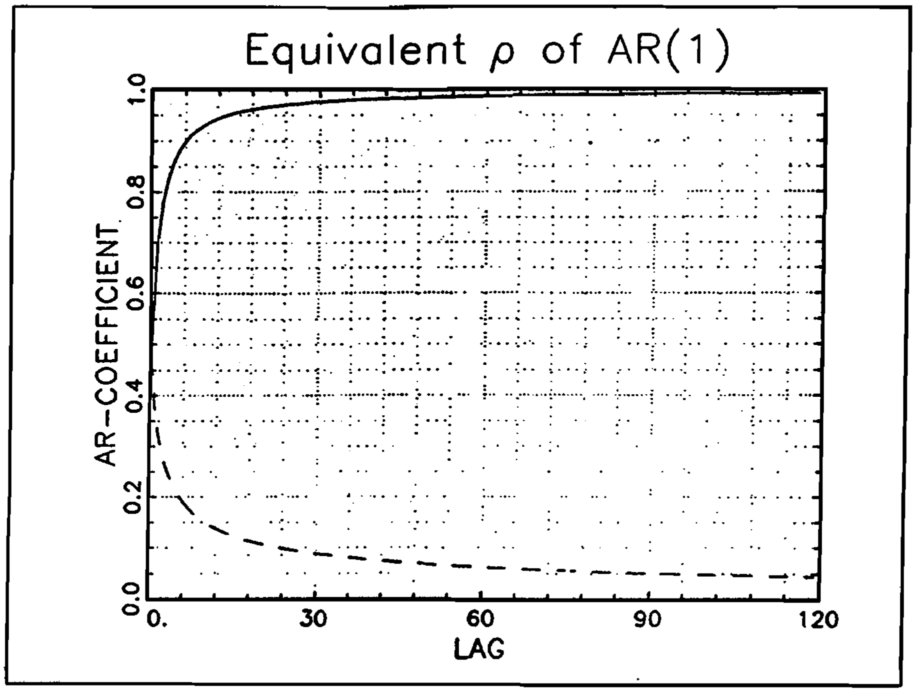
Figure 2A



Impulse response function [solid line] of the fractionally differenced time series $X_t = (1-L)^{-d}\epsilon_t$ for differencing parameter $d = 0.475$. For comparision, the impulse-response function of an AR(1) with autoregressive parameter 0.90 is also plotted [dashed lines].

Source: Haubrich and Lo (1989).

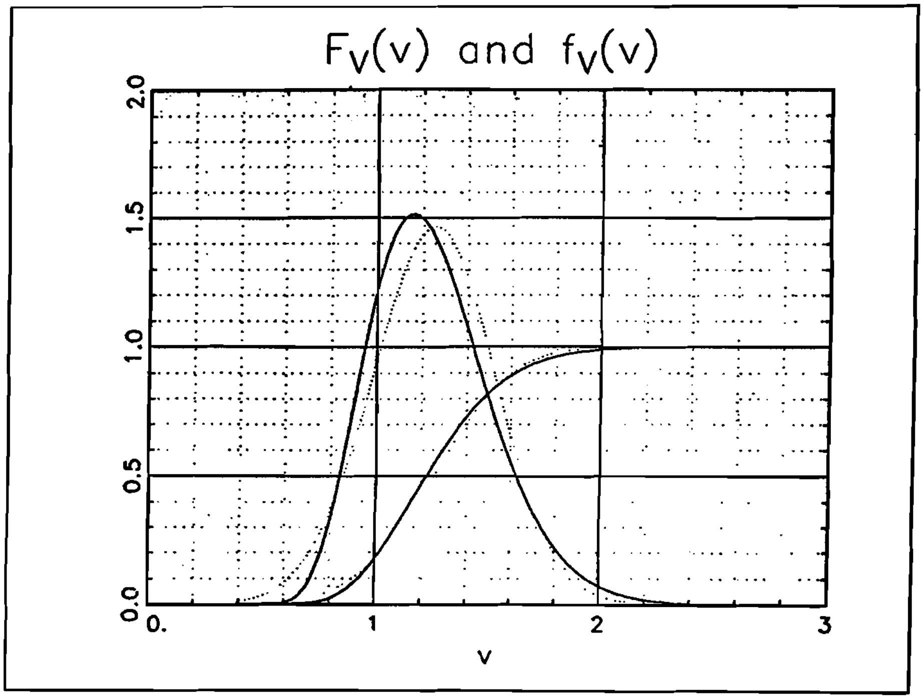
Figure 2B



Values of an AR(1)'s autoregressive parameter required to generate the same k-th order autocorrelation as the fractionally differenced series $X_t = (1-L)^{-d}\epsilon_t$ for differencing parameter $d = 0.475$ [solid line]. Formally, this is simply the k-th root of the fractionally differenced series' impulse-response function [dashed line]. For large k, the autoregressive parameter must be very close to unity.

Source: Haubrich and Lo (1989).

Figure 3



Distribution and density function of the range V of a Brownian bridge. Dashed curves are the normal distribution and density functions with mean and variance equal to those of V.

Source: Haubrich and Lo (1989).