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EXTENSION OF GRANGER CAUSALITY IN MULTIVARIATE TIME SERIES MODELS

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Abstract

This paper presents an example of a situation where Granger causality does not exist but an extended definition of causality does. The extended definition of causality is discussed, along with methods to determine its existence in multivariate time series models.

Key words: Granger causality, multivariate time series.

I. Introduction

The concept of Granger causality (Granger 1969) has become widely used in discussing relationships among variables. Some relevant references to Granger causality are Sims (1972), Haugh (1972), Pierce (1977), and Pierce and Haugh (1977). Generally, Granger causality has been discussed in terms of bivariate models.

This paper proposes an extension of Granger causality when more than two variables are used in a multivariate time series model and it is necessary to consider more than one-period-ahead forecasts.

Granger causality more appropriately may be called one-period-ahead forecasting ability. Variable z is said to "Granger cause" variable y, with respect to a given information set that includes z and y, if forecasts of present y are more accurate when using past values of z than when not doing so, all other available information being used in either case. The measure of accuracy usually used in the definition of Granger causality is the mean square error of one-period-ahead forecasts.

This idea can be expressed as follows: Let $\bar{A}_t$ be the given information set (including at least z_t and y_t), $\bar{A}_t = \{A_s: s < t\}$ $\sigma^2(y|B)$ be the minimum mean square error from forecasting y (one period ahead) given the information set B, and $\overline{A_t - z_t}$ be the set $\bar{A}_t$ without z. Then z is said to Granger cause y if

$$\sigma^2(y|\bar{A}) < \sigma^2(y|\overline{A-z}).$$

Thus, Granger causality refers to only one-period-ahead forecasts.

When forecasting for more than one period ahead, it is necessary to know whether Granger causality would include all possible causality. situations (in terms of forecasting ability). In section II is an example of a multivariate model for three variables (x, y, and z). This model demonstrates that while z may not Granger cause y, the two-period-ahead forecasts using z have a smaller forecast error than the forecasts not using z. Thus, Granger causality does not encompass all situations where one would conclude that some type of causality exists (in terms of forecasting ability). In section II, we also introduce an extension of Granger causality that includes the multiperiod, multivariate situation.

II. Extension of Granger Causality

When dealing with only two variables, Pierce (1975) proves that if better L-period-ahead forecasts for any L > 2 are produced by the addition of past z, then z must Granger cause y with respect to the set (z_t, y_t). However, as the following example demonstrates, this is not necessarily true for systems with more than two variables:

$$
\begin{aligned}
(1) \qquad & y_t = \phi_{12} x_{t-1} + a_{1t} \\
& x_t = \phi_{23} z_{t-1} + a_{2t} \\
& z_t = a_{3t},
\end{aligned}
$$

where the a_1, a_2, and a_3 are mutually orthogonal white noise processes with variances σ_1^2, σ_2^2, σ_3^2, respectively. In this model, $A_t = (y_t, x_t, z_t)$.

The minimum mean-square-error, one-period-ahead forecasts for this model are

$$
\begin{aligned}
(2) \qquad & \hat{y}_{t-1}(1) = \phi_{12} x_{t-1} \\
& \hat{x}_{t-1}(1) = \phi_{23} z_{t-1} \\
& \hat{z}_{t-1}(1) = 0,
\end{aligned}
$$

where $\hat{w}_{t-1}(1)$ is the one-period-ahead forecast of w at time $t-1$. These forecasts have mean square errors of σ_1^2, σ_2^2, and σ_3^2, respectively. (See Tiao and Box (1981) for a general discussion of how to calculate forecasts from these types of models and their mean square errors.)

If z is not included in the model, then the appropriate bivariate model can be derived from the model given in equation (1) by matching the variances, covariances, and cross covariances of y and x implied by this model. This reduced model is given by

$$
\begin{aligned}
(3) \qquad & y_t = \phi_{12} x_{t-1} + a_{1t} \\
& x_t = a'_{2t},
\end{aligned}
$$

where $\sigma'^2_2 = \phi_{23}^2 \sigma_3^2 + \sigma_2^2$.

The minimum mean-square-error, one-period-ahead forecasts from this model are

$$\hat{y}_{t-1}(1) = \phi_{12}x_{t-1}$$

$$\hat{x}_{t-1}(1) = 0,$$

with mean square errors of σ_1^2 and $\phi_{23}^2\sigma_3^2 + \sigma_2^2$, respectively.

We thus have shown that z does not Granger cause y (the mean-square-forecast error is σ_1^2 in both cases), but that z does Granger cause x ($\sigma_2^2 < \phi_{23}^2\sigma_3^2 + \sigma_2^2$). Similarly, we can show that x Granger causes y but not z. Also, y does not Granger cause x or z. The causality chain is thus an example of indirect causality (Tjostheim 1981) between z and y:

$$z \rightarrow x \rightarrow y,$$

where $\rightarrow$ means Granger causes.

When we examine the two-period-ahead forecasts, the result is different. The two-period-ahead forecasts from model 1 are

(4) $$\hat{y}_{t-1}(2) = \phi_{12}\ \hat{x}_{t-1}(1) = \phi_{12}\ \phi_{23}\ z_{t-1}$$
$$\hat{x}_{t-1}(2) = \phi_{23}\ \hat{z}_{t-1}(1) = 0$$
$$\hat{z}_{t-1}(2) = 0,$$

with mean square errors of $\sigma_1^2 + \phi_{12}^2\sigma_2^2$, $\sigma_2^2 + \phi_{23}^2\sigma_3^2$, and σ_3^2, respectively. (See Tiao and Box (1981) for a general discussion of how to calculate multiple-period forecasts and their mean square errors.)

For model 3, the two-period-ahead forecasts are given by

(5) $$\hat{y}_{t-1}(2) = \phi_{12}\ \hat{x}_{t-1}(1) = 0$$
$$\hat{x}_{t-1}(2) = 0,$$

with mean square forecast errors of $\sigma_1^2 + \phi_{12}^2(\sigma_2^2 + \phi_{23}^2 a_3^2)$ and $\sigma_2^2 + \phi_{23}^2\sigma_3^2$, respectively.

Thus, the mean square forecast error for y two periods ahead is less in the model including z (a + $\phi_{12}^2\sigma_2^2$) than it is in the model not including z ($\sigma_1^2 + \phi_{12}^2\sigma_2^2 + \phi_{12}^2\phi_{23}^2\sigma_3^2$). This illustrates the principle that even if a variable z does not Granger cause another variable y, z may be useful in forecasting y more than one period ahead.

This motivates the following extension of Granger causality. A variable z is said to cause another variable y, with respect to the set A, if

(6) $\sigma^2(y(L)|\bar{A}) < \sigma^2(y(L)|\overline{A-z})$ for any $L > 0$,

where $y(L)$ is the L-period-ahead forecast of y at time t-1. We call this type of causality <u>L-period causality</u>, where L is the smallest value so that inequality 6 is true. Thus, Granger causality is one-period causality.

The concept of L-period.causality is not the same as the idea that z is related to y with an L-period lag. Consider the model

$$y_t = \phi_{12} x_{t-1} + \phi_{13} z_{t-j} + a_{1t}$$
$$x_t = a_{2t}$$
$$z_t = a_{3t},$$

for some $j > 1$. In this model, y is related to z with .a j-period lag, but z Granger causes y; the value of j is immaterial. This also illustrates the idea that Granger causality does not necessarily involve only one-period lags.

III. Determining Multiperiod Causality

Given a multivariate time series model., we wish to determine what patterns of causality are represented by the model. One method of doing this (as demonstrated in the example in section II) is to compare the mean square forecast errors from the reduced models resulting from deleting one and only one variable with those of the full model for different forecast lengths. One advantage in doing this is to learn how much the mean square errors change. For example, we saw in section II that the two-period-ahead mean square error for y was reduced by $\phi_{12}^2\phi_{23}^2\sigma^2$ when z was included in the given model.

Consider the following model for n variables:

$$\begin{bmatrix} K & J \\ M & N \end{bmatrix} \begin{bmatrix} W \\ v \end{bmatrix} = \begin{bmatrix} \eta \\ \varepsilon \end{bmatrix},$$

where K (n-1 x n-1), J (n-1 x 1), M (1 x n-1), and N (1 x 1) are polynomial matrixes in B (where B is the backshift operator, i.e., $Bv_t = v_{t-1}$); W is the vector of n-1 variables excluding v; and η (n-1 x 1) and ε (1 x 1) are the corresponding error terms. If the variable v is omitted from this model, the resulting model is given by

$$(7) \qquad (K-JN^{-1}M)W = \eta-JN^{-1}\varepsilon.$$

(See Quenouille 1957, p. 43.) The autoregressive operators will be given by the right-hand side of equation (7). The moving average operators will have to be determined by combining the two sources of error η and . Once the submodels are determined, the mean square forecast errors for the submodels for different forecast lengths can be compared with the corresponding quantities of the full model to determine L-period causality. This method also could be used to determine the effect of Granger causality on the mean square error for one-period-ahead forecasts.

One disadvantage of this method is the number of submodels that must be determined. In general, if there are n variables in the model, then we must determine n-1 submodels, each of which has n-1 variables. The determination of these submodels is difficult when there are more than three variables in the model.

We now hypothesize an additional method of determining L-period causality. The method is presented without proof, but it is intuitively appealing. The hypothesis is that y is L-period caused by z for some L if there exists a chain of Granger causalities between z and y. If each Granger causality in the system involves only one lag, then we hypothesize that L is equal to the length of the minimum-length chain between z and y. However, if some of the lags involved in the Granger causality chain are longer than one, then L may be larger than this minimum, depending on the position of the longer lags. For example, consider the following model:

(8)
$$y_t = \phi_{12} x_{t-j} + a_{1t}$$
$$x_t = \phi_{23} z_{t-k} + a_{2t}$$
$$z_t = a_{3t}.$$

In section II, we saw that if $j = k = 1$, then z two period causes y. In general, it can be shown that for this model z $(1 + j)$ period causes y independently of k. Thus, the value of L depends not only on the length of the Granger causal chain but also on the lags involved and their location in the chain.

In terms of forecasting ability, L-period causality occurs when each variable in the chain is better forecast in the Granger sense using the previous variable. The L-period forecast is thus a better forecast because each variable in the chain is better forecast. That is, the L-period forecast of y depends on the forecast of x, which depends on the forecast of w, ... which depends on lagged z. In the example of equation (1), y is two-period caused by z, because the two-period-ahead forecast of y depends on the one-period-ahead forecast of x, which depends on lagged z.

The advantage of this method over the previous method is its ease of use. Determination of Granger causality is fairly easy (Tjöstheim 1981). Once the Granger causalities are ascertained, it is trivial to determine the chains of causalities. However, this method does not provide an indication of how much the L-period-ahead mean square error is reduced. Thus, we may have L-period causality with no practical significance. This would probably be true when L becomes large.

IV. Summary

Because Granger causality is determined only in terms of one-period forecasts, and because it is often necessary to forecast more than one period ahead, an extension of Granger causality is necessary in multivariate models.. We have presented an example that illustrates this idea and a proposal for an extension of Granger causality that addresses this problem. This extension involves L-period forecasting ability.

That is, the ability to forecast a variable for L periods ahead is improved by using another variable versus not using it in the same sense of Granger causality for one period ahead. We have provided methods for determining L-period causality when a model is known.

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