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Abstract

We consider a sequence of random length M of independent absolutely continuous observations $X_i,\ 1 \leq i \leq M$, where M is geometric, X_1 has cdf G, and $X_i,\ i \geq 2$, have cdf F. Let N be the number of upper records and $R_n, n \geq 1$, be the nth record value. We show that N is free of F if and only if $G(x) = G_0(F(x))$ for some cdf G_0 and that if $E(|X_2|)$ is finite so is $E(|R_n|)$ for $n \geq 2$ whenever $N \geq n$ or $N = n$. We prove that the distribution of N along with appropriately chosen subsequences of $E(R_n)$ characterize F and G, and along with subsequences of $E(R_n - R_{n-1})$ characterize F and G up to a common location shift. We discuss some applications to the identification of the wage offer distribution in job search models.

Key Words: ***Moment sequences; Number of records; Record spacings; Geometric distribution; Müntz-Szász theorem; Titchmarsh convolution theorem; Job search models.***

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Abbreviated Title: Characterizations in a record model

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1. Introduction

Let $\{X_i,\ i \geq 1\}$ be a sequence of independent random variables. Suppose that M is a positive integer-valued random variable independent of the X_i, and assume that only $\{X_i, 1 \leq i \leq M\}$ are observed. Define $L(1) = 1$ and $L(n) = \min\left\{k : X_k > X_{L(n-1)}\right\}$ for $n > 1$, and $R_n = X_{L(n)}$ for $n \geq 1$. Then R_1 is the initial record (sometimes called the trivial record), and R_n for $n \geq 2$ represent the upper record values from the sequence $\{X_i,\ i \geq 1\}$. The total number of records we observe is given by $N = \max\{j : L(j) \leq M\}$ and is itself a random variable.

When the X_i are identically distributed, this model is called the *random record model* (see, e.g., Arnold et al., 1998, p. 224). When we further assume that M has a geometric distribution, i.e. $\Pr(M = m) = q^{m-1}p$ for $m \geq 1$, where $0 < p < 1$ and $q = 1 - p$, we have a geometric random record (GRR) model. Nagaraja and Barlevy (2003) derived several characterization results for the GRR model using record moments. In this paper, we consider a variation of the GRR model in which the initial observation, X_1, has a potentially different distribution from remaining observations $\{X_i,\ i \geq 2\}$. We refer to this as a *GRR model with a non-identically distributed initial record* or a *modified GRR model.* Our purpose in this paper is to determine whether there exist analogous characterization results for this alternative formulation and to discuss some applications of this variation concerning identification of job search models.

Formally, let X_1 be distributed with continuous cumulative distribution function (cdf) G, and $\{X_i,\ i \geq 2\}$ be independent and identically distributed (i.i.d.) with continuous cdf F.

Define a mapping Γ from the set of continuous distribution functions into itself so that $G = \Gamma(F)$. This notation allows us to view the model as being parameterized by a single cdf F. We impose the following assumptions on Γ:

Assumption 1: the probability measure implied by $G = \Gamma(F)$ is absolutely continuous with respect to the probability measure implied by F.

Assumption 2: the composite function $G\left(F^{-1}(u)\right)$ is absolutely continuous in $u \in (0, 1)$.

The first assumption implies that the support of G must always form a subset of the support of F. The second assumption implies that without loss of generality we can assume X_1 has a well-defined density function, since we can always normalize $F(x) = x$. We denote this density function $g(x; F(\cdot)) = \frac{d}{dx} G(x; F(\cdot))$.

Remark 1: The identity mapping $\Gamma : F \to F$ satisfies Assumptions 1 and 2. Our model thus includes the GRR model as a special case.

Remark 2: Our formulation is itself a special case of the Pfeifer (1982) model, in which the distribution of the underlying observations changes after each record is set. Here, the distribution changes only after the first record, and the distribution of the first record $G = \Gamma(F)$ is required to satisfy Assumptions 1 and 2. Although Pfeifer assumes $M = \infty$, Bunge and Nagaraja (1991) subsequently generalized Pfiefer's model to allow the number of observations M to be random.

Remark 3: Our assumptions do not require Γ to be one-to-one, as illustrated by Example 2 below. However, Assumption 1 implies Γ cannot assign a single G to all cdf's F. Thus, G cannot be free of F.

Here are some examples of functions $G(x)$ that satisfy Assumptions 1-2. The motivation for these examples will become clear in Section 6, when we discuss how the model can be applied to estimate job search models.

1. $G(x; F(\cdot)) = \dfrac{F(x)}{1 + \kappa(1 - F(x))}$ for some constant κ
2. $G(x; F(\cdot)) = F(x)/z$ if $F(x) \le z$ and 1 if $F(x) > z$ for some constant $z \in (0, 1)$
3. $G(x; F(\cdot)) = \dfrac{\int_{-\infty}^{x} H(w)\, dF(w)}{\int_{-\infty}^{\infty} H(w)\, dF(w)} = \dfrac{\int_0^{F(x)} H\left(F^{-1}(u)\right) du}{\int_0^1 H\left(F^{-1}(u)\right) du}$ where $H(\cdot)$ is a cdf.

In Example 1, $G(x) = \sum_{i=1}^{\infty} pq^{i-1} F(x)^i$, where $p = (1+\kappa)^{-1}$, or X_1 has the same distribution as the maximum of a random (geometric) number of i.i.d. random variables distributed like X_2. In Example 2, G arises from F by the truncation of its upper tail, and in Example 3, G has the form of a weighted distribution. Note that in the first two examples, $G = \Gamma(F)$ assumes the form $G_0(F(x))$ for some function G_0, i.e. the cdf G evaluated at x depends on $F(x)$ but not on the value of F at any point other than x. This is not true for the last example.

We show in Section 2 of the paper that $\Gamma(F) = G_0(F(x))$ if and only if the distribution of the number of observed records N is independent of F. In Section 3, we focus on mappings Γ where $\Gamma(F) = G_0(F(x))$, and show that under an additional assumption on Γ, the distribution F is characterized by subsequences of the following moments:

a. $E(R_n \mid N \geq n)$

b. $E(R_n \mid N = n)$

c. $E(R_n - R_{n-1} \mid N \geq n)$

d. $E(R_n - R_{n-1} \mid N = n)$.

In demonstrating this result, we appeal not only to the Müntz-Szász theorem, which is often invoked in moment-based characterization theorems (see Kamps, 1998), but also to a convolution theorem due to Titchmarsh (1926). In Section 4, we consider arbitrary mappings Γ that satisfy Assumptions 1 and 2, and provide characterizations of F and G using subsequences of these moments together with the distribution $\Pr(N = n)$. All of these results are premised on a fixed distribution for M. In Section 5, we derive conditions that jointly characterize F and G as well as the distribution for M. Section 6 then discusses how our results can be used to non-parametrically identify the wage offer distribution in job search models when wage data can only be measured with noise.

2. Characterization results for Γ and N

We begin with results that characterize the mapping Γ. Our first result shows that the number of observed records N is identical for all continuous cdf's F if and only if Γ is such that $G = \Gamma(F)$ evaluated at x can be expressed as a function of $F(x)$.

Proposition 1: In the modified GRR model, the number of observed records N is independent of F if and only if $G(x; F(\cdot)) = G_0(F(x))$ for some absolutely continuous cdf G_0 with support $[0, 1]$.

Proof: Building on Bunge and Nagaraja (1991) and Nagaraja and Barlevy (2003), we can express the likelihood of exactly n observed records with values r_1 through r_n as

$$h(r_1, ..., r_n \ \cap N = n) = \frac{(1-q)\, g(r_1; F(\cdot))}{1 - qF(r_1)} \prod_{i=2}^{n} \frac{qf(r_i)}{1 - qF(r_i)}.$$

Next, we integrate out r_2 through r_n to get

$$h(r_1 \ \cap N = n) = \frac{1}{(n-1)!} \frac{1-q}{1-qF(r_1)} \left(\ln \left(\frac{1-qF(r_1)}{1-q} \right) \right)^{n-1} g(r_1; F(\cdot)).$$

Hence, $\Pr(N = n)$ can be expressed as

$$\Pr(N = n) = \frac{1}{(n-1)!} \int_{-\infty}^{\infty} \frac{1-q}{1-qF(r_1)} \left[\ln \left(\frac{1-qF(r_1)}{1-q} \right) \right]^{n-1} g(r_1; F(\cdot))\, dr_1.$$

Suppose $G(x; F(\cdot)) = G_0(F(x))$ where $G_0(\cdot)$ is an absolutely continuous function. We want to show that $\Pr(N = n)$ is independent of $F(\cdot)$. Since G_0 is absolutely continuous, it has a related density function $g_0(x) = \frac{d}{dx} G_0(x)$. This implies

$$g(r_1; F(\cdot)) = \frac{d}{dx} G_0(F(x)) \bigg|_{x=r_1} = g_0(F(r_1))\, f(r_1).$$

Substituting this in and using the change of variables $u = F(r_1)$, we find

$$\Pr(N = n) = \frac{1}{(n-1)!} \int_0^1 \frac{1-q}{1-qu} \left[\ln \left(\frac{1-qu}{1-q} \right) \right]^{n-1} g_0(u)\, du$$

which is indeed independent of $F(\cdot)$.

Next, suppose $\Pr(N = n)$ is independent of $F(\cdot)$. We want to show this implies $G(x; F(\cdot)) = G_0(F(x))$ where $G_0(\cdot)$ is an absolutely continuous cdf. Given Assumption 1, we can rewrite $\Pr(N = n)$ using the change of variables $u = F(r_1)$ so that

$$\Pr(N = n) = \frac{1}{(n-1)!} \int_0^1 \frac{1-q}{1-qu} \left[\ln \left(\frac{1-qu}{1-q} \right) \right]^{n-1} \frac{g\left(F^{-1}(u)\right)}{f\left(F^{-1}(u)\right)} du. \tag{1}$$

Since $\{\Pr(N = n), n \geq 1\}$ does not depend on the distribution of $F(\cdot)$, then for any two distributions $F_1(\cdot)$ and $F_2(\cdot)$, we have for $n = 1, 2, 3, ...$

$$\int_0^1 \frac{1-q}{1-qu} \left[\ln \left(\frac{1-qu}{1-q} \right) \right]^{n-1} \frac{g\left(F_1^{-1}(u); F_1(\cdot)\right)}{f_1\left(F_1^{-1}(u)\right)} du$$
$$= \int_0^1 \frac{1-q}{1-qu} \left[\ln \left(\frac{1-qu}{1-q} \right) \right]^{n-1} \frac{g\left(F_2^{-1}(u); F_2(\cdot)\right)}{f_2\left(F_2^{-1}(u)\right)} du.$$

Let $F_1(\cdot) = F(\cdot)$ be any continuous distribution function, and let $F_2(\cdot)$ to be the uniform distribution, i.e. $F_2^{-1}(u) = u$ and $f_2(u) = 1$ for all $u \in (0, 1)$. Let us further define

$h(u) = g(u;u)$, i.e. the density function $g(\cdot)$ evaluated when $F(\cdot)$ is uniform. By the Müntz-Szász theorem, it follows that for almost all $u \in (0,1)$,

$$\frac{g\left(F^{-1}(u);F(\cdot)\right)}{f\left(F^{-1}(u)\right)} = h(u).$$

Since $G(x;F_1(\cdot)) = \int_{-\infty}^{x} g(y;F_1(\cdot))\,dy$, we have

$$\begin{aligned}
G(x;F(\cdot)) &= \int_{-\infty}^{x} g(y;F(\cdot))\,dy \\
&= \int_{-\infty}^{x} h(F(y))\,f(y)\,dy \\
&= \int_{0}^{F(x)} h(z)\,dz \\
&\equiv G_0(F(x)).
\end{aligned}$$

Since $G(\cdot)$ is a cdf, it follows that $G_0(\cdot)$ is non-decreasing, $G_0(0) = 0$, and $G_0(1) = 1$. Hence, G_0 is a cdf with support $[0,1]$. Absolute continuity of G_0 is immediate. ■

The probability mass function (pmf) of N in (1) implies $N-1$ has a mixed Poisson distribution. An interesting property of this pmf, stated below, will be useful for us later.

Lemma 1: In the modified GRR model, the sequence $\{\Pr(N = n_j), j \geq 1\}$ such that $\sum_{j=1}^{\infty} n_j^{-1} = \infty$ uniquely determines the pmf of N.

Proof: Consider two modified GRR models with cdfs F_1, G_1 and F_2, G_2. If $\Pr(N = n_j)$ remains the same for these two models, from (1) we have

$$\int_0^1 \frac{1-q}{1-qu}\left[\ln\left(\frac{1-qu}{1-q}\right)\right]^{n_j-1}\left[\frac{g_1\left(F_1^{-1}(u)\right)}{f_1\left(F_1^{-1}(u)\right)} - \frac{g_2\left(F_2^{-1}(u)\right)}{f_2\left(F_2^{-1}(u)\right)}\right]du = 0.$$

Set $t = \ln\left(\dfrac{1-qu}{1-q}\right)$ and rewrite the equation above as

$$\int_0^{-\ln(1-q)} h_1(t)t^{n_j-1}dt = 0.$$

It follows from the Müntz-Szász theorem that $h_1(t) = 0$ almost everywhere or

$$\frac{g_1\left(F_1^{-1}(u)\right)}{f_1\left(F_1^{-1}(u)\right)} = \frac{g_2\left(F_2^{-1}(u)\right)}{f_2\left(F_2^{-1}(u)\right)} \tag{2}$$

for almost all $u \in (0,1)$. Again, upon appealing to (1) we conclude that $\Pr(N = n)$, and thus $\Pr(N \geq n)$, remain the same for all $n \geq 1$ under the two models. ■

The mapping Γ can also be characterized by the dependence structure of the record indicators derived from $\{X_i, i \geq 1\}$. Define the m-th record indicator I_m as a random variable that takes on a value of 1 if X_m is a record, i.e. if $X_m > \max\{X_1, ..., X_{m-1}\}$, and 0 otherwise. Previous work by Nevzorov (1986) has shown that if X_i are independent random variables and X_i has cdf

$$F_i(x) = \{F(x)\}^{\alpha(i)}, \tag{3}$$

for some common F, then $\{I_m,\ m \geq 2\}$ are independent and I_m is Bernoulli with success probability $\alpha(m)/\{\sum_{i=1}^{m} \alpha(i)\}$. Conversely, Nevzorov shows that if the supports of X_i are not disjoint and $\{I_2, ...,\ I_{n-1}\}$ and I_n are independent for *any* F_n, then (3) holds for $m = 1, \ldots, n-1$. (see also Arnold et al (1998), p. 219). In our model, (3) implies that

$$G(x) = \Gamma(F(x)) = \{F(x)\}^{\alpha} \tag{4}$$

for some $\alpha > 0$. Our next proposition establishes that Γ satisfies (4) if and only if the record indicator I_2 is independent of the record indicators $\{I_3, ..., I_m\}$ for all m, conditional on at least m observations.

Proposition 2: In a random record model with a non-identically distributed initial record,

$$\Pr(I_3 = 1, \ldots, I_m = 1 \mid M \geq m) = \Pr(I_3 = 1, \ldots, I_m = 1 \mid I_2 = 1, M \geq m) \tag{5}$$

for all $m = m_j (\geq 3), j \geq 1$ such that $\sum_j m_j^{-1} = \infty$, if and only if (4) holds.

Proof: The given condition (5) can be expressed as

$$\Pr(I_2 = 1, \ldots, I_m = 1) = \Pr(I_3 = 1, \ldots, I_m = 1) \Pr(I_2 = 1).$$

Without loss of generality we take $F(x) = x$. Upon conditioning on X_1, the LHS can be simplified to

$$\frac{1}{(m-1)!} \int_0^1 (1-x)^{m-1} dG(x) \quad \text{or} \quad \frac{1}{(m-2)!} \int_0^1 (1-x)^{m-2} G(x) dx.$$

Further $\Pr(I_3 = 1, \ldots, I_m = 1) = \text{LHS} + \Pr(I_2 = 0, I_3 = 1, \ldots, I_m = 1)$. The second term, upon conditioning on X_1, can be written as

$$\frac{1}{(m-2)!} \int_0^1 (1-x)^{m-2} x g(x) dx.$$

Let $c = \Pr(I_2 = 1)(= \int_0^1 G(x)dx)$. Then (5) reduces to the condition

$$\int_0^1 (1-x)^{m-2}[(1-c)G(x) - cxg(x)]dx = 0, \quad m = m_j,\ j \geq 1. \tag{6}$$

By the Müntz-Szász theorem, it follows that for almost all $x \in (0,1)$,

$$\frac{g(x)}{G(x)} = \frac{(1-c)}{c}\frac{1}{x}.$$

This differential equation clearly shows that (4) holds with $\alpha = (1-c)/c$.

Conversely, if (3) holds, with $F(x) = x, G(x) = x^{\alpha}$, then set $c = (1+\alpha)^{-1}$. It follows that $(1-c)G(x) = cxg(x)$ for all $x \in (0,1)$. Thus (6) holds for all $m \geq 1$, or (5) holds. ■

Corollary: If (5) holds, then (4) holds, and in turn the record indicators $\{I_m, m \geq 2\}$ are all independent. Moreover, the distribution of the number of observed records N must be independent of F.

3. Characterization results for a special case

We now turn to characterization results for F. We begin by deriving expressions for the relevant moments we use, and provide conditions for these moments to exist. We then consider the special case where $\Gamma(F) = G_0(F(x))$ and provide a characterization result.

Consider record moments that condition on the event $N \geq n$. Using the expression for the likelihood $h(r_1, ...r_n \cap N \geq n)$ from Bunge and Nagaraja (1991) and integrating out r_2 through r_{n-2} yields the following expression for the joint likelihood of r_1, r_{n-1}, and r_n when $N \geq n$ for $n \geq 3$:

$$h(r_1, r_{n-1}, r_n \cap N \geq n) = \frac{1}{(n-3)!}\left[-\ln\left(\frac{1-qF(r_{n-1})}{1-qF(r_1)}\right)\right]^{n-3}\frac{qg(r_1)}{1-qF(r_1)}\frac{qf(r_{n-1})}{1-qF(r_{n-1})}f(r_n).$$

Limiting attention to $n \geq 3$, we derive the following expression for $E(R_n \mid N \geq n)$:

$$\begin{aligned} E(R_n \mid N \geq n) &= \frac{1}{P(N \geq n)}\int_{-\infty}^{\infty}\int_{-\infty}^{r_{n-1}}\int_{r_{n-1}}^{\infty} r_n h(r_1, r_{n-1}, r_n \cap N \geq n)\, dr_n dr_1 dr_{n-1} \\ &= \int_0^1\int_0^{u_{n-1}} \frac{\left[-\ln\left(\frac{1-qu_{n-1}}{1-qu_1}\right)\right]^{n-3}}{(n-3)!P(N \geq n)}\frac{g\left(F^{-1}(u_1)\right)}{f\left(F^{-1}(u_1)\right)} \\ &\qquad \cdot\phi_F(u_{n-1})\frac{qdu_1}{1-qu_1}\frac{qdu_{n-1}}{1-qu_{n-1}} \end{aligned} \tag{7}$$

where

$$\phi_F(u_{n-1}) = \int_{u_{n-1}}^1 F^{-1}(u_n)\, du_n. \tag{8}$$

Similarly, the expected record spacing $E\left(R_n - R_{n-1} \mid N \geq n\right)$ when $n \geq 3$ can be expressed as

$$E\left(R_n - R_{n-1} \mid N \geq n\right) = \int_0^1 \int_0^{u_{n-1}} \frac{\left[-\ln\left(\frac{1-qu_{n-1}}{1-qu_1}\right)\right]^{n-3}}{(n-3)!P\left(N \geq n\right)} \frac{g\left(F^{-1}\left(u_1\right)\right)}{f\left(F^{-1}\left(u_1\right)\right)} \cdot \phi_F\left(u_{n-1}\right) \frac{qdu_1}{1-qu_1} \frac{qdu_{n-1}}{1-qu_{n-1}} \tag{9}$$

where

$$\phi_F\left(u_{n-1}\right) = \int_{u_{n-1}}^1 \left[F^{-1}\left(u_n\right) - F^{-1}\left(u_{n-1}\right)\right] du_n. \tag{10}$$

In the same fashion, we can use the expression for $h\left(r_1, ..., r_n \cap N = n\right)$ derived above and integrate out r_2 through r_{n-2} to obtain the joint likelihood of r_1, r_{n-1}, and r_n when $N = n$, where $n \geq 3$:

$$h\left(r_1, r_{n-1}, r_n \cap N \geq n\right) = \frac{1}{(n-3)!}\left[-\ln\left(\frac{1-qF\left(r_{n-1}\right)}{1-qF\left(r_1\right)}\right)\right]^{n-3} \frac{qg\left(r_1\right)}{1-qF\left(r_1\right)} \frac{qf\left(r_{n-1}\right)}{1-qF\left(r_{n-1}\right)} f\left(r_n\right).$$

Using this, one can deduce that

$$E\left(R_n \mid N = n\right) = \int_0^1 \int_0^{u_{n-1}} \frac{\left[-\ln\left(\frac{1-qu_{n-1}}{1-qu_1}\right)\right]^{n-3}}{(n-3)!\Pr\left(N = n\right)} \frac{g\left(F^{-1}\left(u_1\right)\right)}{f\left(F^{-1}\left(u_1\right)\right)} \cdot \phi_F\left(u_{n-1}\right) \frac{qdu_1}{1-qu_1} \frac{qdu_{n-1}}{1-qu_{n-1}} \tag{11}$$

where

$$\phi_F\left(u_{n-1}\right) = \int_{u_{n-1}}^1 F^{-1}\left(u_n\right) \frac{(1-q)\, du_n}{1-qu_n}, \tag{12}$$

and likewise

$$E\left(R_n - R_{n-1} \mid N = n\right) = \int_0^1 \int_0^{u_{n-1}} \frac{\left[-\ln\left(\frac{1-qu_{n-1}}{1-qu_1}\right)\right]^{n-3}}{(n-3)!P\left(N = n\right)} \frac{g\left(F^{-1}\left(u_1\right)\right)}{f\left(F^{-1}\left(u_1\right)\right)} \cdot \phi_F\left(u_{n-1}\right) \frac{qdu_1}{1-qu_1} \frac{qdu_{n-1}}{1-qu_{n-1}} \tag{13}$$

where

$$\phi_F\left(u_{n-1}\right) = \int_{u_{n-1}}^1 \frac{1-q}{1-qu_n} \left[F^{-1}\left(u_n\right) - F^{-1}\left(u_{n-1}\right)\right] du_n \tag{14}$$

All four moment sequences above can thus be expressed as an integral of a common term multiplying a function $\phi_F\left(u_{n-1}\right)$ that varies with the particular moment at hand.

We now provide a sufficient condition for the above moments to exist. Here we use the fact that for any random variable Y, $E\left(Y\right)$ exists if and only if $E\left(|Y|\right) < \infty$.

Proposition 3: If $E(|X_2|) < \infty$, then

a. $E(|R_n| \mid N \geq n)$
b. $E(|R_n| \mid N = n)$

exist for all $n \geq 2$.

Proof: Suppose first that $n \geq 3$. From (7),

$$\begin{aligned} E(|R_n| \mid N \geq n) &= \int_0^1 \int_0^{u_{n-1}} \int_{u_{n-1}}^1 \frac{\left[-\ln\left(\frac{1-qu_{n-1}}{1-qu_1}\right)\right]^{n-3}}{(n-3)!P(N \geq n)} \\ &\quad \cdot \frac{g\left(F^{-1}(u_1)\right)}{f\left(F^{-1}(u_1)\right)} \left|F^{-1}(u_n)\right| du_n \frac{q du_1}{1-qu_1} \frac{q du_{n-1}}{1-qu_{n-1}}. \end{aligned}$$

Since u_1 and u_{n-1} both lie in $[0,1]$,the above expression

$$\begin{aligned} &\leq \frac{[-\ln(1-q)]^{n-3}}{(n-3)!P(N \geq n)} \left(\frac{q}{1-q}\right)^2 \int_0^1 \int_{u_{n-1}}^1 \left\{\int_0^{u_{n-1}} \frac{g\left(F^{-1}(u_1)\right)}{f\left(F^{-1}(u_1)\right)} du_1\right\} \left|F^{-1}(u_n)\right| du_n du_{n-1} \\ &= \frac{[-\ln(1-q)]^{n-3}}{(n-3)!P(N \geq n)} \left(\frac{q}{1-q}\right)^2 \int_0^1 \int_{u_{n-1}}^1 G\left(F^{-1}(u_{n-1})\right) \left|F^{-1}(u_n)\right| du_n du_{n-1} \\ &\leq \frac{[-\ln(1-q)]^{n-3}}{(n-3)!P(N \geq n)} \left(\frac{q}{1-q}\right)^2 \int_0^1 \int_0^1 G\left(F^{-1}(u_{n-1})\right) \left|F^{-1}(u_n)\right| du_n du_{n-1} \\ &\leq \frac{[-\ln(1-q)]^{n-3}}{(n-3)!P(N \geq n)} \left(\frac{q}{1-q}\right)^2 E(|X_2|). \end{aligned}$$

As long as $E(|X_2|) < \infty$, then $E(|R_n| \mid N \geq n)$ exists. A similar argument applies for the case where $n = 2$. Extending the argument for $E(|R_n| \mid N = n)$ is straightforward. ∎

Remark 4: For $n \geq 2$, the existence of the n-th record moment does not depend on whether $E(X_1)$ exists. This is because for $n \geq 2$, the relevant moment is always conditioned on an event in which $M \geq n$ and $\max\{X_2, ..., X_M\} > X_1$. As demonstrated in Nagaraja and Barlevy (2003), if M is geometric with success probability p, then $E(E(|\max\{X_2, ..., X_M\}| \mid M)) < p^{-1}E(|X_2|)$. Thus, we are conditioning on the event that X_1 is exceeded by a random variable whose mean is finite. Even if X_1 does not have a well-defined unconditional mean, conditioning on the event that its value is exceeded by a random variable with a finite mean suffices to ensure that $E(X_1 \mid N \geq n)$ is finite.

Equipped with these preliminaries, we turn to characterizing F from moment sequences. However, we first need to impose an additional assumption on the range of Γ.

Assumption 3: For any cdf F, there exists an $\varepsilon > 0$ such that $g\left(F^{-1}(u); F(\cdot)\right) = \left.\frac{d}{dx}\Gamma(F)\right|_{x=F^{-1}(u)}$ is positive for almost all $u \in (0, \varepsilon)$.

This assumption implies that $G^{-1}(0) = F^{-1}(0)$. This assumption is clearly necessary: if two distributions $F_1 \neq F_2$ differ only below $G^{-1}(0)$, they would necessarily yield identical record moment sequences.

We first focus on a special case, namely where $G(x; F(\cdot)) = G_0(F(x))$ for some absolutely continuous cdf $G_0 :[0,1] \to [0,1]$. Recall from Proposition 1 that this is true if and only if $\Pr(N = n)$ does not depend on F. We return to the more general case in the next section.

Proposition 4: Suppose Assumptions 1-3 are satisfied, $E(|X_2|) < \infty$, and $G(x; F(\cdot)) = G_0(F(x))$ for some absolutely continuous cdf $G_0 :[0,1] \to [0,1]$.

a. If two distributions F_1 and F_2 give rise to either the same sequence $E(R_n \mid N \geq n_j)$ or the same sequence $E(R_n \mid N = n_j)$ where $\sum_j n_j^{-1} = \infty$, then $F_1(x) = F_2(x)$ for almost all x;

b. If two distributions F_1 and F_2 give rise to either the same sequence $E(R_n - R_{n-1}|N \geq n_j)$ or the same sequence $E(R_n - R_{n-1} \mid N = n_j)$ where $\sum_j n_j^{-1} = \infty$, then there exists a c such that $F_1^{-1}(x) = F_2^{-1}(x) + c$ for almost all x.

Proof: Since $\Gamma(F) = G_0(F(x))$, it follows that $g(x; F(\cdot)) = g_0(F(x)) f(x)$, which implies $\frac{g\left(F^{-1}(u_1)\right)}{f\left(F^{-1}(u_1)\right)} = g_0(u_1)$. Substituting in, any of the moments above can be expressed as

$$\frac{1}{(n-3)!P(N \geq n)} \int_0^1 \int_{u_1}^1 g_0(u_1) \left[-\ln\left(\frac{1-qu_{n-1}}{1-qu_1}\right)\right]^{n-3} \phi_F(u_{n-1}) \frac{qdu_{n-1}}{1-qu_{n-1}} \frac{qdu_1}{1-qu_1}$$

where $\phi_F(\cdot)$ depends on the particular moment in question, i.e. either (8) or (10) in case (a) and either (12) or (14) in case (b). Changing variables according to

$$\begin{aligned} t &= -\ln(1-qu_{n-1}) \Rightarrow dt = \frac{qdu_{n-1}}{1-qu_{n-1}}, u_{n-1} = \frac{1-e^{-t}}{q} \\ s &= -\ln(1-qu_1) \Rightarrow ds = \frac{qdu_1}{1-qu_1}, u_1 = \frac{1-e^{-s}}{q} \end{aligned}$$

and setting $c = -\ln(1-q)$ allows us to rewrite the above expression as

$$\frac{1}{(n-3)!P(N \geq n)} \int_0^c \int_{t=s}^c g_0\left(\frac{1-e^{-s}}{q}\right) \phi_F\left(\frac{1-e^{-t}}{q}\right) (t-s)^{n-3} \, dtds.$$

We change variables yet again by setting $\omega = t - s$ to rewrite the above as

$$\frac{1}{(n-3)!P\left(N \geq n\right)} \int_{\omega=0}^{c} \int_{t=\omega}^{c} g_0 \left(\frac{1 - e^{-(t-\omega)}}{q} \right) \phi_F \left(\frac{1 - e^{-t}}{q} \right) \omega^{n-3} dt d\omega.$$

Define

$$\eta_F\left(\omega\right) = \int_{t=\omega}^{c} g_0 \left(\frac{1 - e^{-(t-\omega)}}{q} \right) \phi_F \left(\frac{1 - e^{-t}}{q} \right) dt.$$

Let $F_1(\cdot)$ and $F_2(\cdot)$ denote two continuous cdf's that give rise to the same subsequence of moments. Since $G(x; F(\cdot)) = G_0(F(x))$ implies $\Pr(N \geq n)$ is the same for all $F(\cdot)$, it follows that for all $n \geq 3$,

$$\int_{\omega=0}^{c} \eta_{F_1}\left(\omega\right) \omega^{n-3} d\omega = \int_{\omega=0}^{c} \eta_{F_2}\left(\omega\right) \omega^{n-3} d\omega$$

By the Müntz-Szász theorem, it further follows that $\eta_{F_1}(\omega) = \eta_{F_2}(\omega)$ for almost all $\omega \in (0, c)$, i.e.

$$\int_{t=\omega}^{c} g_0 \left(\frac{1 - e^{-(t-\omega)}}{q} \right) \phi_{F_1} \left(\frac{1 - e^{-t}}{q} \right) dt = \int_{t=\omega}^{c} g_0 \left(\frac{1 - e^{-(t-\omega)}}{q} \right) \phi_{F_2} \left(\frac{1 - e^{-t}}{q} \right) dt \quad (15)$$

for almost all $\omega \in (0, c)$.

We next argue that (15) implies $\phi_{F_1}\left(\frac{1 - e^{-t}}{q}\right) = \phi_{F_2}\left(\frac{1 - e^{-t}}{q}\right)$ for almost all $t \in (0, c)$. It will suffice to prove that if

$$\int_{t=\omega}^{c} g_0 \left(\frac{1 - e^{-(t-\omega)}}{q} \right) \phi\left(t\right) dt = 0 \text{ for almost all } \omega \in (0, c) \quad (16)$$

then $\phi(t) = 0$ for almost all $t \in (0, c)$. Appealing to a change in variables $w = c - t$ and $z = c - \omega$, (16) can be transformed into the following integral equation:

$$\int_{0}^{z} a\left(z - w\right) b\left(w\right) dw = 0 \text{ for almost all } z \in (0, c) \quad (17)$$

where $a(x) = g_0\left(\frac{1 - e^{-x}}{q}\right)$ and $b(x) = \phi(c - x)$. Applying Theorem VII in Titchmarsh (1926) [or from Theorem 151 in the more accessible reference Titchmarsh (1948, p. 324-5)], there exists a c^* such that $a(x) = 0$ for all $x \in (0, c^*)$ and $b(x) = 0$ for all $x \in (0, c - c^*)$. But Assumption 3 implies that there exists an $\varepsilon > 0$ such that $g_0(z) > 0$ for almost all $z \in (0, \varepsilon)$, which in turn implies that $a(z) > 0$ for almost all $z \in (0, \varepsilon)$. Hence, c^* must equal 0, implying $b(z) = 0$ for almost all $z \in (0, c)$. But then $\phi(t) = b(c - t) = 0$ for almost all $t \in (0, c)$, as claimed.

Lastly, we need to show that the statement of the proposition follows from the fact $\phi_{F_1}((1-e^{-t})/q) = \phi_{F_2}((1-e^{-t})/q)$ for almost all $t \in (0,c)$. Consider case (a); then

$$\int_{\frac{1-e^{-t}}{q}}^{1} \left[F_1^{-1}(u) - F_2^{-1}(u)\right] du = 0$$

or

$$\int_{\frac{1-e^{-t}}{q}}^{1} \left[\frac{F_1^{-1}(u) - F_2^{-1}(u)}{1-qu}\right] du = 0$$

for almost all $t \in (0,c)$. But from Taylor (1965, p. 415), this implies that the function inside the integral is equal to 0 almost surely, which implies $F_1^{-1}(u) = F_2^{-1}(u)$ almost surely as claimed. In case (b), with ϕ_F given in (10), the fact that $\phi_{F_1}(t) = \phi_{F_2}(t)$ for almost all $t \in (0,c)$ implies that for almost all $u \in (0,1)$,

$$\int_u^1 \left[F_1^{-1}(u_n) - F_1^{-1}(u)\right] du_n = \int_u^1 \left[F_2^{-1}(u_n) - F_2^{-1}(u)\right] du_n$$

or with ϕ_F in (14) we get

$$\int_u^1 \left[\frac{F_1^{-1}(u_n) - F_1^{-1}(u)}{1-qu_n}\right] du_n = \int_u^1 \left[\frac{F_2^{-1}(u_n) - F_2^{-1}(u)}{1-qu_n}\right] du_n$$

Nagaraja and Barlevy (2003) already showed that this implies there exists a constant c such that $F_1^{-1}(u) = F_2^{-1}(u) + c$. ■

Remark 5: From the proof above, we can further deduce what happens when we relax Assumption 3, i.e. when we assume that $G_0^{-1}(0) > 0$. By the Titchmarsh convolution theorem, for any solution $b(w)$ to (17), there exists a value c^* such that $a(x) = 0$ for all $x \in (0, c^*)$ and that $b(x) = 0$ for all $x \in (0, c - c^*)$. However, without Assumption 3, we can only conclude that $c^* \leq -\ln\left(1 - qG_0^{-1}(0)\right)$. Consequently, we can deduce that $\phi(t) = b(c-t) = 0$ for almost all $t \in \left(-\ln\left(1 - qG_0^{-1}(0)\right), c\right)$, and hence that $\phi_{F_1}((1-e^{-t})/q) = \phi_{F_2}((1-e^{-t})/q)$ for almost all $t \in \left(-\ln\left(1 - qG_0^{-1}(0)\right), c\right)$. In case (a) it would therefore follow that $F_1(x) = F_2(x)$ for almost all $x > F^{-1}\left(G_0^{-1}(0)\right)$, and in case (b) it would follow that $F_1^{-1}(u) = F_2^{-1}(u) + c$ for almost all $u > G_0^{-1}(0)$. In other words, one can generalize Proposition 4 to imply that the moment sequences in the statement of the proposition uniquely characterize the distribution F over the range $\left(F^{-1}\left(G_0^{-1}(0)\right), \infty\right)$.

4. Characterization results for the general case

We now move to the general case of any arbitrary function $G(x; F(\cdot))$ which satisfies Assumptions 1-3. In this case, it may no longer be true that record moments alone characterize the distribution F. However, record moments and the distribution of the number of records together do characterize F.

Proposition 5: Suppose Assumptions 1-3 are satisfied, and $E(|X_2|) < \infty$.

a. If two distributions F_1 and F_2 give rise to the same sequences $E(R_n \mid N \geq n_j)$ and $\Pr(N = n_j)$ where $\sum_j n_j^{-1} = \infty$, then $F_1 = F_2$ and $G_1 = G_2$ almost surely

b. If two distributions F_1 and F_2 give rise to the same sequences $E(R_n - R_{n-1} \mid N \geq n_j)$ and $\Pr(N = n_j)$ where $\sum_j n_j^{-1} = \infty$, then $F_1^{-1}(x) = F_2^{-1}(x) + c$ and $G_1^{-1}(x) = G_2^{-1}(x) + c$ for some constant c.

Proof: Since both F_1 and F_2 give rise to the same sequence $\Pr(N = n_j)$ where $\sum_j n_j^{-1} = \infty$, from Lemma 1 and (2) we conclude that $\Pr(N \geq n)$ also match for $n \geq 1$, and for almost all $u \in (0,1)$,

$$\frac{g_1\left(F_1^{-1}(u)\right)}{f_1\left(F_1^{-1}(u)\right)} = \frac{g_2\left(F_2^{-1}(u)\right)}{f_2\left(F_2^{-1}(u)\right)}.$$

Let us write this common function as $g_0(u)$. In contrast to the previous section, $g_0(u)$ now depends on the sequence $\Pr(N = n_j)$ as opposed to a stand-alone function.

Recall that all four moment sequences above can be written as

$$\int_0^1 \int_0^{u_{n-1}} \frac{\left[-\ln\left(\frac{1-qu_{n-1}}{1-qu_1}\right)\right]^{n-3}}{(n-3)!P(N \geq n)} g_0(u_1)\,\phi_F(u_{n-1})\,\frac{qdu_1}{1-qu_1}\frac{qdu_{n-1}}{1-qu_{n-1}}$$

for appropriately defined $\phi_F(\cdot)$. Using the change of variables

$$\begin{aligned} t &= -\ln(1-qu_{n-1}) \Rightarrow ds = \frac{qdu_{n-1}}{1-qu_{n-1}}, u_{n-1} = \frac{1-e^{-t}}{q} \\ s &= -\ln(1-qu_1) \Rightarrow dt = \frac{qdu_1}{1-qu_1}, u_1 = \frac{1-e^{-s}}{q} \\ c &= -\ln(1-q) \end{aligned}$$

we can rewrite this expression as

$$\int_0^c \int_{t=s}^c g_0\left(\frac{1-e^{-s}}{q}\right)\phi_F(t)\,\frac{(t-s)^{n-3}}{(n-3)!P(N \geq n)}dtds$$

Setting $\omega = t - s$, we can further rewrite this expression as

$$\int_{\omega=0}^c \int_{t=\omega}^c g_0\left(\frac{1-e^{-(t-\omega)}}{q}\right)\phi_F(t)\,\frac{\omega^{n-3}}{(n-3)!P(N \geq n)}dtd\omega$$

Let us define

$$\eta_F(\omega) = \int_{t=\omega}^{c} g_0\left(\frac{1-e^{-(t-\omega)}}{q}\right)\phi_F(t)\,dt$$

Let F_1 and F_2 denote two continuous cdf's that give rise to the same sequences. Define N_1 as the number of records when $X_2 \sim F_1$ and N_2 as the number of records when $X_2 \sim F_2$. If F_1 and F_2 give rise to the same moment sequences, then

$$\frac{1}{(n-3)!P(N_1 \geq n)}\int_{\omega=0}^{c}\eta_{F_1}(\omega)\,\omega^{n-3}d\omega = \frac{1}{(n-3)!P(N_2 \geq n)}\int_{\omega=0}^{c}\eta_{F_2}(\omega)\,\omega^{n-3}d\omega$$

Since $\Pr(N_1 \geq n) = \Pr(N_2 \geq n)$, it follows that

$$\int_{\omega=0}^{c}\eta_{F_1}(\omega)\,\omega^{n-3}d\omega = \int_{\omega=0}^{c}\eta_{F_2}(\omega)\,\omega^{n-3}d\omega$$

and then by the Müntz-Szász theorem $\eta_{F_1}(\omega) = \eta_{F_2}(\omega)$ almost surely, i.e.

$$\int_{t=\omega}^{c} g_0\left(\frac{1-e^{-(t-\omega)}}{q}\right)\phi_{F_1}(t)\,dt = \int_{t=\omega}^{c} g_0\left(\frac{1-e^{-(t-\omega)}}{q}\right)\phi_{F_2}(t)\,dt$$

As in the proof of Proposition 4, we rely on the Titchmarsh convolution theorem to establish that $\phi_{F_1}(t) = \phi_{F_2}(t)$ almost surely, from which we conclude that $F_1 = F_2$ in case (a) and $F_1^{-1}(u) = F_2^{-1}(u) + c$ in case (b).

Next, in case (a), we use the fact that $F_1(x) = F_2(x)$ for almost all x and the fact (from Lemma 1) that

$$\frac{g_1\left(F_1^{-1}(u)\right)}{f_1\left(F_1^{-1}(u)\right)} = \frac{g_2\left(F_2^{-1}(u)\right)}{f_2\left(F_2^{-1}(u)\right)}$$

for almost all u to conclude that

$$g_1\left(F_1^{-1}(u)\right) = g_2\left(F_2^{-1}(u)\right)$$

for almost all $u \in (0,1)$. Hence,

$$G_1(x) = \int_{F_1^{-1}(0)}^{x} g_1(x)\,dx = \int_{F_2^{-1}(0)}^{x} g_2(x)\,dx = G_2(x)$$

as claimed.

In case (b), we use the fact that $F_1^{-1}(u) = F_2^{-1}(u) + c$ for almost all u, to conclude that

$$\frac{g_1\left(F_2^{-1}(u)+c\right)}{f_1\left(F_2^{-1}(u)+c\right)} = \frac{g_2\left(F_2^{-1}(u)\right)}{f_2\left(F_2^{-1}(u)\right)}$$

But for almost every $u \in (0,1)$, it must also follow that $f_1\left(F_2^{-1}(u)+c\right) = f_2\left(F_2^{-1}(u)\right)$. Hence, we have

$$\frac{g_1\left(F_2^{-1}(u)+c\right)}{f_2\left(F_2^{-1}(u)\right)} = \frac{g_2\left(F_2^{-1}(u)\right)}{f_2\left(F_2^{-1}(u)\right)}.$$

This implies that for almost every $u \in (0,1)$,

$$g_1\left(F_2^{-1}(u)+c\right) = g_2\left(F_2^{-1}(u)\right)$$

and hence that

$$G_1(x+c) = \int_{F_2^{-1}(0)+c}^{x+c} g_1(x)\,dx = \int_{F_2^{-1}(0)}^{x} g_2(x)\,dx = G_2(x)$$

which implies $G_1^{-1}(u) = G_2^{-1}(u) + c$. This completes the proof. ■

By a similar argument, one can show that the proposition above remains true if we condition on the event that $N = n$ rather than on the event that $N \geq n$.

Remark 6: If Γ is known, then once we identify F, we can also recover $G = \Gamma(F)$. But Proposition 5 implies that G is itself characterized by the sequences $E(R_n \mid N \geq n)$ and $\Pr(N = n)$. Hence, we can test certain conjectures on Γ by checking whether the distribution $\Gamma(F)$ at the F we identify is the same as the G directly implied by the moment sequences and the distribution of N.

5. Characterization results across GRR models

The modified GRR model we study can be summarized by a triple $\{\Gamma, F, q\}$. So far, we have implicitly focused on results that characterize F within a given model. That is, for a given q and Γ, we showed that there is at most one F for which the model is consistent with a given sequence of record moments and a given distribution for N. In this section, we ask whether it is possible to characterize the model itself as opposed to the distribution F within a given model. We show that if two models $\{\Gamma_1, F_1, q_1\}$ and $\{\Gamma_2, F_2, q_2\}$ yield the same record moments and the same distribution of the number of records, then $q_1 = q_2$, $F_1 = F_2$ almost surely, and $\Gamma_1(F_1) = \Gamma_2(F_2)$ almost surely. In other words, the sequences considered in Proposition 5 characterize not only F and G but also the distribution of M.

Proposition 6: Suppose two models $\{\Gamma_1, F_1, q_1\}$ and $\{\Gamma_2, F_2, q_2\}$ both satisfy Assumptions 1-3, and $E(|X_2|) < \infty$ in both models. Let $G_1 = \Gamma_1(F_1)$ and $G_2 = \Gamma_2(F_2)$ and assume $\sum_j n_j^{-1} = \infty$.

a. If $\{\Gamma_1, F_1, q_1\}$ and $\{\Gamma_2, F_2, q_2\}$ give rise to the same sequence $\Pr(N = n_j)$, then $q_1 = q_2$

and (2) holds, i.e.,

$$\frac{g_1\left(F_1^{-1}(u)\right)}{f_1\left(F_1^{-1}(u)\right)}=\frac{g_2\left(F_2^{-1}(u)\right)}{f_2\left(F_2^{-1}(u)\right)}$$

for almost all $u \in (0,1)$.

b. If, in addition to the condition in (a), the two GRR models give rise to the same sequence $E\left(R_n \mid N \geq n_j\right)$, then $G_1 = G_2$ almost surely, and $F_1 = F_2$ almost surely.

c. If, in addition to the condition in (a), the two GRR models give rise to the same sequence
$E\left(R_n - R_{n-1} \mid N \geq n_j\right)$, then there exists a c such that $G_1^{-1}(x) = G_2^{-1}(x) + c$ almost surely, and $F_1^{-1}(x) = F_2^{-1}(x) + c$ almost surely.

Proof: We prove (a). The remaining two claims then follow from Proposition 5.

Since both $\{G_1, F_1, q_1\}$ and $\{G_2, F_2, q_2\}$ give rise to the same sequence $\Pr(N = n_j)$, then for $j = 1, 2, 3, \ldots$

$$\int_0^1 \frac{1-q_1}{1-q_1 u}\left[\ln\left(\frac{1-q_1 u}{1-q_1}\right)\right]^{n_j-1}\frac{g_1\left(F_1^{-1}(u)\right)}{f_1\left(F_1^{-1}(u)\right)}du$$
$$=\int_0^1 \frac{1-q_2}{1-q_2 u}\left[\ln\left(\frac{1-q_2 u}{1-q_2}\right)\right]^{n_j-1}\frac{g_2\left(F_2^{-1}(u)\right)}{f_2\left(F_2^{-1}(u)\right)}du.$$

Set $t = \ln\left((1-q_1 u)/(1-q_1)\right)$ and $\ln\left((1-q_2 u)(1-q_2)\right)$ respectively on the left hand side and right hand side above. We can then rewrite the equation above as

$$\int_0^{-\ln(1-q_1)} \frac{1-q_1}{q_1}\frac{g_1\left(F_1^{-1}\left(\frac{1-(1-q_1)e^t}{q_1}\right)\right)}{f_1\left(F_1^{-1}\left(\frac{1-(1-q_1)e^t}{q_1}\right)\right)}t^{n-1}dt$$
$$=\int_0^{-\ln(1-q_2)} \frac{1-q_2}{q_2}\frac{g_2\left(F_2^{-1}\left(\frac{1-(1-q_2)e^t}{q_2}\right)\right)}{f_2\left(F_2^{-1}\left(\frac{1-(1-q_2)e^t}{q_2}\right)\right)}t^{n-1}dt.$$

Let us rewrite this equation as

$$\int_0^{-\ln(1-q_1)} h_1(t)\, t^{n-1}dt = \int_0^{-\ln(1-q_2)} \frac{q_1}{q_2}\frac{1-q_2}{1-q_1}h_2(t)\, t^{n-1}dt.$$

We now proceed to prove the claim by contradiction. Suppose wlog that $q_2 > q_1$. Then

$$-\ln(1-q_2) > -\ln(1-q_1).$$

Define

$$\widehat{h}_1(t) = \begin{cases} h_1(t) & \text{if } t \leq -\ln(1-q_1) \\ 0 & \text{if } t \in (-\ln(1-q_1), -\ln(1-q_2)). \end{cases}$$

Then we have

$$\int_0^{-\ln(1-q_2)} \widehat{h}_1(t)\, t^{n-1} dt = \int_0^{-\ln(1-q_2)} \frac{q_1}{q_2}\frac{1-q_2}{1-q_1} h_2(t)\, t^{n-1} dt$$

for all $n = 1, 2, 3, ...$ By Müntz-Szász, it follows that

$$\widehat{h}_1(t) = \frac{q_1}{q_2}\frac{1-q_2}{1-q_1} h_2(t)$$

for almost all $t \in (0, -\ln(1-q_2))$, which implies $h_2(t) = 0$ for almost all t in $(-\ln(1-q_1), -\ln(1-q_2))$. But this implies $g_2(x) = 0$ for almost all x in $\left[F_2^{-1}(0),\ F_2^{-1}((q_2-q_1)/[q_2(1-q_1)])\right]$, which violates Assumption 3.

Given $q_1 = q_2 \equiv q$, the fact that both $\{G_1, F_1, q\}$ and $\{G_2, F_2, q\}$ give rise to the same sequence $\Pr(N = n_j)$ implies, in view of Lemma 1, that (2) holds. ■

6. Application

Finally, we discuss how the results of this paper can be used to identify the wage offer distribution in job search models. In particular, we show that the offer distribution is identified in a larger class of models than was previously demonstrated in work by Nagaraja and Barlevy (2003) and Barlevy (2005).

Consider the following model of job search, which is frequently used in labor economics. (The literature on job search is too vast; for a survey of previous work on the identification and estimation of these models, see Eckstein and Van den Berg (2005).) At any point in time, a worker can be either employed or unemployed. While unemployed, all workers receive a fixed dollar amount X^* per unit time (which may be zero). This amount can reflect unemployment benefits, as well as the monetary value of the leisure she enjoys while not working. Employed and unemployed workers encounter employers at a constant rate λ per unit time. Each time a worker encounters an employer, the latter offers her a wage of X that is drawn independently from a continuous offer distribution F. The worker must then choose whether to stay on her current job (alternatively, remain unemployed) or accept the new offer and change employers (alternatively, exit unemployment). In addition, a worker can lose her job, an event that occurs at constant rate δ whenever she is employed. When a worker loses her job, she cannot recall any of her past offers, and instead becomes unemployed. Workers are assumed to maximize their earnings. Hence, the optimal strategy for an employed worker is to only accept offers that surpass her current wage. Similarly, while unemployed, the worker should only accept offers that exceed X^*. We assume $F^{-1}(0) \geq X^*$, i.e. all employers offer at least X^* (otherwise their offers would never be accepted).

Let M denote the number of job offers a worker receives between intervening spells of unemployment, and index the offers according to the order in which they arrive so that X_i denotes the i-th offer since the worker was last unemployed. Barlevy (2005) shows that M will have a geometric distribution, i.e. $\Pr(M = m) = q^{m-1}p$ where $p = \delta(\delta + \lambda)^{-1}$ and $q = 1-p$. Given the worker's strategy, the wages on the jobs the worker accepts corresponds to records from the sequence $\{X_i, 1 \leq i \leq M\}$. In the typical datasets economists use, workers are only queried on the jobs they work on, not on job offers they received but turned down. Thus, we assume that the only available data consists of $\{R_n, 1 \leq n \leq N\}$, not the original observations $\{X_i, 1 \leq i \leq M\}$ or even the number of observations M. A question of interest for economists is whether this data can identify the offer distribution F.

Since $R_1 = X_1$, the distribution F is obviously identified from the empirical distribution of wages of workers on the first job. However, a key obstacle in taking the above model to data is that empirically a considerable number of workers voluntarily move into lower wage jobs, in direct violation of the model. To resolve this discrepancy, economists have argued that wages in the data are a noisy version of wages in the model, i.e. we observe not R_n but $R_n + \varepsilon_n$ for some random variable ε_n where $E(\varepsilon_n) = 0$. The ε_n can be viewed as measurement error, but alternative interpretations for this term have been offered (see Barlevy (2005) for a discussion). Once we assume that we only observe $R_n + \varepsilon_n$, we can no longer identify F from the distribution of X_1. Previous work, as summarized in Eckstein and Van den Berg (2005), resorted to parametric assumptions on F and the distribution of ε to proceed with estimation. By contrast, Nagaraja and Barlevy (2003) and Barlevy (2005) argued that characterization results for the GRR model imply that F is identified non-parametrically, since one can still recover $E(R_n)$ from noise-ridden data.

However, in order to apply this identification result, we need to keep track of all jobs between spells of unemployment so that we can determine which record number n each job represents. Unfortunately, this is not possible in many datasets. In particular, many surveys collect data on workers that are already employed. For those workers, we have no way of classifying which record number to assign to the jobs we observe for them. Although we could wait until the worker is next unemployed, unemployment is often a sufficiently low probability event that a large part of the data would have to be thrown out. The results of this paper suggest a way to incorporate data for workers who are already employed. In particular, we know from previous work on search models, e.g. Burdett and Mortensen (1998), that the economy described by this model converges in the limit to a steady state in which the fraction of all employed workers who earn a wage of x or less is equal to

$$G(x) = \frac{F(x)}{1 + \lambda/\delta\,(1 - F(x))} \tag{18}$$

Moreover, the number of offers the worker receives starting from any job continues to have a geometric distribution. Thus, as long as the economy we consider is at its steady state, the wages on the jobs we observe for a randomly chosen employed worker will correspond to records from a sequence $\{X_i, 1 \leq i \leq M\}$ where M has a geometric distribution, $X_1 \sim G$ as defined in (18), and $X_2, ..., X_M \sim F$. Since $G = G_0(F(x))$, we can appeal to Proposition 4 to argue that average wages or average wage changes identify the wage offer distribution F. For example, if the average wage gains of workers is constant regardless of how many jobs they have changed since the first job we observe them on, the wage offer distribution must be exponential.

More generally, our results can be applied whenever the distribution of the wage on the first job we observe for a worker differs from the offer distribution F. For example, some surveys focus on the poor, and use the initial earnings of a worker as a criterion for selection into the survey. In this case, even if we could track workers from their very first job out of unemployment, the distribution of wages on the worker's first job would correspond to

$$G(x) = \begin{cases} F(x)/z & \text{if } F(x) < z \\ 1 & \text{else} \end{cases} \tag{19}$$

where z reflects the percentile of the threshold wage workers must earn within the wage offer distribution to qualify for the survey. Once again, we can appeal to Proposition 4 to argue that average wages or average wage changes identify the offer distribution F.

Discrepancies between the wage on the first job we observe for a worker and the offer distribution $F(\cdot)$ are not confined to sampling issues. Suppose we could track workers from the first job out of unemployment and that no wages were censored. However, suppose the amount workers earn while unemployed varies across workers. For example, they might enjoy leisure differently, or they might earn different unemployment benefits (which is not unreasonable given these often depend on what the worker earned on his last job before becoming unemployed). Let $H(x)$ denote the fraction of workers whose X^* is x or less, and suppose $H^{-1}(0) \leq F^{-1}(0)$. Workers whose X^* will hold out for a higher wage before they accept a job offer. The wage on the first job out of unemployment for a worker chosen at random from H is now given by

$$G(x; F(\cdot)) = \frac{\int_{-\infty}^{x} H(w)\, dF(w)}{\int_{-\infty}^{\infty} H(w)\, dF(w)} = \frac{\int_{0}^{F(x)} H\left(F^{-1}(u)\right) du}{\int_{0}^{1} H\left(F^{-1}(u)\right) du}. \tag{20}$$

Workers continue to draw offers from F at rate λ, so the wages of a worker chosen at random between two consecutive unemployment spells will correspond to records from a sequence $\{X_i, 1 \leq i \leq M\}$ where M has a geometric distribution, $X_1 \sim G$ as defined in (20), and $X_2, ..., X_M \sim F$. Since $G(x; F(\cdot))$ cannot be represented as $G_0(F(x))$ for some function $G_0(\cdot)$, we must appeal to Proposition 5 to argue that average wages or average wage changes, together with the distribution of the number of jobs workers hold between unemployment spells, identify F. If the distribution of reservation wages H is itself unknown, Proposition 5 implies we can also identify the distribution of wages of workers on their first job G. It is easily demonstrated that given F and G, one can recover H. Thus, when workers have different reservation wages, not only is the common offer distribution F they face still identified, but so is the distribution of X^* across workers. Thus, we could infer the distribution of how much workers value leisure from the extent of job mobility we observe for them once they become employed. Lastly, Proposition 6 tells us that we do not need to know the ratio λ/δ in advance to identify F, since we can recover it from data on N, i.e. the distribution of how many jobs workers hold between consecutive unemployment spells. For an empirical implementation of these ideas using panel data on young workers, see Barlevy and Nagaraja (2005).

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