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REVISED
March 2011

WP 2007-12

A Nonparametric Analysis of Black-White Differences in Intergenerational Income Mobility in the United States

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Abstract

Lower intergenerational income mobility for blacks is a likely cause behind the persistent inter-racial gap in economic status in the US. However, few studies have analyzed black-white differences in intergenerational income mobility and the factors that determine these differences. This is largely due to the absence of appropriate methodological tools. We develop nonparametric methods to estimate the effects of covariates on two measures of mobility. We first consider the traditional transition probability of movement across income quantiles. We then introduce a new measure of upward mobility which is the probability that an adult child's relative position exceeds that of the parents. Conducting statistical inference on these mobility measures and the effects of covariates on them requires nontrivial modifications of standard nonparametric regression theory since the dependent variables are nonsmooth functions of marginal quantiles or relative ranks. Using NLSY data, we document that blacks experience much less upward mobility across generations than whites. Applying our new methodological tools, we find that most of this gap can be accounted for by differences in cognitive skills during adolescence.

Keywords: intergenerational mobility, upward mobility, nonparametric regression, Hadamard differentiability, black-white mobility gap; JEL codes: C14, D31

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1 Introduction

A topic of long standing interest among social scientists is the persistent disadvantage in economic status faced by blacks in the US many generations after the end of slavery and several decades after the elimination of state sanctioned segregation. The fact that the racial gap is so highly persistent suggests that blacks in the US may have low rates of upward intergenerational economic mobility. While there is a vast literature on the black-white earnings gap and a growing number of studies on intergenerational mobility (IGM) in the US, very few studies have examined differences in the rates of IGM between racial groups in the US and the underlying sources behind those differences. Such an analysis could potentially provide insight into why there is such persistence in racial inequality and whether this may reflect greater inequality of opportunity for blacks. A better understanding of the factors behind differences in IGM could also inform policies intended to address racial gaps such as early life interventions or affirmative action policies. This paper studies racial differences in relative income mobility over generations, using US data from the National Longitudinal Survey of Youth (NLSY) which contains large intergenerational samples of both blacks and whites.

The dearth of studies on racial differences in IGM is due, in part, to the fact that most recent research in this area has focused on using one particular measure, the intergenerational elasticity (IGE) which is simply the regression coefficient obtained by regressing (log) child's permanent income on (log) parents' permanent income.¹ The IGE provides a measure of income persistence, and 1 minus the IGE is widely used as a measure of relative mobility. However, the IGE cannot be used to compare mobility differences between population subgroups with respect to the *entire* distribution. For example, the IGE for blacks only describes the rate at which earnings among black children regress to the black mean –not the mean of the entire distribution.

An alternative approach is to calculate transition probabilities to describe the rates of movement across specific quantiles of the distribution over a generation. Since transition probabilities can measure the movements of blacks across the income distribution of the entire population comprising both blacks and whites, one can make meaningful statements concerning racial differences in mobility.

However, a difficulty arises with transition probabilities if one wants to estimate rates

¹The intergenerational correlation (IGC), used by many researchers, is qualitatively similar and the two measures are equivalent when the variance in income is unchanging across generations.

of IGM *conditional* on (continuous) covariates like test scores. For the IGE, it is straightforward to measure effects of covariates, one simply needs to include them along with their interactions with parents' income as additional regressors and the statistical theory is straightforward. In contrast, a formal statistical method for using covariates in transition matrices is lacking. The development of such a methodology would allow one to investigate the underlying mechanisms behind black-white differences in IGM. For example, it is often hypothesized that inadequate parental investment in children's human capital could lead to reduced mobility. Therefore it is natural to consider the association between IGM and measures of human capital such as education and test scores. Previous research using the NLSY has shown that cognitive skills during adolescence as measured by percentile scores on the Armed Forces Qualifying Test (AFQT) can account for black-white gaps in *levels* of educational attainment and adult wages (Cameron and Heckman, 2001; Neal and Johnson, 1996); so it would be useful to examine if this result extends to measures of *changes in relative position* across generations.

In order to address the above void in the literature, we develop in this paper a non-parametric statistical methodology for analyzing conditional transition probabilities. The relevant inference theory for *marginal* transition probabilities was previously developed in Formby, Smith and Zheng (2004). When the relevant covariates are discrete, one can simply apply their results within each covariate category to conduct inference on conditional transitions. But with continuously distributed covariates, the parameters of interest are infinite-dimensional and thus nonparametric smoothing methods are warranted.²³ Using a standard parametric model for conditional mobility (e.g., a probit) is problematic here because it is unclear what type of joint distribution of errors will imply a probit form for transition probabilities; in particular, a bivariate normal error distribution will not. Furthermore, a probit will be shown below to produce misleading qualitative conclusions in our empirical analysis.

²The AFQT percentile scores take on values from 1 to 99. Given our sample size, analysis by racial group within each percentile of AFQT will be very imprecise. Furthermore, small differences in AFQT percentiles, unlike differences in race, are unlikely to imply big changes in functional relationships. Therefore, treating AFQT score as a continuously distributed covariate will be the natural and correct approach.

³Trede (1998) has shown how continuous covariates can be included in analyzing a class of inequality reducing functionals that have been used to measure *intra-generational* mobility but not for measures based on transition probabilities.

While transition probabilities can be effectively used to compare relative mobility across subgroups, its substantive drawback is its overtly disaggregate nature, i.e., there are an infinite number of transition probabilities depending on which quantile is chosen to be the threshold for the sons, and a summary measure of mobility across *relative* income positions would be useful for consolidating the information provided in transition matrices. Therefore, we introduce a new alternative measure of upward mobility. Specifically, we condition on families where the father’s income is at or below a particular percentile (say, the median) and then estimate the probability that the income rank of a son randomly picked from such a family exceeds his parents’ rank in the prior generation. It is a single easily interpretable summary measure and its value does not depend on arbitrary discretization of income distributions. Further, as explained below, it is more robust than the transition probability to heterogeneity of income within the reference group of families that we started with. We conduct statistical inference on both the level of this mobility measure and the effects of covariates on that level, using NLSY data. Establishing the relevant distribution theory requires some methodological innovation because IGM measures involve outcome variables that contain non-smooth functions of initially estimated rank-type functionals. Standard nonparametric regression theory is therefore inadequate for this purpose.

Our first set of empirical results document the large racial disparity in transition probabilities out of the bottom of the income distribution. For example, we find that blacks are 26 percent less likely to move out of the bottom quartile than are whites. We then show that black-white differences in mobility are much smaller when based on our measure of upward mobility. This is because (i) our measure captures the fact that many blacks exhibit small upward movements in relative ranks which are ignored by transition probabilities and (ii) because black income levels below any given threshold tend to be smaller than white income levels, blacks need much larger absolute gains to surpass a common percentile threshold, leading them to have lower transition probability.

Our most striking finding, however, is that when we (nonparametrically) control for AFQT scores, most of the racial gap in IGM disappears— whether measured via transition probability or upward mobility. This finding suggests that policy makers should potentially place particular emphasis on interventions earlier in life that may determine cognitive ability differences by adolescence rather than strategies targeting adult labor markets. Indeed, growing evidence shows that black-white differences in tests scores can

be strongly affected by environmental influences (e.g. Neal and Johnson (1996); Hansen, Heckman, and Mullen 2004; Chay, Guryan, and Mazumder 2009) and do not simply reflect innate differences. Our results also extend the current literature which has focused on *levels* of economic status, by showing that test scores can also potentially explain racial differences in upward *movements* in economic status over generations.

We offer a few caveats to our study. Like most of the existing IGM literature, our analysis is essentially descriptive in nature and one should be cautious in attaching causal interpretations to them. Additionally, all measures used in the present paper are based on relative positions and do not pertain to movements out of absolute thresholds such as from under the poverty line.

The rest of the paper proceeds as follows: section 2 presents a review of some related papers on black-white differences in IGM; section 3 describes and discusses the parameters of interest; section 4 states the asymptotic distribution theory required for conducting statistical inference on these measures; section 5 describes the NLSY data; section 6 presents the empirical results and section 7 concludes. All technical proofs as well as a technical lemma (lemma 1) on Hadamard differentiability are collected in the appendix. Throughout the paper the symbol ":= " will denote equality by definition.

2 Related Literature

Research on intergenerational mobility (IGM) has been primarily motivated by the notion of equality of opportunity. Societies in which economic status in one generation is largely determined by economic status in the prior generation are thought to be less fair and provide less opportunity. Measures of mobility such as the intergenerational elasticity and transition probabilities are designed to capture the degree of persistence in economic status.⁴ Black and Devereux (2010) and Solon (1999) provide extensive surveys of research on IGM.

There is also an extensive literature on black-white differences in economic status. Fryer (2010) provides a review of this literature. These studies have typically had little to say about IGM in part, because of the lack of appropriate measures for studying group differences but also because most intergenerational samples of blacks are too small

⁴The interpretation of mobility measures as indices of "equality of opportunity" has been critiqued by several authors (e.g., Van de Gaer et al, 2001, Roemer (2004), Swift (2005), Jencks and Tach (2006)).

for inference. A few studies have addressed the racial dimension of the intergenerational transmission of status but did not estimate the rate of *income* mobility. For example Datcher (1981) using the PSID regressed adult outcomes on family background characteristics, separately by race and sex and finds that black families are not as successful as white families in translating parental economic gains into offspring achievement. Corcoran and Adams (1997) also use the PSID and compare how the probability of escaping poverty differs by race and sex and how these estimates are affected by adding covariates. They find that black children living in poverty are substantially more likely to be poor as adults and that structural economic factors during childhood (e.g. local unemployment rates) play a key role.

Hertz (2005) was the first to use transition probabilities to directly measure intergenerational income mobility by race. Using the PSID, Hertz estimated very large racial differentials in the probability of leaving the bottom quartile with blacks substantially less upwardly mobile. Using probit models, Hertz, found that these racial differences could not be explained by parental income or education. However, the use of a probit is not econometrically well-grounded since no natural joint distribution (such as a bivariate normal or log-normal) of underlying income variables implies a probit form for transition probabilities.⁵ Incidentally, in his study, Hertz provides some motivation for our construction of a summary measure of upward mobility. He points out that with transition matrices “the problem ... is that there is no best way to summarize their contents”.⁶

3 Parameters of interest

We first describe the parameters of interest based on transition probabilities and then those related to our new measure of upward mobility.

⁵Given the relatively smaller samples in the PSID and possible concerns about the quality of the data on blacks, it would be useful to produce similar estimates using the NSLY. Although Hertz makes maximum use of the PSID by including the oversample of poorer households there is some concern about the validity of the sample due to issues related to availability of the initial sampling frame (Lee and Solon, 2009). In addition there has been substantial attrition of black families in the PSID.

⁶Hertz (2008) develops an estimator that takes into account both the within- and between- group effects and finds that blacks have less mobility. However, it is not clear how to interpret the scale of the measure, whether it has desirable properties and can be used for producing conditional estimates.

3.1 Conditional transition probabilities

Let $F_0(\cdot)$ and $F_1(\cdot)$ denote the c.d.f. of the overall income distribution for fathers and sons, respectively. Then, the transition probability measures the probability that a son is at or above the s th quantile of F_1 , conditional on his father being between the s_1 th and s_2 th quantiles of $F_0(\cdot)$, i.e.,

$$\theta(s, (s_1, s_2)) = \frac{\Pr[F_1(Y_1) \geq s, s_1 \leq F_0(Y_0) \leq s_2]}{\Pr[s_1 \leq F_0(Y_0) \leq s_2]}. \quad (1)$$

Notice that $\theta(s, (s_1, s_2))$ can be decomposed by levels of discrete and continuous covariates X such as age and education of the father and/or the son as

$$\begin{aligned} & \theta(s, (s_1, s_2)) \\ = & \int \Pr[F_1(Y_1) \geq s | s_1 \leq F_0(Y_0) \leq s_2, X = x] dF(x | s_1 \leq F_0(Y_0) \leq s_2) \\ : & = \int \theta(x; s, (s_1, s_2)) dF(x | s_1 \leq F_0(Y_0) \leq s_2), \end{aligned}$$

where

$$\theta(x; s, (s_1, s_2)) = \Pr[F_1(Y_1) \geq s | s_1 \leq F_0(Y_0) \leq s_2, X = x], \quad (2)$$

is labeled the "conditional transition probability". Notice that F_1 and F_0 in (2) still refer to unconditional income distributions in the two generations. To interpret (2) directly, consider a numerical example. Suppose X denotes whether a father is black, $s = 0.5$, $s_1 = 0.4$, $s_2 = 0.6$. Suppose there are a total of 100 father-son pairs within the (0.4, 0.6) quantiles of the entire income distribution of the fathers' generation. Of these, suppose 40 are black and 60 are white. Suppose 10 of the 40 black sons earn above the s th percentile (in the sons' generation) and 30 of the 60 white sons do the same. Then, the overall transition probability is $\theta(s, (s_1, s_2)) = (10+30)/100=0.4$. The conditional transition probability among blacks is $\theta(black; 0.5, (0.4, 0.6)) = 10/40=0.25$ and that among whites is $\theta(white; 0.5, (0.4, 0.6)) = 30/60=0.5$. The white minus black difference is 0.25, suggesting higher mobility among whites.

One can analogously define transition probability conditional on more covariates, e.g., race and father's college attainment, and use it to measure the black-white difference in transition at each value of father's education:

$$\tilde{\theta}_B(z; s, (s_1, s_2)) - \tilde{\theta}_W(z; s, (s_1, s_2)),$$

where

$$\begin{aligned}\tilde{\theta}_B(z; s, (s_1, s_2)) &= \Pr[F_1(Y_1) \geq s | s_1 \leq F_0(Y_0) \leq s_2, Z = z, Black = 1] \\ \tilde{\theta}_W(z; s, (s_1, s_2)) &= \Pr[F_1(Y_1) \geq s | s_1 \leq F_0(Y_0) \leq s_2, Z = z, Black = 0].\end{aligned}\tag{3}$$

To interpret (3), refer to the previous numerical example and let $Z = 1$ denote that the father went to college. Now suppose that among the 40 black fathers, 10 went to college and 30 did not. Of the 10 black sons whose fathers went to college, suppose 6 surpassed the s th quantile, and suppose that of the 30 black fathers who did not attend college, only 4 sons surpassed the s th percentile. Similarly, suppose that among the 60 white fathers, 50 went to college of whom 25 sons surpassed the the s th quantile and among 10 white fathers who did not attend college, 5 sons surpassed the s th percentile. Then $\tilde{\theta}_W(\text{College}; 0.5, (0.4, 0.6)) = 25/50 = 0.5$ and $\tilde{\theta}_B(\text{College}; 0.5, (0.4, 0.6)) = 6/10 = 0.6$ and the difference is -0.1. These hypothetical numbers would suggest that although blacks in the (0.4, 0.6) class are less mobile than whites on average, the college educated blacks among them are in fact more mobile than college educated whites. In section 6 we will explore racial differences in conditional transition probabilities nonparametrically using actual data on sons' education and AFQT percentile scores.

3.2 Upward mobility

We formally introduce our new measure of upward mobility in this section. We first present the analytic expressions and then discuss the substantive features which make our measure both intuitively appealing and analytically different from measures based on transition probabilities.

Our direct measure of upward mobility is simply the probability that the son's percentile rank in the overall income distribution of his generation exceeds that of his parents' in the income distribution of the parents' generation by a fixed amount. Let Y_0, Y_1 denote father and son's income with respective marginal c.d.f.'s F_0 and F_1 . Then for fixed $0 < s_1 < s_2 < 1$, we define upward mobility for families between s_1 th and s_2 th quantiles by an extent $\tau \in [0, 1 - s_2]$ as

$$v(\tau, s_1, s_2) = \Pr(F_1(Y_1) - F_0(Y_0) > \tau | s_1 \leq F_0(Y_0) \leq s_2).\tag{4}$$

One can alternatively define upward mobility by conditioning on $F_0(Y_0) = s$ rather than $s_1 \leq F_0(Y_0) \leq s_2$, and thus avoid aggregation bias due to income heterogeneity

within the interval (s_1, s_2) . However, conditioning on $F_0(Y_0) = s$ would require additional smoothing since $F_0(Y_0)$ would be continuously distributed. By making the length of the interval (s_1, s_2) small, we would both avoid this smoothing and yet remove some of the aggregation bias. Furthermore, defining the reference group to be $s_1 \leq F_0(Y_0) \leq s_2$ is also consistent with how transition probabilities have been traditionally defined in the literature and thus enables direct comparison and contrast between the two measures.⁷

In analogy with (2) above, one may introduce covariates X into the analysis and define conditional upward mobility at $X = x$ as

$$\begin{aligned} & v_c(x; \tau, s_1, s_2) \\ = & \Pr(F_1(Y_1) - F_0(Y_0) > \tau | s_1 \leq F_0(Y_0) \leq s_2, X = x). \end{aligned} \quad (5)$$

The idea is that we start with all families where the father was between the s_1 th and s_2 th percentile. This ensures that all the corresponding sons have equal “space to move up”. With these families constituting our population, we then evaluate the extent of upward mobility for different groups defined by values of X . Below, we derive the statistical distribution theory for estimates of $v(\tau, s_2, s_1)$ and $v_c(x; \tau, s_2, s_1)$. In section 6 we will contrast overall upward mobility among blacks versus whites and then analyze how controlling for relevant covariates affects this difference.

Contrasting upward mobility with transition probability: A key feature of $v(\tau, s_1, s_2)$ is that it counts the sons’ small upward movements in relative positions from their fathers’ which are ignored by transition probabilities. Comparing upward mobility

$$v(\tau, s_1, s_2) = \Pr(F_1(Y_1) > F_0(Y_0) + \tau | s_1 \leq F_0(Y_0) \leq s_2)$$

and the transition probability

$$\theta(s_2, (s_1, s_2)) = \Pr[F_1(Y_1) > s_2 | s_1 \leq F_0(Y_0) \leq s_2],$$

one can see that unlike transition probability, $v(\tau, s_1, s_2)$ is counting those sons whose ranks exceeded their fathers’ by τ but did not necessarily exceed s_2 . The resulting magnitude of difference between the two measures, however, is an empirical question and depends on the joint distribution of (Y_0, Y_1) . Below, we graph these differences for the the lognormal case.

⁷Conditioning $F_0(Y_0) = s$, will require averaging over a subjectively determined bandwidth around s , and thus entertain a certain amount of income heterogeneity in finite samples. In addition, the inference theory will become far more complicated. So we do not pursue this option here.

An advantage of our upward mobility measure can be readily seen when comparing mobility between, say, whites versus blacks. Suppose we condition on fathers with incomes at or below the median income in the fathers' generation., i.e., $s_1 = 0$ and $s_2 = 0.5$. Suppose this interval is $(0, \$40k)$. Then it is reasonable to expect that black incomes are concentrated in the lower end of $(0, \$40k)$ and white incomes in the upper end (see fig. 5 below which compares the C.D.F. of whites and black parental income in the bottom quintile). Now, the transition probability counts only those sons who exceed the median income for the sons' generation so that black sons have to make a much larger income gain than white sons to be counted as having made progress. Our upward mobility measure corrects this by requiring all sons to have advanced by (at least) the same amount τ w.r.t. their fathers' percentile rank in order for them to be counted as having progressed.

The extent of movement τ controls how much we want to include small movements in relative position. With $\tau = 0$, every positive movement is counted, however small. As τ rises, we count only larger movements in relative position but keep this *extent* of movement the same for all subgroups of the population. This implies, however, that we are applying different *absolute* thresholds to sons of different subgroups, e.g., black sons will be counted as having moved up even if their rank has not crossed an absolute threshold as long as they have made sufficient progress relative to their own fathers. In contrast, the transition probability keeps the absolute threshold constant, and hence counts wealthier subgroups with smaller extent of movements but ignores some poorer subgroups with larger extent of movement from their parents' rank.

As we will show in section 6, black-white differences in mobility (c.f., figure 4) are in fact, quite different, depending on which measure is used. Specifically, we find that whites appear to be much more upwardly mobile relative to blacks when measured by the transition probability of moving out of a given quantile. The white-black difference in mobility, however, is much smaller when measured in terms of our upward mobility index. The key reason for this is that many black sons make relatively small upward movements which are missed by $\theta(s_2, (s_1, s_2))$ but are captured by $v(\tau, s_1, s_2)$. Incomes of white fathers tend to be larger than that of black fathers (figure 5) below virtually all fixed percentile thresholds so that black sons need a larger increase in absolute income to be counted by the transition probability measure.

Lognormal illustrations: To gain some concrete insight into these measures, denote incomes for father and son by (Y_0, Y_1) and assume that incomes are jointly log-normal

with $\text{corr}(\ln Y_0, \ln Y_1) = \rho \geq 0$. Then calculation of (2) and (5) in this lognormal model (note that these measures are invariant to mean and variance changes) yields, for example,

$$v(\tau, 0, s) = \frac{1}{s} E \left[1(Z < \Phi^{-1}(s)) \times \Phi \left(\frac{\rho Z - \Phi^{-1}(\Phi(Z) + \tau)}{\sqrt{1 - \rho^2}} \right) \right],$$

$$\theta(s, (0, s)) = \frac{1}{s} E \left[1(Z < \Phi^{-1}(s)) \times \Phi \left(\frac{\rho Z - \Phi^{-1}(s)}{\sqrt{1 - \rho^2}} \right) \right],$$

where $Z \sim N(0, 1)$ and Φ, ϕ denote its c.d.f. and density, respectively. Note that ρ represents the intergenerational correlation that is typically estimated to be in the 0.2 to 0.5 range for industrialized countries. Appendix Figure A1 plots the transition probability (black line), upward mobility (gray line) and the difference between them (dotted line) as functions of ρ , for $\tau = 0.05$ (panel A), 0.20 (panel B) and 0.50 (panel C). The graphs show that: (i) both mobility measures decline from 1 to 0 as ρ rises from very negative (fortune reversal) to very positive (high persistence); (ii) when τ is closer to 0 as in panel A, the upward mobility measure is including smaller gains in ranks and is thus higher than the transition probability for fixed ρ ; (iii) for any fixed τ , the difference between upward mobility and transition probability is close to zero when ρ is close to -1, rises with ρ , reaches a peak, then declines as ρ increases.

When the population is a mixture of several subgroups, e.g., blacks and whites, and the difference in the two measures becomes analytically intractable. However, one can surmise from the single group analysis that if blacks are poorer than whites within every quantile class but have similar values of ρ , then the difference between the upward mobility measure and the transition probability for blacks will be higher than the corresponding difference for whites. So the white-black difference in transition probabilities will be higher than that for upward mobility. This effect may be either enhanced or mitigated if the intergenerational correlation in income for whites exceeds that for blacks (note that in Figure A1, the difference in the two measures first rises and then falls as ρ increases). The intergenerational correlation in our data for blacks and whites are quite similar at around 0.25 to 0.3 for each.⁸ Therefore our empirical finding of higher inter-racial differences in transition probability is most likely due to the first reason, viz., the inter-racial differences in income distribution below the reference threshold. This suggests that upward mobility

⁸These values are lower than what is typically found in the literature. This is mostly because the sample is not conditioned on having a minimum number of years of parent income (e.g. 5 year averages) as in Solon (1992) to address attenuation bias.

is a useful alternative, or at least complementary, measure to consider rather than relying on just transition probabilities for the quantitative assessment of income mobility.

Alternative definition of mobility: One can alternatively define overall mobility based on transition matrices after incorporating effects of covariates. Consider a transition matrix based on an arbitrary M -class discretization of the marginal distributions of Y_0 and Y_1 : $\tilde{\Theta} = \left\{ \tilde{\theta}(j, k) \right\}_{j, k=1, \dots, M}$. Then Shorrocks's (1978) measure of *unconditional* mobility is given by

$$M_1 = \frac{K - \text{trace}(\tilde{\Theta})}{K - 1} = 1 - \frac{\sum_{j=1}^K \tilde{\theta}(j, j) - 1}{K - 1}.$$

One can incorporate covariates into the above formula and define

$$M_1(x) = 1 - \frac{\sum_{j=1}^K \tilde{\theta}(j, j; x) - 1}{K - 1} \quad (6)$$

where

$$\tilde{\theta}(j, j; x) = \Pr(\zeta_j \leq Y_1 \leq \zeta_{j+1} | \xi_j \leq Y_0 \leq \xi_{j+1}, X = x),$$

and ζ_j and ξ_j denote the j th marginal quantiles of Y_1, Y_0 , respectively. Given the simple linear relation (6), inference on $M_1(x)$ will follow straightforwardly from inference on $\tilde{\theta}(j, j; x)$. However, this measure will depend crucially on the discretization employed which is clearly an undesirable feature. Altering the above formulas to allow for a continuous transition matrix seem complicated⁹ and we leave that to future research.

3.3 Measurement Error

Researchers working on earnings mobility have paid particular attention to measurement error in sons' and fathers' earnings in the context of intergenerational regressions (c.f., Haider and Solon, 2006).¹⁰ It is interesting to note that all our measures of mobility are based on the relative positions of individuals in the population. So if ranks of individuals are preserved despite measurement errors, then our measures will not be affected by the fact that we have erroneous earnings measures. One specific example of this is where reported earnings are a monotone function of true earnings— i.e. if for two people 1 and

⁹The problem is that $\int_0^1 \theta(s, s) ds$ is not a probability, unlike $\sum_{j=1}^K \tilde{\theta}(j, j)$.

¹⁰We are aware of only one study that has examined the effect of measurement error in the context of transition probabilities. O'Neill et al (2007) show that measurement error can induce a modest bias in transition probabilities compared to regressions and that this bias may vary at different points of the distribution.

2, true incomes satisfy $y_1^* > y_2^*$, then their reported incomes satisfy $y_1 > y_2$. This can be easily consistent with non-classical measurement error, i.e., $y - y^*$ being negatively correlated with true earnings y^* (Bound et al (1994)). In this case, all our measures based on reported y will be identical to those based on y^* . With more general types of measurement error, using time averaged incomes or earnings, as is common in the IGM literature, can partially mitigate the effect of purely random measurement error in addition to providing more reliable estimates for permanent income. This is the approach we follow in the application below. Finally our measures of son's earnings are taken around the age of 40 when life-cycle bias (Haider and Solon (2006)) is minimized.

Based on the parameters (2) and (5), one can define the corresponding marginal effects by differentiating them with respect to the regressor values and/or summarize these by density-weighted average derivatives, a la Powell, Stoker and Stock (1989). For brevity, we do not pursue these quantities here.

4 Estimation and distribution theory

We now turn to estimation of the parameters and derivation of their asymptotic properties. Note that we have defined 4 parameters above, viz., (1), (2), (4), (5). Formby, Smith and Zheng (2004) had analyzed only (1) and so, in what follows, we will derive the distribution theory for the other three. The analysis of (2) requires slight modification of standard kernel regression theory. We only provide an outline of proof by pointing out the modifications needed. But the estimators of (4) and (5) are fundamentally harder to analyze owing to the presence of $\hat{F}_1(\cdot)$ and $\hat{F}_0(\cdot)$ in the definition of the dependent variables. Our analysis of these will rely crucially on the idea of Hadamard differentiability and we will use Hoeffding's inequality to control the errors involved in the estimation of $\hat{F}_1(\cdot)$ and $\hat{F}_0(\cdot)$.

Given that the support of income variables can be taken to be bounded below, without loss of generality, we will assume that the supports of Y_0 and Y_1 are subsets of $[1, \infty)$. Note that all our mobility measures are based on quantiles and so fixed location shifts in either variable does not affect any of the measures. For fixed s, t , denote the s th quantile of Y_1 by ζ_1 and the t th quantile of Y_0 by ζ_0 with corresponding estimates by $\hat{\zeta}_1, \hat{\zeta}_0$. For transition probability, W will denote the indicator $1\{Y_1 \leq \zeta_1, Y_0 \leq \zeta_0\}$ and for upward mobility, W will denote $1(F_1(Y_1) - F_0(Y_0) > \tau)$ and V will denote $1\{F_0(Y_0) \leq s\} := 1\{Y_0 \leq \zeta_0\}$ in

both cases.

4.1 Conditional transition probability

We will first state the distribution theory for estimating the conditional transition probability, c.f., (2)

$$\theta(x; s, t) = \Pr[Y_1 \geq \zeta_1 | Y_0 \leq \zeta_0, X = x] = 1 - \frac{\Pr[Y_1 \leq \zeta_1, Y_0 \leq \zeta_0 | X = x]}{\Pr[Y_0 \leq \zeta_0 | X = x]}. \quad (7)$$

Now, (7) can be estimated by

$$\hat{\theta}(x; s, t) = 1 - \frac{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) \mathbf{1}(Y_{1i} \leq \hat{\zeta}_1, Y_{0i} \leq \hat{\zeta}_0)}{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) \mathbf{1}(Y_{0i} \leq \hat{\zeta}_0)}, \quad (8)$$

where $K(\cdot)$ is a d -dimensional kernel and σ_n is a sequence of bandwidths. Let $f(y_0, y_1 | x)$ denote the density of (Y_0, Y_1) conditional on $X = x$ and define

$$\begin{aligned} \phi(x, \zeta_0, \zeta_1) &= \Pr\{Y_1 \leq \zeta_1, Y_0 \leq \zeta_0 | X = x\} = \int_1^{\zeta_1} \int_1^{\zeta_0} f(y_0, y_1 | x) dy_0 dy_1, \\ \hat{\phi}(x, \hat{\zeta}_0, \hat{\zeta}_1) &= \frac{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) \mathbf{1}(Y_{1i} \leq \hat{\zeta}_1, Y_{0i} \leq \hat{\zeta}_0)}{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right)}, \\ \hat{\phi}(x, \zeta_0, \zeta_1) &= \frac{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) \mathbf{1}(Y_{1i} \leq \zeta_1, Y_{0i} \leq \zeta_0)}{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right)}. \end{aligned}$$

The asymptotic distribution for conditional (on covariates) transition probabilities are based on the following proposition. We state this and subsequent propositions in terms of a d -dimensional X all of whose components are continuously distributed. For discrete covariates, the analysis is identical to that for the marginal (i.e. unconditional) measures.

Proposition 1 *Suppose that conditions **NW1-5** in the appendix part A are satisfied. Assume further that for $X = x$, (Y_0, Y_1) admits a nonnegative joint density w.r.t. the Lebesgue measure everywhere on the joint support. Then we have*

$$(n\sigma_n^d)^{1/2} \left(\hat{\phi}(x, \hat{\zeta}_0, \hat{\zeta}_1) - \hat{\phi}(x, \zeta_0, \zeta_1) \right) = o_p(1).$$

Proof. See appendix part A. ■

The implication is that the distribution of $\hat{\phi}(x, \hat{\zeta}_0, \hat{\zeta}_1)$ and that of the infeasible estimator $\hat{\phi}(x, \zeta_0, \zeta_1)$ are identical. The argument is based on the observation that $(\hat{\zeta}_0, \hat{\zeta}_1)$ converges at the parametric \sqrt{n} rate but $\hat{\phi}(x, \zeta_0, \zeta_1)$ converges to $\phi(x, \zeta_0, \zeta_1)$ slower than \sqrt{n} -rate and a standard equicontinuity argument can then be used to handle the non-smoothness of $1(Y_1 \leq \zeta_1, Y_0 \leq \zeta_0)$ in the ζ 's. Note also that through our assumptions, we have used an "undersmoothed" estimator to achieve bias reduction and omitted bounded moment assumptions on the errors because the dependent variable and $\phi(\cdot, \cdot, \cdot)$ lie in $[0, 1]$.

An exactly analogous proposition with a virtually identical proof applies to

$$\hat{\psi}(x, \hat{\zeta}_0) = \frac{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) 1(Y_{0i} \leq \hat{\zeta}_0)}{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right)}.$$

Returning to (8), we note that $\hat{\theta}(x; s, t) = \hat{\phi}(x, \hat{\zeta}_0, \hat{\zeta}_1) / \hat{\psi}(x, \hat{\zeta}_0)$ and its asymptotic distribution follows by the standard delta method and the Cramer-Wold device.

4.2 Marginal upward mobility

We now consider the notationally simpler version of $v(\tau, s)$ defined in (4)¹¹

$$v(\tau, s) = \Pr(F_1(Y_1) - F_0(Y_0) > \tau | F_0(Y_0) \leq s) \quad (9)$$

which can be estimated by

$$\hat{v}(\tau, s) = 1 - \frac{\frac{1}{n} \sum_{i=1}^n 1\left(\hat{F}_1(y_{1i}) \geq \hat{F}_0(y_{0i}) + \tau\right) 1\left(\hat{F}_0(y_{0i}) \leq s\right)}{s} \quad (10)$$

where

$$\hat{F}_1(y_{1i}) = \frac{1}{n} \sum_{j \neq i} 1(y_{1j} \leq y_{1i}).$$

We will now state the asymptotic distribution of $\hat{v}(\tau, s)$. Let $F(\cdot, \cdot)$ denote the joint c.d.f. of (Y_0, Y_1) with corresponding joint density $f(\cdot, \cdot)$. Then for fixed s, τ , one may view $v(\tau, s)$ as a functional $v(F)$. We can therefore estimate it by $v(\hat{F})$, where \hat{F} denotes the usual empirical c.d.f.. We will obtain a large sample distribution of $v(\hat{F})$. The key step is showing that the functional $F \mapsto v(F)$ is smooth in the Hadamard sense,

¹¹One can move from (9) to (4) using simple subtractions.

with a derivative at F given by a linear functional $v'_F(\cdot)$.¹² This is done in lemma 1 in appendix part B and the relevant tail conditions and the proof are stated there as well. If one assumes that the joint density of (Y_0, Y_1) is bounded away from zero on a compact support, then the proof is considerably simpler. This assumption would, however, exclude families with "abnormally" high and low earnings in either generation—which is typically where the density will be close to zero – and this is clearly undesirable. So we establish Hadamard differentiability under more general tail conditions on the joint density and its partial derivatives.

Given this lemma, the asymptotic distribution of $\hat{v} = v(\hat{F})$ can be derived as follows. Let \rightsquigarrow denote standard weak convergence of distribution functions and define the Gaussian process \mathcal{G} by $\sqrt{n}(\hat{F} - F) \rightsquigarrow \mathcal{G}$. Then from lemma 1 and the functional delta method, we have that

$$\sqrt{n}(\hat{v} - v_0) \xrightarrow{d} v'_F(\mathcal{G}),$$

whence $v'_F(\mathcal{G})$ will be distributed as a univariate zero mean normal. See technical appendix part B for the exact form of this distribution.

It is well-known that the bootstrap provides consistent approximations to the asymptotic distribution of the sample c.d.f. process $\sqrt{n}(\hat{F} - F)$.¹³ Using the Hadamard differentiability result of our lemma 1, it follows, via the functional delta method for the bootstrap in probability (c.f., van der Vaart and Wellner (1996), theorem 3.9.11), that bootstrapping will lead to consistent approximation of the distribution of the estimator of \hat{v} . We summarize the above discussion in the following proposition, where "regularity conditions A and B" refer to those in lemma 1 in the appendix part B and \hat{v}^* denotes the bootstrap version of \hat{v} .

Proposition 2 *Under regularity conditions A and B, the bootstrap distribution of $\sqrt{n}(\hat{v}^* - \hat{v})$ will consistently estimate the distribution of $\sqrt{n}(\hat{v} - v_0)$.*

Proof. Follows from Hadamard differentiability of the map $F \mapsto v(F)$ (lemma 1 in appendix B) by applying the functional delta method. See Appendix part B for details.

¹²The concept of Hadamard differentiability has been used before in the context of analyzing features of *univariate* income distributions, c.f. Bhattacharya (2007) and Barrett and Donald (2000). The results obtained here involve more complicated functionals of bivariate distribution functions and are not related to the results in the above papers.

¹³For a textbook treatment, see theorem 3.6.1 part (iii) in van der Vaart and Wellner (1996) and its discussion on page 346 of the same text.

■

In the application discussed below, we use the bootstrap to approximate standard errors for the marginal upward mobility by race (table 2) and for mobility by race and parent income (table 4). We also provide a histogram for the bootstrap distribution (figure 3) and summarize some descriptive measures pertaining to the distribution, such as moments, skewness and kurtosis (table 3). Standard tests fail to reject normality of the distribution, as is to be expected, given the Gaussian form of the ingredients of $v'_F(\mathcal{G})$.

4.3 Conditional upward mobility

Recall from (5) that conditional upward mobility is given by

$$v_c(\tau, s; x) = \Pr(F_1(Y_1) - F_0(Y_0) > \tau | F_0(Y_0) \leq s, X = x).$$

Its natural estimates then is

$$\begin{aligned} & \hat{v}_c(\tau, s; x) \\ = & \frac{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) \mathbf{1}\left(\hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \hat{F}_0(Y_{0i}) \leq s\right)}{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) \mathbf{1}\left(\hat{F}_0(Y_{0i}) \leq s\right)}. \end{aligned}$$

Recall the definitions $W := \mathbf{1}(F_1(Y_1) - F_0(Y_0) > \tau)$ and $V := \mathbf{1}\{F_0(Y_0) \leq s\}$.

Proposition 3 *Suppose the data (X_i, Y_{1i}, Y_{0i}) for $i = 1, \dots, n$, are i.i.d. and assumptions **NW1-4**, **NW5'** in the technical appendix part C hold. Then, we have that*

$$\begin{aligned} & (n\sigma_n^d)^{1/2} [\hat{v}_c(\tau, s; x) - v_c(\tau, s; x)] \\ = & \frac{1}{E(V|X=x)} \times (n\sigma_n^d)^{1/2} \times \left\{ \hat{E}(W|X=x) - E(W|X=x) \right\} \\ & - \frac{E(W|X=x)}{\{E(V|X=x)\}^2} \times (n\sigma_n^d)^{1/2} \times \left\{ \hat{E}(V|X=x) - E(V|X=x) \right\} \\ & + R_n(x), \end{aligned}$$

where $\hat{E}(W|X=x)$ and $\hat{E}(V|X=x)$ denote respectively the Nadaraya-Watson regression estimates of W and V on X and the remainder $R_n(x)$ is $o_p(1)$.

Asymptotic normality now follows by standard arguments for Nadaraya-Watson regressions, e.g., Bierens (1994), theorem 10.2.1 whose conditions are implied by assumptions **NW1-4**, **NW5'**.

Proof. Appendix part C ■

In our empirical application, we will want to compare the entire curve of the AFQT-conditioned mobility for blacks with that of whites. Consequently, we need to construct uniform confidence bands on these (difference in) regression curves, using the decomposition in proposition 3. This corresponds to testing the null hypothesis that the black-white difference in mobility conditional on $AFQT = x$ is zero for every x . Based on proposition 3 and strengthening $R_n(x) = o_p(1)$ to $\sup_x |R_n(x)| = o_p(1)$ (see appendix part D), we can apply theorem 4.3.1 of Hardle, 1990 to the empirical process $(n\sigma_n^d)^{1/2} [\hat{v}_c(\tau, s; \cdot) - v_c(\tau, s; \cdot)]$ and construct uniform 95% confidence bands which will, asymptotically, contain the true curve $v_c(\tau, s; x)$ with probability 95%. The details of this construction and applicability of Hardle's theorem are outlined in the appendix, part D.

5 Data

We use the National Longitudinal Survey of Youth (NLSY) which starts with a sample of individuals who were between the ages of 14 and 21 as of December 31st, 1978. Respondents were interviewed annually through 1994 and every other year since then. To avoid dealing with issues related to labor force participation, we focus only on men in this study. The NLSY has largely not been used for intergenerational analysis because parents of respondents were not interviewed. However, the parents of sons who were living at home with their children during the first few years of the survey did report total family income. Therefore, for those sons we use their average family income for any available years between 1978 and 1980. We also measure the sons' average annual earnings as adults for any of the available years covering 1996, 1998, 2000 and 2002.¹⁴ This allows us to include individuals even if income data is missing in some years for either generation. The time averaging also provides a better measure of permanent income in both generations. Our sample restrictions lead to a sample of 2766 white and black men.

A key covariate for our analysis is the Armed Forces Qualifying Test (AFQT). All individuals in the NLSY were given the AFQT test in 1980 as part of the renorming of the test.¹⁵ For education we use years of completed schooling by age 26.

¹⁴All income variables are deflated to 1978 dollars using the CPI-U.

¹⁵Following Neal and Johnson (1996) we use the 1989 version of the percentile score.

6 Results

Our estimates use the two measures described earlier: transition probabilities and upward mobility. We show both unconditional estimates and estimates conditional on AFQT.

6.1 Marginal probabilities

6.1.1 Upward transition probabilities

We begin by showing estimates of transition probabilities. We have simplified the notation from (1) to use a common cutoff, s , in both generations. To facilitate comparisons with the upward mobility measure we have introduced in this paper, we also consider transition probabilities where the son must surpass the quantile by the amount τ , viz., $\Pr [F_1(Y_1) > s + \tau | F_0(Y_0) \leq s]$. Confidence intervals for these are calculated using analogs of the analytical formulae in Formby, Smith and Zheng (2004).

The results are shown in Table 1. In the first set of three columns we produce separate estimates for whites, blacks, and the white-black difference for the baseline case where $\tau = 0$. In the subsequent sets of columns we allow τ to vary from 0.1 to 0.3. In each row we condition on parent income being below the s percentile where s goes from 0.05 to 0.5 in increments of 0.05. The racial differences are striking. For example, the baseline transition probability out of the bottom quartile is 71 percent for whites but only 45 percent for black, or a 26 percentage point difference. We plot the transition probabilities for whites and blacks along with the pointwise 95 percent confidence intervals in Figure 1. As is evident in the chart, except for those at the very bottom of the distribution (below the 5th percentile), blacks are significantly less likely to surpass the quantile thresholds.

Interestingly, the white-black difference in the transition probability out of the bottom quartile does not change very much as we allow τ to vary.¹⁶ When we condition on parents that are at or below the median and allow τ to be large (0.2 to 0.3) then we find that the interracial mobility gap begins to narrow to a smaller, but still significant, 10 percentage point difference.

¹⁶For example the racial gap in the probability of rising from the bottom quartile to at least the 45th percentile (i.e. $\tau = 0.2$) is 23 percentage points.

6.1.2 Upward mobility

We now show an analogous set of estimates of our upward mobility measure

$$\Pr(F_1(Y_1) - F_0(Y_0) > \tau | F_0(Y_0) \leq s)$$

for whites and blacks and the white-black difference in Table 2. We now find much smaller racial differences in our baseline case ($\tau = 0$). For example, among white men whose family income during their youth was below the 25th percentile, 84 percent achieved a higher percentile than their parents. The comparable figure for black men is 76 percent implying a difference of about 8 percentage points. Figure 2 plots the results along with pointwise 95% confidence bands in Figure 2. To calculate pointwise confidence intervals for mobility v , we compute the sample analog \hat{v} and then draw 200 bootstrap resamples from our sample. The use of the bootstrap is justified via the functional delta method, discussed above.¹⁷ For each bootstrap resample, we calculate the corresponding estimate, v^* and the statistic $t^* = \sqrt{n} |v^* - \hat{v}|$. We then calculate z^* , the 95th percentile of t^* , and use $\left(\hat{v} - \frac{z^*}{\sqrt{n}}, \hat{v} + \frac{z^*}{\sqrt{n}}\right)$ as our confidence interval. We calculate the standard errors shown in the table, σ_v , by taking the standard deviation of v^* . The histogram for the bootstrap distribution of $(v^* - \hat{v})$ is plotted in figure 3 for the case of whites where $s = 0.25$ and $\tau = 0.2$. We report the summary statistics (e.g. skewness, kurtosis and tests for normality) for various values of s and τ in Table 3. Because the histograms do not look perfectly symmetric, we also calculated equal-tailed confidence intervals. Since we found no consistent pattern in the relative size of the confidence intervals between the symmetric and the equal-tailed ones, we chose to report the symmetric ones.

As figure 2 shows, aside from those whose family income was at or below the fifth percentile, whites experience greater upward mobility than blacks but not nearly as much as implied by transition probabilities. The gap in most cases, however, is statistically significant as is shown in figure 4 where we plot the white minus black difference for both the transition probability and the upward mobility along with confidence bands.

Among poorer families there are many blacks who exceed their parents rank in the distribution but do not surpass them by enough to move across specific quantiles. As discussed in section 3, the fact that the white distribution of parent income lies to the

¹⁷While studentization may be preferable before bootstrapping for higher order refinements, it is quite challenging to simulate the distribution stated in theorem 1 and so we simply use the unstudentized version.

right of blacks over most of the support, makes it more likely that whites will surpass the quantile thresholds more easily. This is illustrated in Figure 5 which plots the c.d.f.’s of the parent income distribution for both blacks and whites. This implies that if blacks and whites below the threshold experienced equal sized percentile gains, then the transition probabilities would generally be higher for whites.¹⁸

The remaining columns of Table 2 show the comparable results as τ varies from 0.1 to 0.3. In each case, the magnitude of the black white difference is generally between 15 and 25 percentage points and does not change too much as s changes. These results are comparable to the upward transition probability results in Table 1 and suggest that the two measures produce roughly similar results for larger values of τ .

Thus far we have used progressively larger samples as we increase s . This “cumulative” approach could obscure patterns that might arise if we focused more finely on specific parts of the income distribution. In addition, the fact that the white distribution lies to the right of the black distribution suggests that blacks may have a built in advantage with respect to upward mobility using cumulative samples since they have more “room” to rise. To address this we recalculated measures by using non-overlapping ranges (s_1 to s_2) for parent income that move progressively up the income distribution. Table 4 which presents these results, demonstrates that much of the rapid upward mobility experienced by blacks is concentrated at the very bottom of the distribution. For example, among families between the 21st and 25th percentile, upward mobility is 28 percentage points more rapid for whites than blacks. Overall, these results suggest that by most measures, the *extent* of upward mobility among blacks is vastly lower than for whites.

6.2 Conditional probabilities

Estimates conditional on key covariates can potentially shed light on the underlying mechanisms behind the intergenerational transmission of economic status and the source of the black-white mobility gap. As noted in section 1, previous studies have taken advantage of the AFQT as a measure of cognitive skills to study black-white differences in levels.¹⁹

¹⁸However, in other results (not shown) we also find that the magnitude of the percentile gains for blacks are actually much lower than for whites.

¹⁹Neal and Johnson (1996) have shown that the black-white wage gap among adults can largely be explained by pre-market skills as proxied by AFQT scores during adolescence. O’Neill, Sweetman and Van de Gaer (2006) show that equalization of cognitive skill gaps does not fully account for the black-white gap at the low end of the distribution. Cameron and Heckman (2001) have shown that the sizable

Unlike these previous studies, our measures of mobility capture movements in the distribution *relative to the parent generation*, so it is not obvious whether mobility gaps will be eliminated the same way that level gaps are once we include AFQT scores as a covariate.

Another difference from previous work is that we account for test scores nonparametrically. We employ Nadaraya-Watson regressions. To do so, we first normalized the regressor to lie between 0 and 1, using maximum and minimum possible values of the AFQT viz., 99 and 1 and then estimated the regressions at 100 points with spacing of 0.01. We used an Epanechnikov kernel and chose the bandwidth σ_n in accordance with assumption NW4 above where $d = 1$.²⁰ For inference, we calculated uniform bands using the analytical formulas from Hardle (1990, algorithm 4.3.2), which are based on Bickel and Rosenblatt (1973).²¹ The latter steps are reproduced in the appendix part D. Uniform, rather than pointwise, confidence bands are necessary because here we are making inference on the entire conditional mobility curve as a function of the conditioning variable AFQT to see how mobility differences vary with AFQT. Therefore we need data-based bands which contain the *entire* true curves with at least a pre-assigned probability. Joining point-wise confidence limits will reduce the coverage probability arbitrarily below the nominal level, leading to wrong confidence statements.

6.2.1 Conditional Transition Probabilities

We estimate the effect of AFQT scores on the probability of leaving the bottom quintile separately by race. Figure 6 shows that conditional on AFQT scores, whites have only slightly higher likelihood of exiting the bottom quintile and that this gap does not vary a great deal across the AFQT distribution. For example at the 25th interval of our normalized AFQT scores, the transition probability for whites is 0.63 and for blacks is 0.61, or a difference of just 2 percentage points. At the 10th interval the gap is about 7 percentage points and at the 75th interval the gap is about 15 percentage points. At no point in the

gap in college enrollment between whites and blacks can largely be accounted for by AFQT scores.

²⁰We experimented with bandwidths around the range $n^{-1/4}$ (moving from $n^{-1/5}$ to $n^{-1/3}$), where n denotes the size of the effective sample (this size varies depending on which parent percentile and race are conditioned on). Our results for conditional mobility were quite stable over this range and so we report the results at the $\sigma_n = n^{-1/4}$ value.

²¹Hardle (1990), theorem 4.3.1 justifies the validity of this algorithm. In the appendix, we show that our conditions NW1-NW4, NW 5' and NW 6 imply the sufficient conditions A1-A5 (page 116, Hardle, 1990) for this theorem to hold.

AFQT distribution can we reject the hypothesis that the transition probabilities are the same. The shape of the regression lines are also similar between blacks and whites for the bottom half of the distribution. In the upper half of the AFQT distribution, however, the slopes differ and the lines fan apart. It is important to note however, that there is relatively little data for blacks in the upper end of the AFQT distribution as is evidenced by the widening confidence intervals.

In figure 7 we contrast the AFQT results with analogous results using the sons' years of education. We find sharp differences even conditional on education, throughout much of the distribution. For example among those with 10 years of schooling, the transition probability out of the bottom quintile for whites is 67 percent while for blacks it is just 45 percent. At the higher end of the education distribution, however, the racial gap converges and at the very top of the distribution, black mobility is actually higher. Our confidence intervals are large so although the differences are quite big over much of the distribution, they are not statistically significant.²²

Finally, the results from using our nonparametric approach are substantively different from simply estimating a probit with AFQT as a covariate, i.e.,

$$\Pr [Y_1 \leq \zeta_1, Y_0 \leq \zeta_0 | X = x] = \Phi (\beta_0 + \beta_1 \times AFQT).$$

This is particularly true for whites at the bottom of the distribution and for blacks at the top of the distribution. In figure 8 we compare the transition probability results for whites in the bottom of the distribution with the results from simply using a probit. As the chart shows, moving from the first percentile of the AFQT distribution to the median nearly doubles the transition probability of leaving the bottom quintile for whites from 0.43 to 0.85 when using the probit. In contrast, the non-parametric estimator implies an increase of only about 27 percentage points from 0.52 to 0.79. This is not surprising because the probit estimate at a point is affected by the outcome at far-off regressor values unlike the nonparametric estimates.

6.2.2 Conditional Upward Mobility

We also estimate the effect of AFQT scores on our measure of upward mobility separately by race. For this exercise, we condition on parent income being at or below the 20th

²²In similar exercises using measures of parent education (not shown) we find broadly similar results. Hertz (2005) also found that parent education cannot explain the black-white mobility gap.

percentile and set $\tau = 0$. The results are shown in Figure 9. In this case the effects on the black-white gap are even more striking as the point estimates imply that upward mobility is virtually identical for blacks and whites in the bottom half of the distribution.

7 Concluding Thoughts

In this paper we develop new analytic tools that allow for an investigation of inter-racial differences in IGM and its underlying sources. Using large intergenerational samples from the NLSY, we document that upward transition probabilities for blacks in the bottom of the income distribution are sharply lower than for whites. We introduce a new measure of upward mobility that overcomes some of the limitations of the transition probability. The new measure is simply the probability that the sons' rank in the distribution exceeds the parents' rank in the prior generation. The baseline upward mobility measure shows a much smaller inter-racial gap in IGM partially because it reflects the fact that many blacks make small gains in rank over generations that are missed by the transition probability. On the other hand if we adjust our upward mobility measure to require rank gains of a certain amount then the two measures paint a more similar picture of low upward mobility for blacks.

We also investigate, using non-parametric methods, how the inter-racial differences in upward mobility are impacted by incorporating the effects of cognitive skills during adolescence as measured by AFQT scores. Remarkably, we find that AFQT scores can account for virtually the entire black-white difference in upward mobility using either measure. This suggests that early life interventions that address pre-market skills may be more effective than policies that target labor markets institutions. It is not yet clear, however, exactly how policy can best address the racial gap in children's test scores. Cognitive skills are influenced by a range of factors including health endowments, parental investment, peer influences and school quality. Recent work by Chay, Guryan and Mazumder (2009) using military applicant data show that much of the apparent narrowing of the black-white test score gap during the 1980s can be attributed to improvements in infant health arising from greater access by Southern blacks to hospitals during the 1960s. Dobbie and Fryer (forthcoming) find that the Harlem Children's Zone, which combines community programs with charter schools, can significantly close black-white achievement gaps. These results among others suggest that there is potential for policy to address the

sharp black-white differences in upward mobility highlighted here.

There are many other aspects of inter-racial differences in IGM which we have not considered. For example, there may be important differences by gender. An analysis of other covariates such as measures of health, family structure, wealth and non-cognitive skills are also important areas for examination. For example, Heckman, Stixrud and Urzua (2006) demonstrate the importance of non-cognitive skills (e.g. dependability, persistence) on socioeconomic outcomes. We have also limited our outcome of interest to labor market earnings and it may be fruitful to analyze patterns in mobility with respect to other measures such as hours worked, wages and total family income. Finally, we have limited our analysis to upward mobility but there may be important inter-racial differences with respect to downward mobility as well.

The methodological innovations of the present paper were primarily motivated by nonparametric empirical analysis of IGM; but they have potentially more general applicability. One can directly use our methods to the analysis of *intragenerational* upward mobility, considered in Kopczuk, Saez and Song (2010), but by adding covariates or comparing differences by industry or occupation. A second potential application pertains to estimating the probability that a high school student finishes in the top 10% of her high school class as a function of covariates. Such students are offered guaranteed admission to the flagship university by many US states, like Texas, Florida and California (c.f., Cullen et al 2011). Note that such relative regressions, which implicitly control for inter-school variations in absolute grades, are different from quantile regressions in that they capture effects of covariates on the relative position in the *marginal* distribution of the dependent variable and not the *conditional* (on the covariate) distribution. More generally, whenever the parameter of interest is a nonparametric regression or a functional thereof but the dependent variable involves preliminary components estimated from the same dataset, possibly in a non-smooth way, the methods developed here will be relevant.

Table 1: Transition Probability Estimates by Race

$$\theta = \text{Prob}(F_1(Y_1) > s + \tau, F_0(Y_0) < s) / \text{Prob}(F_1(Y_0) < s)$$

s	$\tau=0$			$\tau=0.1$			$\tau=0.2$			$\tau=0.3$		
	Whites	Blacks	W-B	Whites	Blacks	W-B	Whites	Blacks	W-B	Whites	Blacks	W-B
0.05	0.978 (0.030)	0.891 (0.025)	0.087 (0.041)	0.849 (0.057)	0.579 (0.043)	0.270 (0.073)	0.704 (0.070)	0.407 (0.044)	0.297 (0.084)	0.593 (0.084)	0.280 (0.040)	0.312 (0.093)
0.10	0.917 (0.030)	0.702 (0.027)	0.216 (0.042)	0.760 (0.046)	0.458 (0.030)	0.302 (0.055)	0.632 (0.053)	0.340 (0.028)	0.292 (0.061)	0.555 (0.054)	0.249 (0.025)	0.306 (0.059)
0.15	0.812 (0.030)	0.616 (0.026)	0.196 (0.042)	0.692 (0.035)	0.423 (0.026)	0.269 (0.045)	0.542 (0.037)	0.309 (0.023)	0.232 (0.046)	0.459 (0.038)	0.212 (0.020)	0.247 (0.046)
0.20	0.752 (0.028)	0.524 (0.025)	0.228 (0.041)	0.618 (0.033)	0.389 (0.025)	0.229 (0.044)	0.496 (0.035)	0.281 (0.022)	0.215 (0.041)	0.379 (0.037)	0.192 (0.019)	0.187 (0.043)
0.25	0.708 (0.025)	0.447 (0.024)	0.261 (0.036)	0.558 (0.026)	0.326 (0.021)	0.232 (0.035)	0.459 (0.027)	0.234 (0.019)	0.225 (0.034)	0.342 (0.029)	0.156 (0.017)	0.186 (0.035)
0.30	0.646 (0.024)	0.403 (0.020)	0.244 (0.034)	0.539 (0.026)	0.290 (0.018)	0.249 (0.033)	0.418 (0.024)	0.200 (0.016)	0.217 (0.028)	0.305 (0.021)	0.131 (0.013)	0.174 (0.025)
0.35	0.583 (0.023)	0.349 (0.020)	0.234 (0.034)	0.478 (0.023)	0.254 (0.018)	0.224 (0.032)	0.366 (0.022)	0.173 (0.015)	0.193 (0.028)	0.257 (0.020)	0.120 (0.013)	0.136 (0.024)
0.40	0.544 (0.019)	0.311 (0.018)	0.233 (0.029)	0.427 (0.019)	0.223 (0.017)	0.203 (0.027)	0.315 (0.018)	0.148 (0.013)	0.167 (0.024)	0.228 (0.018)	0.105 (0.011)	0.122 (0.022)
0.45	0.494 (0.017)	0.262 (0.017)	0.232 (0.024)	0.372 (0.018)	0.180 (0.015)	0.192 (0.025)	0.264 (0.016)	0.123 (0.012)	0.141 (0.020)	0.190 (0.014)	0.080 (0.011)	0.109 (0.020)
0.50	0.428 (0.015)	0.226 (0.015)	0.202 (0.023)	0.320 (0.016)	0.152 (0.012)	0.168 (0.021)	0.227 (0.015)	0.107 (0.011)	0.119 (0.020)	0.147 (0.011)	0.065 (0.009)	0.082 (0.015)

Notes: See text for a description of the estimator. Data are from the NLSY. We use multiyear averages of son's income over 1996-2002 and parent income measured over 1978-1980. Standard errors are in parentheses.

Table 2: Upward Mobility Estimates by Race

$$v = \text{Prob}(F_1(Y_1) - F_0(Y_0) > \tau | F_0(Y_0) \leq s)$$

s	$\tau=0$			$\tau=0.1$			$\tau=0.2$			$\tau=0.3$		
	Whites	Blacks	W-B	Whites	Blacks	W-B	Whites	Blacks	W-B	Whites	Blacks	W-B
0.05	0.977 (0.024)	0.950 (0.018)	0.027 (0.033)	0.904 (0.047)	0.635 (0.044)	0.270 (0.066)	0.745 (0.065)	0.420 (0.045)	0.325 (0.083)	0.614 (0.073)	0.312 (0.040)	0.303 (0.084)
0.10	0.947 (0.022)	0.883 (0.022)	0.065 (0.032)	0.840 (0.035)	0.574 (0.032)	0.266 (0.051)	0.698 (0.047)	0.377 (0.031)	0.321 (0.059)	0.595 (0.053)	0.288 (0.025)	0.307 (0.061)
0.15	0.909 (0.021)	0.835 (0.020)	0.074 (0.029)	0.786 (0.031)	0.567 (0.027)	0.219 (0.042)	0.629 (0.040)	0.390 (0.025)	0.240 (0.047)	0.519 (0.040)	0.281 (0.025)	0.238 (0.048)
0.20	0.871 (0.021)	0.796 (0.017)	0.075 (0.027)	0.755 (0.029)	0.556 (0.024)	0.198 (0.039)	0.592 (0.030)	0.387 (0.022)	0.205 (0.037)	0.485 (0.032)	0.285 (0.020)	0.200 (0.039)
0.25	0.838 (0.021)	0.762 (0.019)	0.076 (0.030)	0.724 (0.024)	0.537 (0.024)	0.187 (0.038)	0.575 (0.028)	0.373 (0.024)	0.202 (0.036)	0.463 (0.028)	0.274 (0.019)	0.188 (0.034)
0.30	0.821 (0.018)	0.734 (0.019)	0.087 (0.027)	0.715 (0.021)	0.521 (0.021)	0.193 (0.033)	0.568 (0.026)	0.360 (0.020)	0.208 (0.036)	0.447 (0.025)	0.262 (0.019)	0.185 (0.035)
0.35	0.786 (0.019)	0.717 (0.017)	0.069 (0.026)	0.668 (0.020)	0.514 (0.023)	0.154 (0.030)	0.537 (0.021)	0.360 (0.019)	0.178 (0.031)	0.415 (0.023)	0.263 (0.016)	0.153 (0.028)
0.40	0.757 (0.018)	0.704 (0.016)	0.052 (0.025)	0.641 (0.017)	0.506 (0.020)	0.135 (0.028)	0.513 (0.020)	0.357 (0.019)	0.156 (0.027)	0.393 (0.019)	0.254 (0.018)	0.139 (0.027)
0.45	0.731 (0.015)	0.687 (0.017)	0.044 (0.024)	0.605 (0.021)	0.495 (0.021)	0.110 (0.032)	0.484 (0.019)	0.350 (0.018)	0.134 (0.028)	0.367 (0.019)	0.248 (0.017)	0.119 (0.026)
0.50	0.695 (0.014)	0.668 (0.018)	0.028 (0.025)	0.578 (0.016)	0.481 (0.020)	0.097 (0.028)	0.457 (0.016)	0.342 (0.018)	0.115 (0.025)	0.342 (0.017)	0.242 (0.015)	0.100 (0.024)

Notes: See text for a description of the estimator. Data are from the NLSY. We use multiyear averages of son's income over 1996-2002 and parent income measured over 1978-1980. Bootstrapped standard errors are in parentheses.

Table 3: Summary statistics of bootstrapped values of $v^* - \hat{v}$

For $s = 0.25$							
	mean	median	skewness	kurtosis	p-value skew test	p-value kurt. test	p-value joint (chi sq)
$\tau = 0$	-0.009	-0.010	0.100	2.787	0.550	0.637	0.746
$\tau = 0.1$	0.000	0.000	-0.091	2.700	0.587	0.417	0.618
$\tau = 0.2$	0.001	0.001	0.118	2.916	0.483	0.982	0.780

For $s = 0.5$							
	mean	median	skewness	kurtosis	p-value skew test	p-value kurt. test	p-value joint (chi sq)
$\tau = 0$	-0.001	-0.002	0.283	3.301	0.097	0.297	0.143
$\tau = 0.1$	-0.001	-0.002	0.066	2.638	0.695	0.285	0.519
$\tau = 0.2$	0.001	0.001	-0.017	2.543	0.918	0.131	0.314

Notes: In all cases $N = 200$. p -values are from using "sktest" command in STATA v10.1

Table 4: Upward Mobility Estimates by Race Using Intervals of Parent Income

$$v = \text{Prob}(F_1(Y_1) - F_0(Y_0) > \tau \mid F_0(Y_0) >= s_1 \ \& \ F_0(Y_0) <= s_2)$$

s_1 to s_2	$\tau=0$			$\tau=0.1$			$\tau=0.2$			$\tau=0.3$		
	Whites	Blacks	W-B	Whites	Blacks	W-B	Whites	Blacks	W-B	Whites	Blacks	W-B
0.01 to 0.05	0.977 (0.024)	0.950 (0.018)	0.027 (0.033)	0.904 (0.047)	0.635 (0.044)	0.270 (0.066)	0.745 (0.065)	0.420 (0.045)	0.325 (0.083)	0.614 (0.073)	0.312 (0.040)	0.303 (0.084)
0.06 to 0.10	0.915 (0.048)	0.813 (0.035)	0.102 (0.059)	0.770 (0.067)	0.511 (0.048)	0.259 (0.083)	0.647 (0.079)	0.332 (0.043)	0.315 (0.093)	0.573 (0.079)	0.263 (0.035)	0.311 (0.090)
0.11 to 0.15	0.847 (0.047)	0.708 (0.051)	0.138 (0.070)	0.698 (0.062)	0.547 (0.053)	0.151 (0.083)	0.518 (0.075)	0.423 (0.051)	0.095 (0.093)	0.395 (0.068)	0.263 (0.050)	0.132 (0.089)
0.16 to 0.20	0.780 (0.058)	0.645 (0.048)	0.134 (0.079)	0.679 (0.067)	0.516 (0.053)	0.162 (0.089)	0.501 (0.070)	0.376 (0.050)	0.124 (0.082)	0.404 (0.066)	0.300 (0.049)	0.104 (0.087)
0.21 to 0.25	0.751 (0.052)	0.473 (0.070)	0.278 (0.092)	0.645 (0.058)	0.376 (0.062)	0.269 (0.089)	0.532 (0.057)	0.256 (0.060)	0.275 (0.082)	0.404 (0.058)	0.186 (0.056)	0.218 (0.083)
0.26 to 0.30	0.755 (0.049)	0.534 (0.059)	0.221 (0.077)	0.677 (0.061)	0.406 (0.061)	0.271 (0.088)	0.542 (0.065)	0.265 (0.051)	0.277 (0.083)	0.388 (0.062)	0.173 (0.050)	0.215 (0.077)
0.31 to 0.35	0.639 (0.072)	0.495 (0.073)	0.144 (0.104)	0.471 (0.063)	0.420 (0.075)	0.051 (0.098)	0.408 (0.060)	0.358 (0.076)	0.050 (0.102)	0.282 (0.061)	0.272 (0.071)	0.010 (0.104)
0.36 to 0.40	0.613 (0.055)	0.489 (0.092)	0.124 (0.113)	0.510 (0.061)	0.371 (0.090)	0.139 (0.117)	0.392 (0.056)	0.313 (0.087)	0.079 (0.113)	0.282 (0.053)	0.110 (0.068)	0.172 (0.090)
0.41 to 0.45	0.578 (0.060)	0.258 (0.096)	0.320 (0.116)	0.385 (0.071)	0.220 (0.090)	0.165 (0.111)	0.307 (0.063)	0.162 (0.088)	0.145 (0.094)	0.213 (0.047)	0.085 (0.057)	0.128 (0.072)
0.46 to 0.50	0.450 (0.055)	0.311 (0.080)	0.138 (0.094)	0.393 (0.064)	0.225 (0.079)	0.168 (0.114)	0.275 (0.053)	0.195 (0.068)	0.080 (0.089)	0.166 (0.041)	0.135 (0.064)	0.031 (0.071)

Notes: See text for a description of the estimator. Data are from the NLSY. We use multiyear averages of son's income over 1996-2002 and parent income measured over 1978-1980. Bootstrapped standard errors are in parentheses.

Figure 1: Transition Probabilities Conditional On Parent Percentile

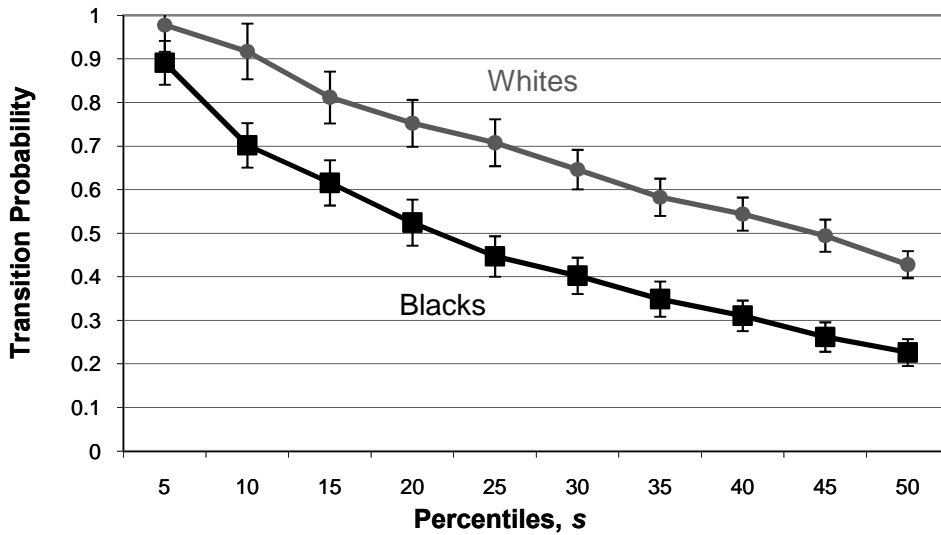


Figure 2: Upward Mobility Conditional On Parent Percentile
 $\Pr(F1 > F0 | F0 \leq s)$

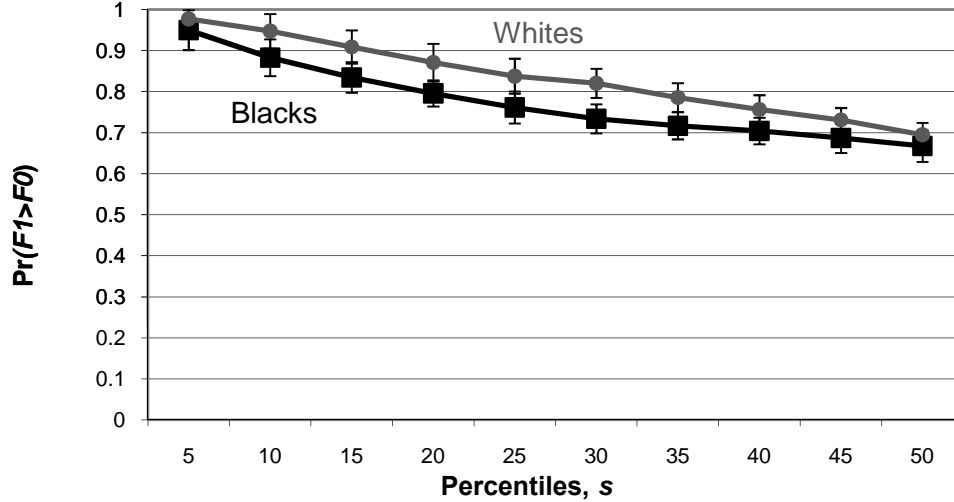
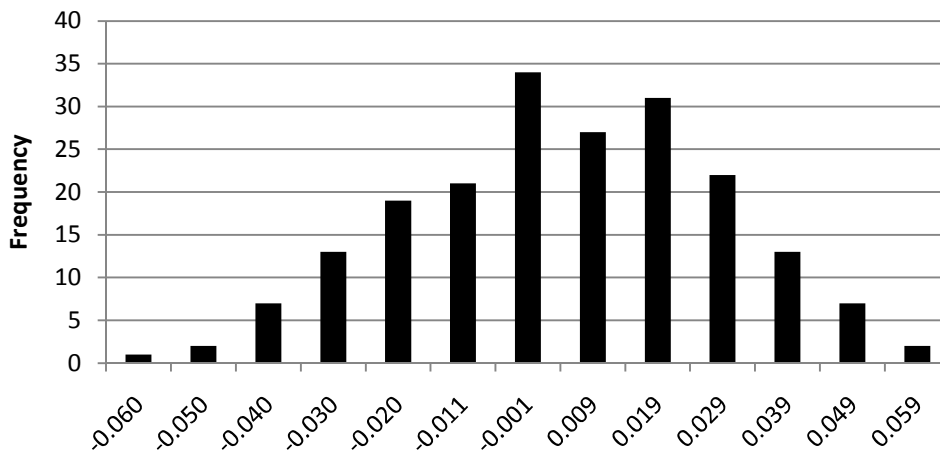


Figure 3: Histogram of $v^* - \hat{v}$



Notes for figures 1 to 3: See text for descriptions of the estimators. Data are from the NLSY. We use multiyear averages of son's income over 1996-2002 and parent income measured over 1978-1980. For upward mobility estimates, bootstrapped 95% pointwise confidence intervals are shown as bands.

Figure 4: Transition Probabilities vs Upward Mobility, Whites - Blacks, Conditional On Parent Percentile

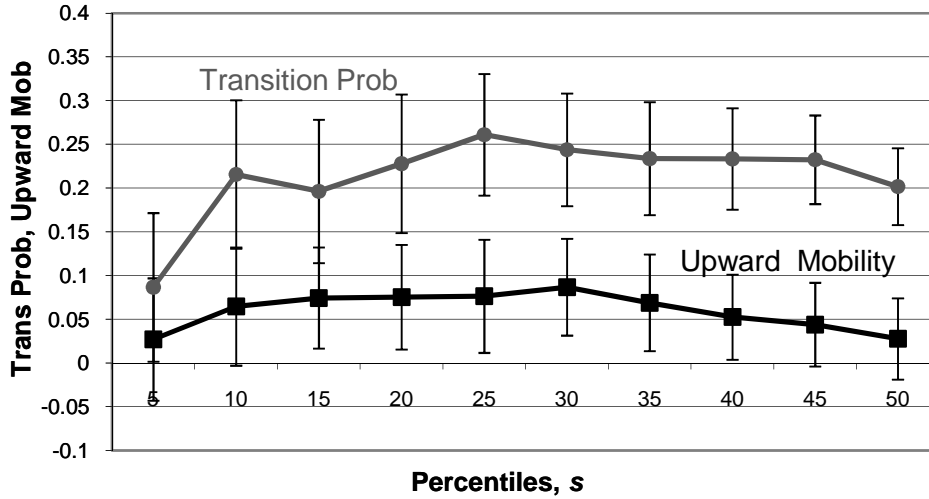


Figure 5: CDF of Parent Income Conditional on Being in the Bottom Quintile, Whites vs Blacks

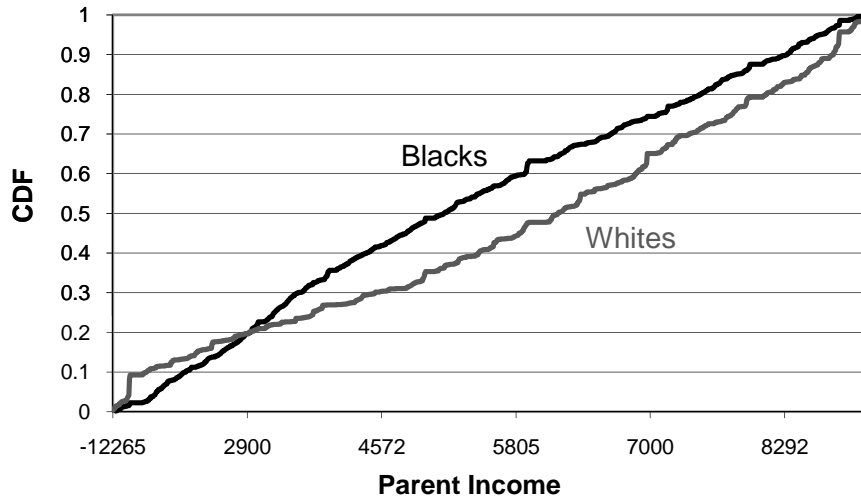
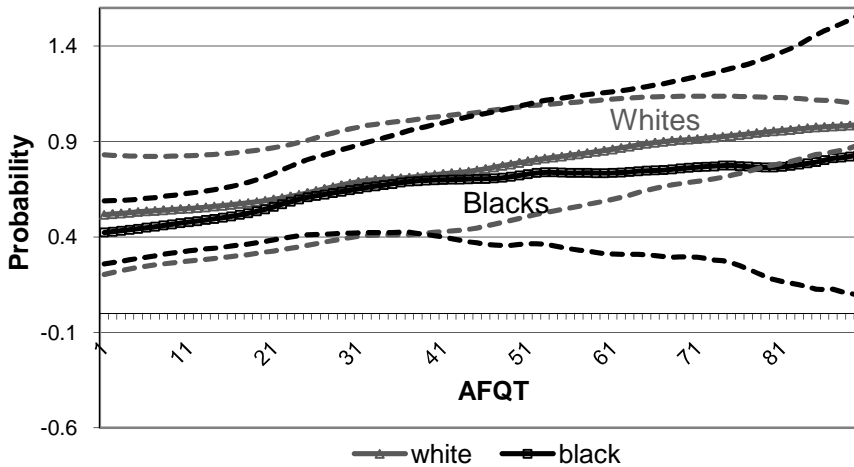


Figure 6: Probability of Leaving Bottom Quintile Conditional on AFQT, Whites vs Blacks



Notes for figures 4 to 6: Data are from the NLSY. We use multiyear averages of son's income over 1996-2002 and parent income measured over 1978-1980. Uniform confidence intervals shown as dashed lines. For details of computation see text.

Figure 7: Transition Probability of Leaving Bottom Quintile Conditional on Ed., Whites vs Blacks

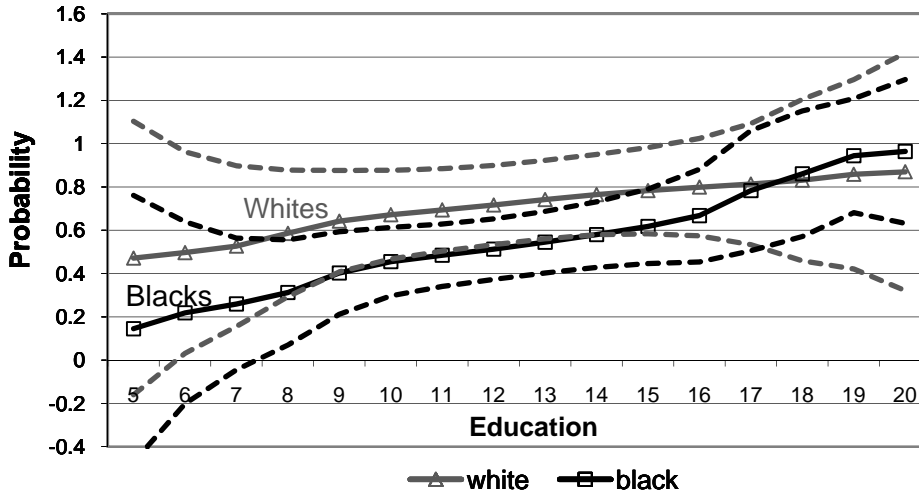


Figure 8: Comparison of Probit and Non-Parametric Estimates of Transition Probability of Whites Leaving Bottom Quintile Conditional on AFQT

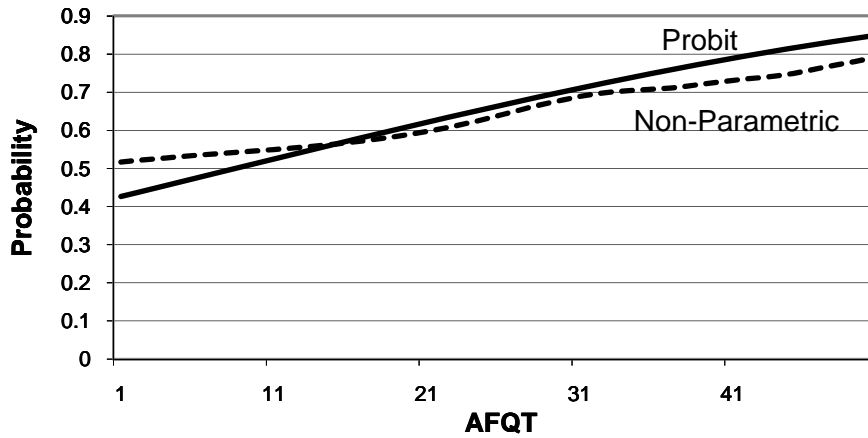
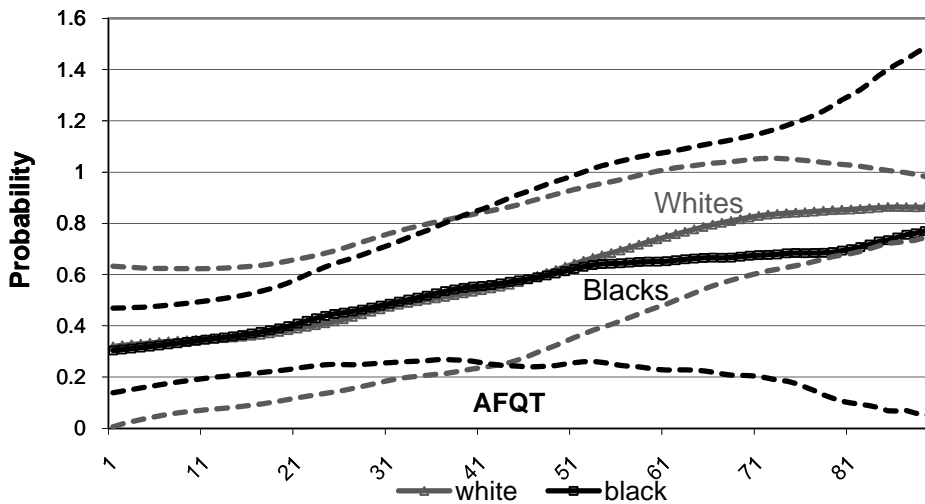
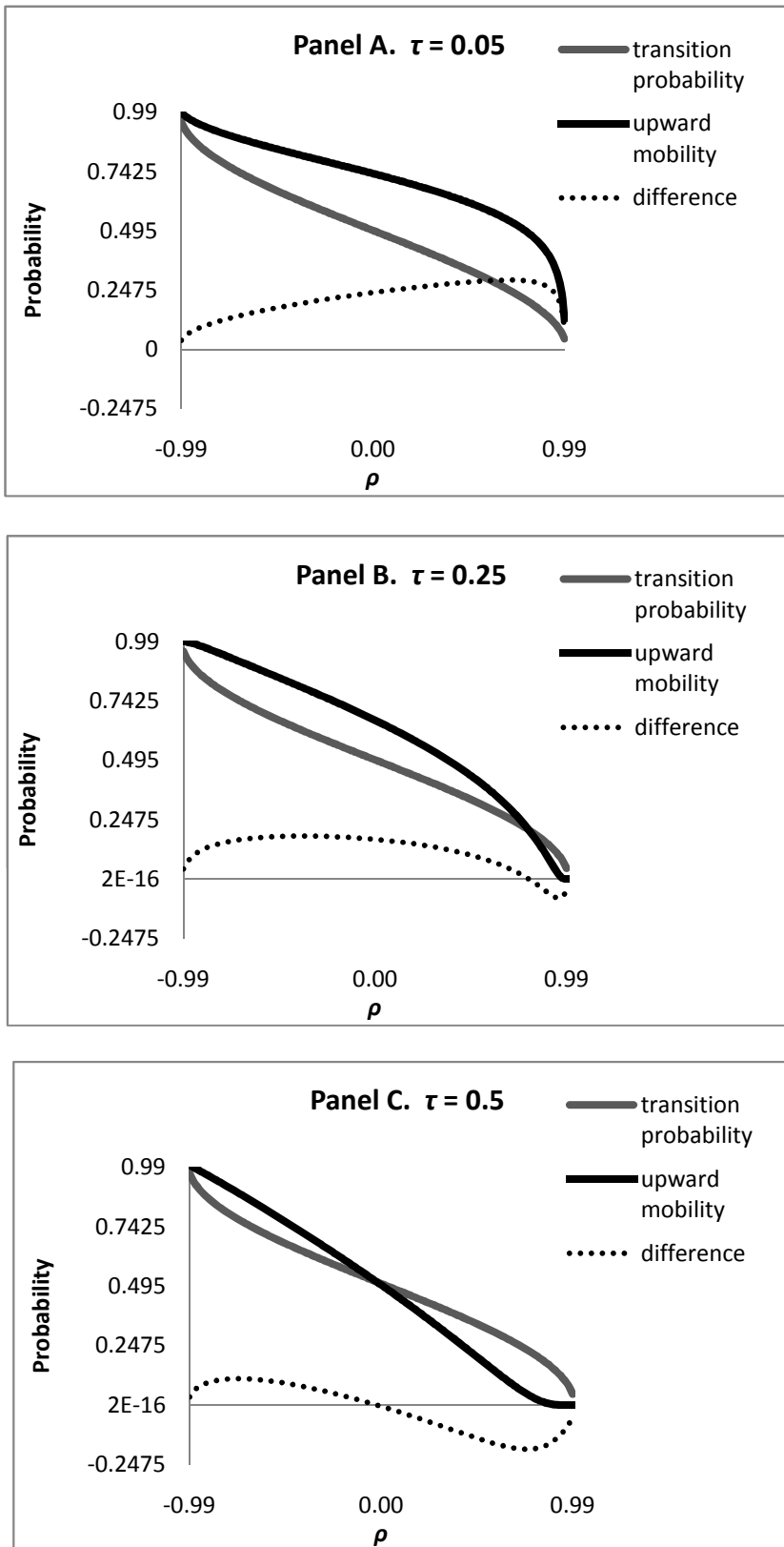


Figure 9: Upward Mobility Conditional on AFQT Scores, Whites vs Blacks



Notes for figures 7 to 9: Data are from the NLSY. We use multiyear averages of son's income over 1996-2002 and parent income measured over 1978-1980. For figures 7 and 9, uniform confidence intervals shown as dashed lines. For details of computation see text.

Figure A1: Comparisons of Transition Probability with Upward Mobility, Lognormal Model



Notes : Figures are based on a simulation using a lognormal model. See text for futer details.

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Appendix with Proofs

In the statements of the results and in their proofs, c will denote a generic positive constant not always having the same value and whenever moments, derivatives or Lebesgue densities are defined, they are implicitly assumed to exist.

Part A:

Proposition 1: We now state a set of general regularity conditions which will imply zero-mean asymptotic normality for the Nadaraya-Watson estimated regression of the unobserved random variable

$$W := 1 \{Y_1 \leq \zeta_1, Y_0 \leq \zeta_0\}$$

on X , evaluated at $X = x$. These conditions are standard (for textbook treatments, see Bierens (1994), theorem 10.2.1 or Pagan and Ullah (1999) theorem 3.5, 3.6) but we state them here to make the subsequent proposition and lemma statements self-contained.

Condition NW

NW1. X is a d -dimensional continuously distributed random variable with Lebesgue density $f(\cdot)$ which is positive at x . For all our applications, $d = 1$.

NW2. The data (X_i, Y_{1i}, Y_{0i}) are i.i.d.

NW3. $K(\cdot)$ is a Borel-measurable, bounded and real-valued kernel function with d -dimensional argument, satisfying (i) $\int K(a) da = 1$, $\int aK(a) da = 0$, $\int a_i^2 K(a) da < \infty$ for $i = 1, \dots, d$, (ii) $\int |K(a)| da < \infty$, (iii) for some $\delta > 0$, $\int |K(a)|^{2+\delta} da < \infty$.

NW4. The bandwidth sequence σ_n satisfies $\lim_{n \rightarrow \infty} \sigma_n = 0$, $\lim_{n \rightarrow \infty} n\sigma_n^d = \infty$ and $\lim_{n \rightarrow \infty} \sigma_n^2 (n\sigma_n^d)^{1/2} = 0$; $\frac{1}{\sigma_n^d} K\left(\frac{a-b}{\sigma_n}\right)$ is uniformly bounded for $a, b \in \text{support}(X)$.

NW5. The functions $f(\cdot)$ and $f(\cdot) \times \phi(\cdot, \zeta_0, \zeta_1)$ and their derivatives up to order 2 are continuous and uniformly bounded.

Then we have

$$(n\sigma_n^d)^{1/2} \left(\hat{\phi}\left(x, \hat{\zeta}_0, \hat{\zeta}_1\right) - \phi\left(x, \zeta_0^0, \zeta_1^0\right) \right) \xrightarrow{d} N\left(0, \frac{v^2(x)}{f(x)} \int K^2(u) du\right),$$

where $v^2(x) = \phi(x, \zeta_0^0, \zeta_1^0) \times (1 - \phi(x, \zeta_0^0, \zeta_1^0))$.

Proof outline for proposition 1:

Proof. First note that the function $\phi(x, \cdot, \cdot)$ is Lipschitz with respect to the Euclidean norm $\|\cdot\|$:

$$|\phi(x, \zeta_0, \zeta_1) - \phi(x, \tau_0, \tau_1)| \leq \|(\zeta_0, \zeta_1) - (\tau_0, \tau_1)\| \delta(x) \quad (11)$$

with $\delta(\cdot)$ uniformly bounded on the support of X . To see this, note that $\nabla\phi(x, a, b) = f_{Y_0, Y_1|X}(a, b|x)$, so that applying the mean-value theorem to the LHS of (11) in conjunction with the NW5 will yield the result.

Now consider the expression

$$\bar{m}(\zeta, x) = \frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{X_i - x}{\sigma_n}\right) 1(Y_{1i} \leq \zeta_1, Y_{0i} \leq \zeta_0)$$

whose expectation is given by

$$\begin{aligned} \bar{m}^*(\zeta, x) &= E_{X_i} \left(\frac{1}{\sigma_n^d} K\left(\frac{X_i - x}{\sigma_n}\right) \phi(X_i, \zeta_0, \zeta_1) \right) \\ &= \int K(u) f(x + u\sigma_n) \phi(x + u\sigma_n, \zeta_0, \zeta_1) du \\ &= f(x) \phi(x, \zeta_0, \zeta_1) + O(\sigma_n^2). \end{aligned} \tag{12}$$

So

$$\begin{aligned} \bar{m}^*(\hat{\zeta}, x) &= f(x) \phi(x, \hat{\zeta}_0, \hat{\zeta}_1) + O(\sigma_n^2) \\ &= f(x) \left[\begin{aligned} &\phi(x, \zeta_0^0, \zeta_1^0) + \phi_0(x, \tilde{\zeta}_0, \tilde{\zeta}_1) (\hat{\zeta}_0 - \zeta_0) \\ &+ \phi_1(x, \tilde{\zeta}_0, \tilde{\zeta}_1) (\hat{\zeta}_1 - \zeta_1) \end{aligned} \right] + O(\sigma_n^2), \end{aligned}$$

where $\tilde{\zeta}_1$ denote value intermediate between $\hat{\zeta}_1$ and ζ_1^0 and similarly, $\tilde{\zeta}_0$. Now,

$$\begin{aligned} &\hat{\phi}(x, \hat{\zeta}_0, \hat{\zeta}_1) - \phi(x, \zeta_0^0, \zeta_1^0) \\ &= \frac{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{X_i - x}{\sigma_n}\right) 1(Y_{1i} \leq \hat{\zeta}_1, Y_{0i} \leq \hat{\zeta}_0)}{\hat{f}(x)} - \phi(x, \zeta_0^0, \zeta_1^0) \\ &= \frac{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{X_i - x}{\sigma_n}\right) \left\{ 1(Y_{1i} \leq \hat{\zeta}_1, Y_{0i} \leq \hat{\zeta}_0) - 1(Y_{1i} \leq \zeta_1, Y_{0i} \leq \zeta_0) \right\}}{\hat{f}(x)} \\ &\quad + \underbrace{\left\{ \frac{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{X_i - x}{\sigma_n}\right) 1(Y_{1i} \leq \zeta_1, Y_{0i} \leq \zeta_0)}{\hat{f}(x)} - \phi(x, \zeta_0^0, \zeta_1^0) \right\}}_{=S_n, \text{ say}} \\ &= \underbrace{\left\{ \frac{[\bar{m}(\hat{\zeta}, x) - \bar{m}^*(\hat{\zeta}, x)] - [\bar{m}(\zeta^0, x) - \bar{m}^*(\zeta^0, x)]}{\hat{f}(x)} \right\}}_{T_{1n}} \\ &\quad + \underbrace{\frac{\bar{m}^*(\hat{\zeta}, x) - \bar{m}^*(\zeta^0, x)}{\hat{f}(x)}}_{T_{2n}} + S_n \end{aligned} \tag{13}$$

Now $(n\sigma_n^d)^{1/2} S_n$, under the assumptions **NW**, will be $O_p(1)$ and zero-mean normal (c.f., Bierens (1994), theorem 10.2.1), viz.,

$$(n\sigma_n^d)^{1/2} S_n \xrightarrow{d} N\left(0, \frac{v^2(x)}{f(x)} \int K^2(u) du\right). \quad (14)$$

Next, using (12) and the fact that $(\hat{\zeta}_0 - \zeta_0)$ and $(\hat{\zeta}_1 - \zeta_1)$ have parametric convergence rates, we get that

$$\begin{aligned} & (n\sigma_n^d)^{1/2} T_{2n} \\ & f(x) \left[\begin{array}{l} \phi_0(x, \tilde{\zeta}_0, \tilde{\zeta}_1) \times \underbrace{(n\sigma_n^d)^{1/2} \times (\hat{\zeta}_0 - \zeta_0)}_{o_p(1)} \\ + \phi_1(x, \tilde{\zeta}_0, \tilde{\zeta}_1) \times \underbrace{(n\sigma_n^d)^{1/2} \times (\hat{\zeta}_1 - \zeta_1)}_{o_p(1)} \end{array} \right] + \underbrace{O_p\left((n\sigma_n^d)^{1/2} \times \sigma_n^2\right)}_{o_p(1), \text{ by NW4}} \\ = & \frac{ \left[\begin{array}{l} \phi_0(x, \tilde{\zeta}_0, \tilde{\zeta}_1) \times \underbrace{(n\sigma_n^d)^{1/2} \times (\hat{\zeta}_0 - \zeta_0)}_{o_p(1)} \\ + \phi_1(x, \tilde{\zeta}_0, \tilde{\zeta}_1) \times \underbrace{(n\sigma_n^d)^{1/2} \times (\hat{\zeta}_1 - \zeta_1)}_{o_p(1)} \end{array} \right]}{\underbrace{\hat{f}(x)}_{=O_p(1)}} \\ = & o_p(1). \end{aligned} \quad (15)$$

The nonstandard term in (13) is T_{1n} and we now demonstrate a stochastic equicontinuity property for it. Letting

$$v_n(\zeta, x) = (n\sigma_n^d)^{1/2} \{\bar{m}(\zeta, x) - \bar{m}^*(\zeta, x)\}$$

the first term satisfies

$$(n\sigma_n^d)^{1/2} T_{1n} = \frac{v_n(\hat{\zeta}, x) - v_n(\zeta^0, x)}{\hat{f}(x)}, \quad (16)$$

and we now show that the numerator of (16) is $o_p(1)$, for each x . Now, for fixed x , the class of functions

$$g_n(W, \zeta) := \frac{1}{\sigma_n^d} K\left(\frac{X-x}{\sigma_n}\right) 1(Y_1 \leq \zeta_1, Y_0 \leq \zeta_0)$$

form a type IV class (c.f. Andrews (1994), equation 5.3) with $p = 2$. This follows from the Lipschitz property (11) and the uniform boundedness of $\frac{1}{\sigma_n^d} K\left(\frac{X-x}{\sigma_n}\right)$. This, in turn, implies that the sequence $v_n(\zeta, x)$ is stochastically equicontinuous. Now, using the same steps as Andrews (1994) leading to his equation (3.8), we conclude that

$$v_n(\hat{\zeta}, x) - v_n(\zeta^0, x) = o_p(1).$$

Put together (13), (14), (15) and (16) together with $\hat{f}(x) = O_p(1)$ to conclude. ■

Part B:

Hadamard differentiability of upward mobility: Let $f(y_0, y_1)$ and $f^{(0)}(y_0, y_1)$ denote respectively the joint density of (Y_0, Y_1) and its derivative w.r.t. the first argument, evaluated at the point (y_0, y_1) . Let $f_1(y_1)$ denote the marginal density of Y_1 and let c denote a generic positive constant. Since all our measures are robust to monotone transformation of the income variables, we will continue assume that the support of the income variables is contained in $[1, \infty)$.

Condition : (Ai) for some $\alpha > 1$, we have $f_1(x) \geq \frac{c}{x^\alpha}$ for x large enough which also implies that $F_1^{-1}(u) < c(1-u)^{\frac{1}{1-\alpha}}$, **(Aii)** $f^{(0)}(y_0, F_1^{-1}(F_0(y_0) + \tau)) \leq \frac{c}{y_0^{\alpha_0}}$ for some $\alpha_0 > 0$, **(Aiii)** for some $\varepsilon > 0$, $1 - F_0(y_0) > cy_0^{\frac{(1+\varepsilon-\alpha_0)(\alpha-1)}{\alpha}}$ and **(Aiv)**²³

$$\int_1^\infty (1 - F_0(y_0))^{\frac{\alpha}{\alpha-1}} f(y_0, F_1^{-1}(F_0(y_0) + \tau)) dy_0 < \infty.$$

It is interesting to note that if the tails of (Y_0, Y_1) have a joint Pareto distribution, then all of these conditions are automatically satisfied. To see this, assume that (Y_0, Y_1) satisfy

$$\Pr(Y_0 \geq y_0, Y_1 \geq y_1) = \frac{1}{(1 + (y_0 - 1) + (y_1 - 1))^\gamma}$$

for all $y_0, y_1 \geq 1$ for some $\gamma > 0$. Then their joint density is given by

$$f(y_0, y_1) = \frac{\gamma(\gamma + 1)}{(1 + (y_0 - 1) + (y_1 - 1))^{\gamma+2}}.$$

Then one may verify that conditions A(i)-A(iv) are satisfied with $\alpha = \gamma + 1$, $\alpha_0 = \gamma + 2$ and $\varepsilon = 1 + \gamma + \gamma(\gamma + 1)$.

An exactly symmetric set of conditions are assumed to hold for the marginal density $f_0(\cdot)$ of Y_0 as well.

(Bi) for some $\beta > 1$, we have $f_0(x) \geq \frac{c}{x^\beta}$ for x large enough which also implies that $F_0^{-1}(u) < c(1-u)^{\frac{1}{1-\beta}}$, **(Bii)** $f^{(0)}(F_0^{-1}(s), y_1) \leq \frac{c}{y_1^{\beta_0}}$ for some $\beta_0 > 0$, **(Biii)** for some $\delta > 0$, we have $1 - F_1(y_1) > cy_1^{\frac{(1+\delta-\beta_0)(\beta-1)}{\beta}}$ and

$$\mathbf{(Biv)} \int_1^\infty (1 - F_1(y_1))^{\frac{\beta}{\beta-1}} f(F_0^{-1}(s), y_1) dy_1 < \infty.$$

²³Condition A(iv) is like a moment condition; recall that for a positive random variable X with marginal c.d.f. $G(\cdot)$ and support A , the quantity $\int_A (1 - G(x)) dx$ equals $E(X)$.

To show that the map $F \mapsto v(F)$ is Hadamard-differentiable, let $\bar{D}[1, \infty)$ denote the space of bivariate c.d.f.'s on $[1, \infty)$, satisfying conditions (Ai)-(Aiv) and (Bi)-(Biv). Denote by D_0 the space of sample paths corresponding to the composite Brownian bridge $[\mathbb{G}_\lambda \circ F]$ where G_λ is a standard Brownian bridge and F is any c.d.f. in $\bar{D}[1, \infty)$. Let $D = \bar{D}[1, \infty) \cup D_0$, equipped with the supremum norm. We want to show Hadamard differentiability of the map $F \mapsto v(F)$ as a map from the normed vector space D to \mathbb{R} . Consider perturbations $F_t(y_0, y_1) = F(y_0, y_1) + tH_t(y_0, y_1) \in \bar{D}[1, \infty)$ with $H_t \rightarrow H \in D_0$, uniformly as $t \rightarrow 0$. We want to show that

$$\left| \frac{v(F_t) - v(F)}{t} - v'_F(H) \right| \rightarrow 0 \text{ as } t \rightarrow 0$$

for a linear functional $v'_F(\cdot)$ which is a map from $\bar{D}[1, \infty)$ to \mathbb{R} .

Lemma 1 *Under conditions (Ai)-(Aiv) and (Bi)-(Biv), the map $F \mapsto v(F)$ from $D \rightarrow \mathbb{R}$, defined as*

$$v(F) = \int_1^{F_0^{-1}(s)} \int_1^{F_1^{-1}(F_0(y_0) + \tau)} f(y_0, y_1) dy_1 dy_0$$

for any fixed s , $\tau \in (0, 1)$ is Hadamard differentiable at F tangentially to D_0 . The derivative at F in the direction H is given by the linear functional $v'_F(\cdot)$ defined as

$$\begin{aligned} v'_F(H) &= \frac{H_0(F_0^{-1}(s))}{f_0(F_0^{-1}(s))} \int_1^{F_1^{-1}(F_0(y_0) + \tau)} f(F_0^{-1}(s), y_1) dy_1 \\ &+ \int_1^{F_0^{-1}(s)} \frac{H_0(y_0) - H_1(F_1^{-1}(F_0(y_0) + \tau))}{f_1(F_1^{-1}(F_0(y_0) + \tau))} f(y_0, F_1^{-1}(F_0(y_0) + \tau)) dy_0 \\ &+ \int_1^{F_0^{-1}(s)} \int_1^{F_1^{-1}(F_0(y_0) + \tau)} dH(y_0, y_1), \end{aligned}$$

where

$$H_0(a) = \lim_{x \rightarrow \infty} H(a, x) \text{ and } H_1(a) = \lim_{x \rightarrow \infty} H(x, a). \quad (17)$$

Proof. Consider perturbations $F_t(y_0, y_1) = F(y_0, y_1) + tH_t(y_0, y_1)$ with $F_{0t}(y_0) = F_0(y_0) + tH_{0t}(y_0)$ and $F_{1t}(y_1) = F_1(y_1) + tH_{1t}(y_1)$ denoting the corresponding marginals. Let $H_t \rightarrow H$ uniformly as $t \rightarrow 0$ and let H_0 and H_1 denote its marginals. We want to show that for a linear functional $v'_F(\cdot)$,

$$\left| \frac{v(F_t) - v(F)}{t} - v'_F(H) \right| \rightarrow 0 \text{ as } t \rightarrow 0. \quad (18)$$

Define

$$\begin{aligned} z_1(y_0) &= F_1^{-1}(F_0(y_0) + \tau), \quad z_{1t}(y_0) = F_{1t}^{-1}(F_{0t}(y_0) + \tau) \\ z_0 &= F_0^{-1}(s), \quad z_{0t} = F_{0t}^{-1}(s). \end{aligned}$$

So we need to show

$$\left| \frac{\int_1^{z_{0t}} \int_1^{z_{1t}(y_0)} f_t(y_0, y_1) - \int_1^{z_0} \int_1^{z_1(y_0)} f(y_0, y_1) dy_1 dy_0}{t} - v'_F(H) \right| \rightarrow 0 \text{ as } t \rightarrow 0.$$

Note that the first term inside $|\cdot|$ can be expanded as

$$\begin{aligned} & \int_1^{z_{0t}} \frac{z_{1t}(y_0) - z_1(y_0)}{t} f(\bar{z}_{1t}(y_0), y_0) dy_0 + \frac{z_{0t} - z_0}{t} \times \int_1^{z_1(y_0)} f(\bar{z}_{0t}, y_1) dy_1 \\ & + \int_1^{z_{0t}} \int_1^{z_{1t}(y_0)} dH_t(y_0, y_1) \\ = & \underbrace{\left[\int_1^{z_{0t}} \frac{z_{1t}(y_0) - z_1(y_0)}{t} [f(y_0, \bar{z}_{1t}(y_0)) - f(y_0, z_1(y_0))] dy_0 \right]}_{T_{10t}} \\ & + \underbrace{\left[\frac{z_{0t} - z_0}{t} \times \int_1^{z_1(y_0)} [f(\bar{z}_{0t}, y_1) - f(z_0, y_1)] dy_1 \right]}_{T_{11t}} \\ & \underbrace{\hspace{10em}}_{T_{1t}} \\ & + \underbrace{\int_1^{z_{0t}} \frac{z_{1t}(y_0) - z_1(y_0)}{t} f(z_0, y_1) dy_1}_{T_{2t}} + \underbrace{\frac{z_{0t} - z_0}{t} \times \int_1^{z_1(y_0)} f(z_0, y_1) dy_1}_{T_{3t}} \\ & + \underbrace{\int_1^{z_{0t}} \int_1^{z_{1t}(y_0)} dH_t(y_0, y_1) - \int_1^{z_{0t}} \int_1^{z_{1t}(y_0)} dH(y_0, y_1)}_{T_{4t}} \\ & + \underbrace{\int_1^{z_{0t}} \int_1^{z_{1t}(y_0)} dH(y_0, y_1) - \int_1^{z_0} \int_1^{z_1(y_0)} dH(y_0, y_1)}_{T_{5t}} + \underbrace{\int_1^{z_0} \int_1^{z_1(y_0)} dH(y_0, y_1)}_{T_6}. \end{aligned}$$

We will show that as $t \rightarrow 0$,

Step 1: $|T_{1t}| \rightarrow 0$

Step 2:

$$T_{2t} \rightarrow \frac{H_0(z_0)}{f_0(z_0)} \int_1^{z_1(y_0)} f(z_0, y_1) dy_1$$

Step 3:

$$T_{3t} \rightarrow \int_1^{z_0} \frac{H_0(y_0) - H_1(z_1(y_0))}{f_1(z_1(y_0))} f(y_0, z_1(y_0)) dy_0$$

Step 4: $|T_{4t}| \rightarrow 0$, $|T_{5t}| \rightarrow 0$.

Then we will have shown (18) with

$$\begin{aligned} v'_F(H) &= \frac{H_0(F_0^{-1}(s))}{f_0(F_0^{-1}(s))} \int_1^{F_1^{-1}(F_0(y_0)+\tau)} f(F_0^{-1}(s), y_1) dy_1 \\ &+ \int_1^{F_0^{-1}(s)} \frac{H_0(y_0) - H_1(F_1^{-1}(F_0(y_0) + \tau))}{f_1(F_1^{-1}(F_0(y_0) + \tau))} f(y_0, F_1^{-1}(F_0(y_0) + \tau)) dy_0 \\ &+ \int_1^{F_0^{-1}(s)} \int_1^{F_1^{-1}(F_0(y_0)+\tau)} dH(y_0, y_1) \end{aligned}$$

which is linear in H .

For steps 1 and 2, we will need the following derivation.

$$\begin{aligned} F_1(z_{1t}(y_0)) + tH_{1t}(z_{1t}(y_0)) &= F_{1t}(z_{1t}(y_0)) = F_{0t}(y_0) + \tau \\ &= F_0(y_0) + \tau + tH_{0t}(y_0) = F_1(z_1(y_0)) + tH_{0t}(y_0) \end{aligned}$$

implying that

$$\begin{aligned} tH_{1t}(z_{1t}(y_0)) - tH_{0t}(y_0) &= F_1(z_1(y_0)) - F_1(z_{1t}(y_0)) \\ &= [z_1(y_0) - z_{1t}(y_0)] f_1(\tilde{z}_{1t}(y_0)) \end{aligned}$$

where for any y_0 and t , $\tilde{z}_t(y_0)$ lies in between $z(y_0)$ and $z_t(y_0)$. Therefore,

$$\frac{z_{1t}(y_0) - z_1(y_0)}{t} = \frac{H_{0t}(y_0) - H_{1t}(z_t(y_0))}{f_1(\tilde{z}_{1t}(y_0))}. \quad (19)$$

Similarly, $F_0(z_0) = s = F_{0t}(z_{0t}) = F_0(z_{0t}) + tH_{0t}(z_{0t})$, whence

$$\frac{z_{0t} - z_0}{t} = \frac{H_{0t}(z_{0t})}{f_0(\tilde{z}_{0t})}. \quad (20)$$

Below, c will denote a generic constant, not always of the same value.

Step 1: By a mean-value theorem argument,

$$\begin{aligned} |T_{10t}| &\leq \int_1^{z_{0t}} \left| \frac{z_{1t}(y_0) - z_1(y_0)}{t} [f(y_0, \bar{z}_{1t}(y_0)) - f(y_0, z_1(y_0))] \right| dy_0 \\ &\leq \int_1^{z_{0t}} \left| \frac{[z_t(y_0) - z(y_0)]^2}{t} f^{(1)}(y_0, \check{z}_{1t}(y_0)) \right| dy_0 \end{aligned}$$

where $f^{(1)}(\cdot, \cdot)$ denotes derivative w.r.t. the second argument and $\check{z}_{1t}(y_0)$ lies in between $z(y_0)$ and $z_t(y_0)$. Using (19), we get

$$|T_{10t}| \leq t \int_1^\infty \left| \frac{[H_{0t}(y_0) - H_{1t}(z_{1t}(y_0))]^2}{f_1^2(\check{z}_{1t}(y_0))} f^{(1)}(y_0, \check{z}_{1t}(y_0)) \right| dy_0.$$

We will show that (i) $[H_{0t}(y_0) - H_{1t}(z_{1t}(y_0))]^2$ is uniformly bounded, (ii) $f_1^2(\check{z}_{1t}(y_0)) \geq \frac{c}{\check{z}_{1t}(y_0)^{2\alpha}} \geq c(1 - F_0(y_0))^{\frac{2\alpha}{1-\alpha}}$ for y_0 large enough and t small enough and (iii) $f^{(1)}(y_0, \check{z}_{1t}(y_0)) \leq \frac{c}{y_0^{\alpha_0}}$ for some $\alpha_0 > 1$. Then we will have

$$|T_{10t}| \leq ct \int_1^\infty \left| \frac{1}{y_0^{\alpha_0} (1 - F_0(y_0))^{\frac{2\alpha}{1-\alpha}}} \right| dy_0 \leq ct \int_1^\infty \frac{1}{y_0^{1+\varepsilon}} dy_0 \rightarrow 0,$$

by A(iii).

To see (i), note that $\{[H_{0t}(y_0) - H_{1t}(z_{1t}(y_0))] - [H_0(y_0) - H_1(z_{1t}(y_0))]\}$ converges uniformly to 0 and $H_0(\cdot)$ and $H_1(\cdot)$ are uniformly bounded.

Next,

$$\begin{aligned} & z_{1t}(y_0) \\ &= F_{1t}^{-1}(F_{0t}(y_0) + \tau) \stackrel{(1)}{\leq} c(1 - F_{0t}(y_0) - \tau)^{\frac{1}{1-\alpha}} = c(1 - F_0(y_0) - tH_{0t}(y_0) - \tau)^{\frac{1}{1-\alpha}} \\ &\leq c'(1 - F_0(y_0) - \tau)^{\frac{1}{1-\alpha}} \\ &\leq c(1 - F_0(y_0))^{\frac{1}{1-\alpha}} \end{aligned} \tag{21}$$

for small enough t , since $\alpha > 1$ and $tH_{0t}(\cdot)$ converges uniformly to 0. Inequality (1) comes from condition **Ai**. Similarly,

$$z_1(y_0) \leq c(1 - F_0(y_0))^{\frac{1}{1-\alpha}} \tag{22}$$

and therefore (ii) follows. Finally (iii) follows from (21), (22) and condition Aii.

Next, for $|T_{11t}|$, we have that

$$\begin{aligned} |T_{11t}| &\leq \left| \frac{z_{0t} - z_0}{t} \times \int_1^{z_1(y_0)} [f(\bar{z}_{0t}, y_1) - f(z_0, y_1)] dy_1 \right| \\ &\leq \left| \frac{[z_{0t} - z_0]^2}{t} \times \int_1^{z_1(y_0)} f^{(0)}(\bar{z}_{0t}, y_1) dy_1 \right| \\ &\leq t \left| \left[\frac{H_{0t}(z_{0t})}{f_0(\check{z}_{0t})} \right]^2 \times \int_1^{z_1(y_0)} f^{(0)}(\bar{z}_{0t}, y_1) dy_1 \right| \stackrel{(2)}{\leq} ct \int_1^{z_1(y_0)} \frac{1}{y_0^{1+\delta}} dy_1 \leq ct \int_1^\infty \frac{1}{y_0^{1+\delta}} dy_1 \end{aligned}$$

for t small enough and some $\delta > 0$. Inequality (2) follows from conditions (Bi), (Bii), (Biii) using arguments analogous to those for T_{10t} . This implies that $|T_{11t}| \rightarrow 0$.

Step 2:

$$\begin{aligned}
& \left| T_{2t} - \frac{H_0(z_0)}{f_0(z_0)} \int_1^{z_1(y_0)} f(z_0, y_1) dy_1 \right| \\
&= \left| \int_1^{z_{0t}} \frac{z_{1t}(y_0) - z_1(y_0)}{t} f(z_0, y_1) dy_0 - \frac{H_0(z_0)}{f_0(z_0)} \int_1^{z_1(y_0)} f(z_0, y_1) dy_1 \right| \\
&= \left| \int_1^{z_{0t}} \frac{H_{0t}(z_{0t})}{f_0(\tilde{z}_{0t})} f(z_0, y_1) dy_1 - \frac{H_0(z_0)}{f_0(z_0)} \int_1^{z_1(y_0)} f(z_0, y_1) dy_1 \right| \\
&\leq \int_1^{z_{0t}} \left| \frac{H_{0t}(z_{0t})}{f_0(\tilde{z}_{0t})} - \frac{H_0(z_0)}{f_0(z_0)} \right| f(z_0, y_1) dy_1 \\
&\stackrel{(1)}{\leq} c \int_1^{z_{0t}} |H_{0t}(z_{0t}) - H_0(z_0)| (1 - F_1(y_1))^{\frac{\beta}{\beta-1}} f(z_0, y_1) dy_1 \\
&\leq c \left(\sup_u |H_{0t}(u) - H_0(u)| + |H_0(z_{0t}) - H_0(z_0)| \right) \int_1^\infty (1 - F_1(y_1))^{\frac{\beta}{\beta-1}} f(z_0, y_1) dy_1 \\
&\rightarrow 0
\end{aligned}$$

as $t \rightarrow 0$, by B(iv). Inequality (1) is a consequence of B(i)-B(iii).

Step 3:

$$\begin{aligned}
& \left| T_{3t} - \int_1^{z_{0t}} \frac{H_0(y_0) - H_1(z_1(y_0))}{f_1(z_1(y_0))} f(y_0, z_1(y_0)) dy_0 \right| \\
&= \left| \int_1^{z_{0t}} \frac{z_{1t}(y_0) - z_1(y_0)}{t} f(z_1(y_0), y_0) dy_0 - \int_1^{z_{0t}} \frac{H_0(y_0) - H_1(z_1(y_0))}{f_1(z_1(y_0))} f(y_0, z_1(y_0)) dy_0 \right| \\
&\leq \int_1^{z_{0t}} \left| \frac{z_{1t}(y_0) - z_1(y_0)}{t} - \frac{H_0(y_0) - H_1(z_1(y_0))}{f_1(z_1(y_0))} \right| f(y_0, z_1(y_0)) dy_0 \\
&= \int_1^\infty \left| \frac{H_{0t}(y_0) - H_{1t}(z_{1t}(y_0))}{f_1(\tilde{z}_{1t}(y_0))} - \frac{[H_0(y_0) - H_1(z_1(y_0))]}{f_1(z_1(y_0))} \right| f(y_0, z_1(y_0)) dy_0 \\
&\stackrel{(0)}{\leq} c \int_1^\infty \left\{ \begin{aligned} & |H_{0t}(y_0) - H_{1t}(z_{1t}(y_0)) - [H_0(y_0) - H_1(z_1(y_0))]| \\ & \times (1 - F_0(y_0))^{\frac{\alpha}{\alpha-1}} f(y_0, z_1(y_0)) \end{aligned} \right\} dy_0 \\
&\leq c \left\{ \begin{aligned} & \sup_{y_0} |H_{0t}(y_0) - H_{1t}(z_t(y_0)) - [H_0(y_0) - H_1(z(y_0))]| \\ & \times \int_1^\infty (1 - F_0(y_0))^{\frac{\alpha}{\alpha-1}} f(y_0, z_1(y_0)) dy_0 \end{aligned} \right\}
\end{aligned}$$

which goes to zero if $\int_1^\infty (1 - F_0(y_0))^{\frac{\alpha}{\alpha-1}} f(z(y_0), y_0) dy_0 < \infty$, which is condition **(Aiv)**.

Note that the inequality $\stackrel{(0)}{\leq}$ follows from step (ii) in the proof of Step 1, above. Finally, since $\int_1^{z_{0t}} \frac{H_0(y_0) - H_1(z_1(y_0))}{f_1(z_1(y_0))} f(y_0, z_1(y_0)) dy_0$ is continuous in z_{0t} , the conclusion follows.

Step 4:

$T_{4t} \rightarrow 0$ since $H_t \rightarrow H$ uniformly and T_{5t} goes to zero by the continuous mapping theorem since paths of an F -Brownian bridge are everywhere continuous with probability 1. ■

Functional delta method and proposition 3: Since $\sqrt{n}(\hat{F} - F) \rightsquigarrow \mathcal{G}$, we have from lemma 1 and the functional delta method that

$$\sqrt{n}(\hat{v} - v_0) \xrightarrow{d} v'_F(\mathcal{G}),$$

whence $v'_F(\mathcal{G})$ will be distributed as a univariate zero mean normal given by

$$\begin{aligned} v'_F(\mathcal{G}) &= \frac{\mathcal{G}_0(F_0^{-1}(s))}{f_0(F_0^{-1}(s))} \int_1^{F_1^{-1}(F_0(y_0)+\tau)} f(F_0^{-1}(s), y_1) dy_1 \\ &\quad + \int_1^{F_0^{-1}(s)} \frac{\mathcal{G}_0(y_0) - \mathcal{G}_1(F_1^{-1}(F_0(y_0) + \tau))}{f_1(F_1^{-1}(F_0(y_0) + \tau))} f(y_0, F_1^{-1}(F_0(y_0) + \tau)) dy_0 \\ &\quad + \int_1^{F_0^{-1}(s)} \int_1^{F_1^{-1}(F_0(y_0)+\tau)} d\mathcal{G}(y_0, y_1), \end{aligned}$$

where \mathcal{G}_0 and \mathcal{G}_1 are stochastic processes defined from \mathcal{G} , analogous to (17), e.g., $\mathcal{G}_0(a)$ is a univariate normal with mean zero and variance $F_0(a) \times [1 - F_0(a)]$. Now we can apply the functional delta method argument to justify consistency of the bootstrap via van der Vaart and Wellner (1996), theorem 3.9.11.

Part C:

Proposition 3: Recall that

$$\begin{aligned} v_c(\tau, s; x) &= \Pr(F_1(Y_1) - F_0(Y_0) > \tau | F_0(Y_0) \leq s, X = x) \\ &= \frac{\Pr(F_1(Y_1) - F_0(Y_0) > \tau, F_0(Y_0) \leq s | X = x)}{\Pr(F_0(Y_0) \leq s | X = x)} := \frac{A(F_0, F_1, x)}{B(F_0, x)}, \end{aligned}$$

is estimated by

$$\begin{aligned} &\hat{v}_c(\tau, s; x) \\ &= \frac{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) \mathbf{1}\left(\hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \hat{F}_0(Y_{0i}) \leq s\right)}{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) \mathbf{1}\left(\hat{F}_0(Y_{0i}) \leq s\right)} / \frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) \\ &: = \frac{\hat{A}_n(\hat{F}_0, \hat{F}_1, x)}{\hat{B}_n(\hat{F}_0, x)}, \end{aligned}$$

where for $l = 0, 1$, $\hat{F}_l(Y_{li}) = \frac{1}{n-1} \sum_{j \neq i} 1(Y_{lj} \leq Y_{li})$ and $K(\cdot)$ is a d -dimensional kernel function with a bandwidth sequence σ_n , satisfying the NW conditions specified in the appendix. The object whose distribution is needed is given by:

$$\hat{v}_c(\tau, s; x) - v_c(\tau, s; x) = \frac{\hat{A}_n(\hat{F}_0, \hat{F}_1, x)}{\hat{B}_n(\hat{F}_0, x)} - \frac{A(F_0, F_1, x)}{B(F_0, x)}.$$

Let

$$\begin{aligned} \hat{f}(x) &= \frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) \\ \hat{A}_n(F_0, F_1, x) &= \frac{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) 1(F_1(Y_{1i}) - F_0(Y_{0i}) > \tau, F_0(Y_{0i}) \leq s)}{\hat{f}(x)}, \\ \hat{B}_n(F_0, x) &= \frac{\frac{1}{n\sigma_n^d} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) 1(F_0(Y_{0i}) \leq s)}{\hat{f}(x)}, \\ W &: = 1(F_1(Y_1) - F_0(Y_0) > \tau, F_0(Y_0) \leq s), \\ V &= 1(F_0(Y_0) \leq s). \end{aligned}$$

Then

$$\begin{aligned} T_n(x) &: = \hat{A}_n(\hat{F}_0, \hat{F}_1, x) - A(F_0, F_1, x) \\ &= \underbrace{\left\{ \hat{A}_n(F_0, F_1, x) - A(F_0, F_1, x) \right\}}_{\tilde{T}_{1n}(x)} + \underbrace{\left\{ \hat{A}_n(\hat{F}_0, \hat{F}_1, x) - \hat{A}_n(F_0, F_1, x) \right\}}_{\tilde{T}_{2n}(x)}. \end{aligned}$$

We will show that $\tilde{T}_{2n}(x)$ is of smaller order of magnitude than $\tilde{T}_{1n}(x)$. This will imply that asymptotically, the distribution of T_n will be that of \tilde{T}_{1n} which is simply the Nadaraya-Watson regression of the unobserved random variable W on X , evaluated at $X = x$, denoted by $\hat{E}(W|X = x)$. The formal result is stated below and its proof appears in the appendix under theorem 2. An exactly analogous result holds for the denominator $\hat{B}_n(\hat{F}_0, x)$.

The following additional assumption is used.

Assumption NW5'. The functions $f(x)$, $E(W|X = x)$ and $f(x) \times E(W|X = x)$ are twice differentiable and the functions and their derivatives up to order 2 are continuous and uniformly bounded. X has bounded support with density bounded away from zero.

Claim: Suppose the data (X_i, Y_{1i}, Y_{0i}) for $i = 1, \dots, n$, are i.i.d. and assumptions **NW1-4**, **NW5'** hold. Then, we have that

$$(n\sigma_n^d)^{1/2} \tilde{T}_{2n}(x) = o_p(1).$$

Proof of claim and thus proposition 3.: Let $T_{2n}(x) = \tilde{T}_{2n}(x) \hat{f}(x)$. We will now show that $E\left(\sqrt{n\sigma_n^d} |T_{2n}(x)|\right) \rightarrow 0$ which will imply that $|T_{2n}(x)| = o_p\left((n\sigma_n^d)^{-1/2}\right)$ and thus establish the result since $\hat{f}(x)$ is bounded in probability by assumption.

First observe that

$$\begin{aligned} & E \left[\left[\begin{array}{l} 1 \left(\hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \hat{F}_0(Y_{0i}) \leq s \right) \\ -1 \left(F_1(Y_{1i}) - F_0(Y_{0i}) > \tau, F_0(Y_{0i}) \leq s \right) \end{array} \right] \right] \\ = & \Pr \left[\left[\begin{array}{l} 1 \left(\hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \hat{F}_0(Y_{0i}) \leq s \right) \\ -1 \left(F_1(Y_{1i}) - F_0(Y_{0i}) > \tau, F_0(Y_{0i}) \leq s \right) \end{array} \right] \neq 0 \right] \\ = & \Pr \left[\begin{array}{l} \left\{ \hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \hat{F}_0(Y_{0i}) \leq s \right\} \\ \cap \left(F_1(Y_{1i}) - F_0(Y_{0i}) > \tau, F_0(Y_{0i}) \leq s \right)^c \end{array} \right] \\ & + \Pr \left[\begin{array}{l} \left\{ \hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \hat{F}_0(Y_{0i}) \leq s \right\}^c \\ \cap \left(F_1(Y_{1i}) - F_0(Y_{0i}) > \tau, F_0(Y_{0i}) \leq s \right) \end{array} \right] \\ \leq & \Pr \left[\left\{ \hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \hat{F}_0(Y_{0i}) \leq s \right\} \cap \left(F_1(Y_{1i}) - F_0(Y_{0i}) \leq \tau \right) \right] \\ & + \Pr \left[\left\{ \hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \hat{F}_0(Y_{0i}) \leq s \right\} \cap \left(F_0(Y_{0i}) > s \right) \right] \\ & + \Pr \left[\left\{ F_1(Y_{1i}) - F_0(Y_{0i}) > \tau, F_0(Y_{0i}) \leq s \right\} \cap \left(\hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) \leq \tau \right) \right] \\ & + \Pr \left[\left\{ F_1(Y_{1i}) - F_0(Y_{0i}) > \tau, F_0(Y_{0i}) \leq s \right\} \cap \left(\hat{F}_0(Y_{0i}) > s \right) \right] \\ \leq & \Pr \left[\hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, F_1(Y_{1i}) - F_0(Y_{0i}) \leq \tau \right] \\ & + \Pr \left\{ \hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) \leq \tau, F_1(Y_{1i}) - F_0(Y_{0i}) > \tau \right\} \\ & + \Pr \left[\hat{F}_0(Y_{0i}) \leq s, F_0(Y_{0i}) > s \right] + \Pr \left[F_0(Y_{0i}) \leq s, \hat{F}_0(Y_{0i}) > s \right]. \end{aligned}$$

Therefore,

$$\begin{aligned}
& E \left(\sqrt{n\sigma_n^d} |T_{2n}(x)| \right) \\
&= \frac{1}{\sqrt{n\sigma_n^d}} \sum_{i=1}^n E \left(K \left(\frac{X_i - x}{\sigma_n} \right) \left| \begin{array}{l} 1 \left(\hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \hat{F}_0(Y_{0i}) \leq s \right) \\ -1 \left(F_1(Y_{1i}) - F_0(Y_{0i}) > \tau, F_0(Y_{0i}) \leq s \right) \end{array} \right| \right) \\
&= \frac{1}{\sqrt{n\sigma_n^d}} \sum_{i=1}^n E_{X_i, Y_{0i}, Y_{1i}} \left\{ \begin{array}{c} K \left(\frac{X_i - x}{\sigma_n} \right) \\ \times E \left(\left| \begin{array}{l} 1 \left(\hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \hat{F}_0(Y_{0i}) \leq s \right) \\ -1 \left(F_1(Y_{1i}) - F_0(Y_{0i}) > \tau, F_0(Y_{0i}) \leq s \right) \end{array} \right| \middle| X_i, Y_{0i}, Y_{1i} \right) \end{array} \right\} \\
&\leq \frac{1}{\sqrt{n\sigma_n^d}} \sum_{i=1}^n E_{X_i, Y_{0i}, Y_{1i}} \left\{ K \left(\frac{X_i - x}{\sigma_n} \right) \Pr \left\{ \left(\begin{array}{l} \hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \\ F_1(Y_{1i}) - F_0(Y_{0i}) \leq \tau \end{array} \right) \middle| X_i, Y_{0i}, Y_{1i} \right\} \right\} \\
&\quad + \frac{1}{\sqrt{n\sigma_n^d}} \sum_{i=1}^n E_{X_i, Y_{0i}, Y_{1i}} \left\{ K \left(\frac{X_i - x}{\sigma_n} \right) \Pr \left\{ \left(\begin{array}{l} \hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) \leq \tau, \\ F_1(Y_{1i}) - F_0(Y_{0i}) > \tau \end{array} \right) \middle| X_i, Y_{0i}, Y_{1i} \right\} \right\} \\
&\quad + \frac{1}{\sqrt{n\sigma_n^d}} \sum_{i=1}^n E_{X_i, Y_{0i}} E \left\{ K \left(\frac{X_i - x}{\sigma_n} \right) \Pr \left\{ \hat{F}_0(Y_{0i}) \leq s, F_0(Y_{0i}) > s \middle| X_i, Y_{0i} \right\} \right\} \\
&\quad + \frac{1}{\sqrt{n\sigma_n^d}} \sum_{i=1}^n E_{X_i, Y_{0i}} E \left\{ K \left(\frac{X_i - x}{\sigma_n} \right) \Pr \left\{ \hat{F}_0(Y_{0i}) > s, F_0(Y_{0i}) \leq s \middle| X_i, Y_{0i} \right\} \right\} \\
&: = S_{1n} + S_{2n} + S_{3n} + S_{4n}, \text{ say.} \tag{23}
\end{aligned}$$

We will show that $S_{1n} \rightarrow 0$ and an exactly analogous proof will show that S_{2n}, S_{3n}, S_{4n} are also $o(1)$.

Now, for fixed X_i, Y_{0i}, Y_{1i} and the fact that e.g. $\hat{F}_1(Y_{1i}) = \frac{1}{n-1} \sum_{j \neq i} 1(Y_{ij} \leq Y_{1i})$, we have that

$$\begin{aligned}
& \Pr \left(\hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, F_1(Y_{1i}) - F_0(Y_{0i}) < \tau \middle| X_i, Y_{0i}, Y_{1i} \right) \\
&= \Pr \left(\begin{array}{c} \hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) - (F_1(Y_{1i}) - F_0(Y_{0i})) > \tau - (F_1(Y_{1i}) - F_0(Y_{0i})), \\ F_1(Y_{1i}) - F_0(Y_{0i}) < \tau \end{array} \middle| X_i, Y_{0i}, Y_{1i} \right) \\
&\leq \exp \left(-2(n-1) (\tau - (F_1(Y_{1i}) - F_0(Y_{0i})))^2 \right) \times 1(F_1(Y_{1i}) - F_0(Y_{0i}) < \tau),
\end{aligned}$$

by Hoeffding's inequality (note that conditional on Y_{1i} , $\hat{F}_1(Y_{1i}) = \frac{1}{n-1} \sum_{j \neq i} 1(Y_{ij} \leq Y_{1i})$ is an average of independent, binary $(0, 1)$ random variables, thus satisfying the hypothesis

of Hoeffding's inequality). Thus, we have that

$$\begin{aligned}
& S_{1n} \\
& \leq \frac{1}{\sqrt{n\sigma_n^d}} \sum_{i=1}^n E_{X_i, Y_{0i}, Y_{1i}} \left[\begin{array}{c} K\left(\frac{X_i - x}{\sigma_n}\right) \\ \times \exp\left(-2(n-1)(\tau - (F_1(Y_{1i}) - F_0(Y_{0i})))^2\right) \\ \times 1(F_1(Y_{1i}) - F_0(Y_{0i}) < \tau) \end{array} \right] \\
& = \frac{n}{\sqrt{n\sigma_n^d}} E_{X, Y_0, Y_1} \left[\begin{array}{c} K\left(\frac{X-x}{\sigma_n}\right) \exp\left(-2(n-1)(\tau - (F_1(Y_1) - F_0(Y_0)))^2\right) \\ \times 1(F_1(Y_1) - F_0(Y_0) < \tau) \end{array} \right] \\
& = \frac{n}{\sqrt{n\sigma_n^d}} E_X \left[K\left(\frac{X-x}{\sigma_n}\right) G_n(X) \right], \text{ where} \\
G_n(x) & = E_{Y_0, Y_1 | X} \left[\begin{array}{c} \exp\left(-2(n-1)(\tau - (F_1(Y_1) - F_0(Y_0)))^2\right) \\ \times 1(F_1(Y_1) - F_0(Y_0) < \tau) | X = x \end{array} \right].
\end{aligned}$$

Continuing with the previous display, we have

$$\begin{aligned}
S_{1n} & \leq \frac{n}{\sqrt{n\sigma_n^d}} E_X \left[K\left(\frac{X-x}{\sigma_n}\right) G_n(X) \right] \\
& = \frac{n\sigma_n^d}{\sqrt{n\sigma_n^d}} \int [K(u) G_n(x + \sigma_n u) f(x + \sigma_n u)] du \\
& = \sqrt{n\sigma_n^d} \int [K(u) G_n(x + \sigma_n u) f(x + \sigma_n u)] du \\
& = f(x) \sqrt{n\sigma_n^d} \int K(u) G_n(x) du + \text{terms of smaller order} \\
& = f(x) \sqrt{n\sigma_n^d} G_n(x) + \text{terms of smaller order.} \tag{24}
\end{aligned}$$

Now, notice that defining $Z = \tau - (F_1(Y_{1i}) - F_0(Y_{0i}))$, $G_n(x)$ is of the form

$$\begin{aligned}
G_n(x) & = E_{Z|X} [\exp(-2(n-1)Z^2) \times 1(Z > 0) | X = x] \\
& \leq c \int \exp(-2(n-1)z^2) f(z|x) dz \\
& \leq c' \int \exp(-2(n-1)z^2) dz \\
& = O(n^{-1/2}) \tag{25}
\end{aligned}$$

by the normal (Gaussian) integral formula. From (24) and (25), it follows that

$$E\left(\sqrt{n\sigma_n^d} |T_{2n}(x)|\right) = O\left(n^{-1/2} \times \sqrt{n\sigma_n^d}\right) = O\left(\sqrt{\sigma_n^d}\right) = o(1).$$

Together with analogous proofs for S_{2n}, S_{3n}, S_{4n} , this implies that $\sqrt{n\sigma_n^d}T_{2n}(x) = o_p(1)$, which is the desired result. ■

Part D:

Uniform confidence bands for conditional mobility (following Hardle (1990), algorithm 4.3.2):

First note that the inequality in step 3 of (25) is uniform in x because X has bounded support. Consequently the "terms of smaller order" in (24) are also small uniformly in x . Given that $f(\cdot)$ is also uniformly bounded by **NW5'**, the conclusion $\sqrt{n\sigma_n^d}T_{2n}(x) = o_p(1)$ can be strengthened to $\sup_x \sqrt{n\sigma_n^d}|T_{2n}(x)| = o_p(1)$. This shows that $(n\sigma_n^d)^{1/2}[\hat{v}_c(\tau, s; x) - v_c(\tau, s; x)]$ as an empirical process, is asymptotically equivalent to linear combinations of the Nadaraya-Watson regression residual processes

$$\begin{aligned} & (n\sigma_n^d)^{1/2} \left\{ \hat{E}(W|X=x) - E(W|X=x) \right\}, \text{ and} \\ & (n\sigma_n^d)^{1/2} \left\{ \hat{E}(V|X=x) - E(V|X=x) \right\}, \end{aligned}$$

which converge weakly to zero-mean Gaussian processes. Consequently, they are amenable to the treatment in Hardle (1990), based on Bickel and Rosenblatt (1973), which develops uniform confidence bands for NW regression curves. We now outline Hardle's construction.

For each sample value x of the conditioning variable X , bandwidth σ_n and kernel $K(\cdot)$, denote estimated density at $X = x$ by

$$\hat{f}(x) = \frac{1}{n\sigma_n} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right).$$

Consider dependent variables $W_i = 1\left(\hat{F}_1(Y_{1i}) - \hat{F}_0(Y_{0i}) > \tau, \hat{F}_0(Y_{0i}) \leq s\right)$ for upward mobility and $W_i = 1\left(Y_{1i} \leq \hat{\zeta}_1, Y_{0i} \leq \hat{\zeta}_0\right)$ for transition probability. Denote regression estimate (predicted value) at $X = x$ by

$$\hat{\mu}(x) = \frac{\frac{1}{n\sigma_n} \sum_{i=1}^n K\left(\frac{x_i - x}{\sigma_n}\right) W_i}{\hat{f}(x)}.$$

Corresponding to Epanechnikov kernel, set

$$\begin{aligned} c_K &= \int_{-1}^1 K^2(u) du = \int_{-1}^1 \frac{9}{16} (1 - u^2)^2 du = 3/5 = 0.6 \\ C_2 &= \frac{\int_{-1}^1 \{K'(u)\}^2 du}{2c_K} = 1.25 \end{aligned}$$

-

$$\delta = \sqrt{2 \ln \left(\frac{1}{\sigma_n} \right)}$$

-

$$\begin{aligned} d_n &= \sqrt{\left(2 \ln \left(\frac{1}{\sigma_n} \right) \right)} + \frac{1}{2 \sqrt{\left(2 \ln \left(\frac{1}{\sigma_n} \right) \right)}} \ln \left(\frac{C_2}{2\pi^2} \right) \\ &= \delta + \frac{1}{2\delta} \ln \left(\frac{C_2}{2\pi^2} \right) \end{aligned}$$

-

$$c_\alpha = -\ln(-0.5 \times \ln(1 - .05)) = 3.66$$

-

$$\omega^2(x) = \frac{1}{n\sigma_n \hat{f}(x)} \sum_{i=1}^n \{W_i - \hat{\mu}(X_i)\}^2 K \left(\frac{X_i - x}{\sigma_n} \right)$$

- Then lower and upper limit of uniform CI are given by

$$\begin{aligned} CLO(x) &= \hat{\mu}(x) - \left\{ \frac{c_\alpha}{\delta} + d_n \right\} \times \frac{\sqrt{\frac{\omega^2(x)}{n\sigma_n \times \hat{f}(x)}}}{c_K} \\ CUP(x) &= \hat{\mu}(x) + \left\{ \frac{c_\alpha}{\delta} + d_n \right\} \times \frac{\sqrt{\frac{\omega^2(x)}{n\sigma_n \times \hat{f}(x)}}}{c_K}. \end{aligned}$$

- Use additional assumption:

NW6: The density of X is bounded away from zero on its support.

- Justification of the above algorithm comes from theorem 4.3.1 of Hardle, 1990. Our assumptions NW will imply conditions A1-A5 of Hardle (1990). To see this note that NW5' implies A1 (note that for a binary outcome Y , the variance $var(Y|x) = E(Y|x) \times (1 - E(Y|x))$ so that differentiability of $E(Y|x)$ implies differentiability of $var(Y|x)$); the Epanechnikov kernel satisfies condition A2, the outcome variables lie in $[0, 1]$ with probability one, implying A3; condition NW6 implies A4 and choice of bandwidth σ_n equal to $n^{-1/4}$ implies A5.

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