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DURATION MODELS: A TAXONOMY

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DURATIONS OF NONDEFAULT-FREE SECURITIES

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FEDERAL RESERVE BANK OF CHICAGO

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I

Duration analysis is now an accepted, if not a necessary, part of every bond portfolio manager's toolkit for analyzing interest rate risk. Most of the duration models currently in use are single-term models. More recently, increasing interest has been shown in two- or more-term duration models. Unfortunately, there is considerable confusion about the meaning of these models and their proper uses. In particular, academics and practitioners alike often confuse the second and higher order terms in models that explain the instantaneous price elasticity of a security with respect to interest rates with the terms in multi-factor models that explain noninstantaneous bond returns. This paper demonstrates that, although they may at times look alike, the higher order terms in price elasticity models are not necessarily the same as the additional factors in bond return models. The failure to recognize this difference may lead to poor predictions and possible rejection of the usefulness of duration-based models [7, 15].

II

Duration is related to the derivative of the price of a stream of future payments $(F_1, F_2, \ldots, F_n)$ with respect to the rates at which these flows are discounted. If the yield to maturity (i) is specified as the discount rate, so that the term structure is flat and any changes in interest rates are the same for all maturities, the first derivative may be transformed into the common Macaulay duration [16]:

$$D = \frac{\sum \frac{tF_t}{(1+i)^t}}{\sum \frac{F_t}{(1+i)^t}} \tag{1}$$

If the term structure is assumed to be generated by other stochastic processes, then this procedure derives other measures of duration [3, 13, 17].

By substituting duration as defined in Equation 1 in the first derivative of the price of a bond with respect to the yield to maturity and rearranging terms, the instantaneous percent price change due to a change in interest rates may be approximated by the following relationship:

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$$\Delta P/P = -D\Delta i/(1 + i) \quad (2)$$

where P = the bond price.

This price elasticity equation maps a straight line with duration as its slope. Because the actual bond price-interest rate relationship for a bond of given coupon rate and term to maturity is curvilinear, Equation 2 provides a good approximation of the percent change in price for a given change in interest rates only for small changes in rates around the interest rate at which the straight line is tangent to the curve. Both the Equation 2 approximation and the actual price-interest rate relationships are plotted in Figure 1. Because percent changes are equivalent to changes in logarithms, Equation 2 may be charted most easily by scaling the axes in logarithms.

As can be seen from Figure 1, the larger is the change in interest rates, the greater is the distance between the actual curvilinear relationship and the straight line. The vertical distance between the two lines is referred to as the "convexity" of the bond and represents the price prediction error for single-term duration models, such as given by Equation 2 [6, 12, 18].[1] The error can be reduced by expanding the Taylor series for the percent change in bond price around the initial interest rate and adding second- and higher-order terms as follows:

$$\frac{\Delta P}{P} = \frac{1}{P}(\frac{dP}{dh})\Delta i + \frac{1}{2!}(\frac{1}{P})\frac{d^2P}{dh^2}(\Delta i)^2 + \cdots \frac{1}{t!}(\frac{1}{P})\frac{d^tP}{dh^t}(\Delta i)^t + \cdots \quad (3A)$$

where $h = (1 + i)$ and the derivatives are evaluated at the initial interest rate. Taking the derivative yields:

$$\frac{\Delta P}{P} = \frac{-\Sigma tF_t(1+i)^{-t}}{P}[\frac{\Delta i}{(1+i)}] + \frac{1/2\Sigma t(t+1)F_t(1+i)^{-t}}{P} \quad (3B)$$

$$[\frac{(\Delta i)^2}{(1+i)^2}] + \cdots$$

Substituting the definition of duration from Equation 1 yields:

$$\frac{\Delta P}{P} = -D\frac{\Delta i}{(1+i)} + 1/2[D + \frac{\Sigma t^2(1+i)^{-t}F_t}{P}]\frac{\Delta i^2}{(1+i)^2} + \cdots \quad (3C)$$

Duration now appears in every term in the equation. It is the only coefficient in the first term. In all other terms, duration comprises only part of the coefficient and is combined with progressively more information in progressively higher order terms. As a result, the greater the number of terms, the closer will be the approximation provided by the duration model to the actual relationship. This is shown in Figure 2 for a two-term duration expansion model. Thus, single-factor duration may be viewed as a one-term or one-moment statistical summary of a series, such as a mean or median. Although the first-term summary measures provides important information about the series, it does not provide complete information. This requires the inclusion of second, third, and higher moment measures, such as variance, skewness, and so on.

Knowledge of convexity and the additional information provided in higher order terms is particularly useful when analyzing the price elasticity characteristics of different bonds having the same duration. Two bonds having the same duration but otherwise different characteristics are unlikely to experience precisely the same percent change in price for the same change in interest rates and the difference will be greater the larger the interest rate change. Differences in price responses may be progressively reduced by computing and sequentially matching the second- and higher-order terms in Equation 3C for the two bonds.

The number of terms to be included in a duration price elasticity equation is determined by the benefit-cost tradeoff for each user. The percent of the actual price change predicted by one- and two-term Taylor series duration models for a series of bonds differing in term to maturity and coupon in different interest rate environments for selected changes in yields to maturity is shown in Table 1. To focus on the basis point change in rates, the Macaulay durations frequently are divided by $(1 + i)$. This measure is referred to as "modified duration". As can be seen from the table, one-term modified duration does not predict perfectly. The error varies in the different scenarios, depending on the characteristics of the bond and the market. One-term duration predicts better: the larger the coupon, the shorter the term to maturity, and the higher the market rate of interest, that is, the shorter is duration [20].[2] Addition of the second term of the Taylor expansion explains almost all of the price change not explained by the first term alone. Thus, even if second-order terms are cost effective relative to the first-order, higher-order terms are unlikely to be cost effective.

III

Duration has also been found useful in explaining bond returns over finite periods of time [9]. Duration return generating models are constructed by differentiating the value of an investment fund accumulated over a particular planning period or time interval with respect to a change in the term structure during the period. Thus, the derived durations reflect the underlying stochastic process driving interest rate movements. The number of factors necessary to explain the returns from any stochastic process are econometrically related to the number of interest rates or points on the term structure that can change, at least partially, independently. One-factor models assume that there is only one exogenous force driving interest rates. Thus, in effect, only one rate moves independently. Changes in all other rates along the term structure are assumed to be perfectly correlated with changes in this rate and move in lock-step fashion with it. Knowledge of the stochastic process that describes changes in any one interest rate on the term structure can then describe changes in the entire term structure. Two-factor models permit two exogenous forces to drive two totally or partially independent rates. Changes in all other rates along the term structure will be combinations of changes in these two rates or in functions of these two rates and are perfectly correlated with changes in a weighted combination of these two rates or their functions. And so on for additional factors.

If one assumes a flat term structure and a stochastic process that changes all interest rates by equal amounts so that the term structure remains flat, i.e., additive shocks, the stochastic process can be explained by a single factor whose coefficient is the Macaulay duration shown in Equation 1. For example, the following return equation using Macaulay duration was derived by Babcock [1, 2]:

$$E(i_j) = i_o + \left(\frac{PL - D_j}{PL}\right) (\hat{i}_r - i_o) \tag{4}$$

where

$E(i_j)$ = expected annual interest return over investor's planning period for a default-free bond with duration Dj

PL = investor's planning period

i_o = current market yield to maturity

$\hat{i}_r$ = predicted reinvestment yield to maturity immediately after purchase

D = Macaulay duration.

As with price elasticity models, different assumptions about the stochastic process lead to different measures of duration.

If one assumes a nonflat term structure and a stochastic process that changes all zero-coupon equivalent interest rates (r_t) by amounts that are proportional to $(1 + r_t)$, the stochastic process is also a single factor process but is associated with a different duration coefficient:

$$D = \frac{\sum \frac{tF_t}{(1 + r_t)^t}}{\sum \frac{F_t}{(1 + r_t)^t}} \tag{5}$$

where t = points on term structure so that $(r_1, r_2, ...r_n)$is the initial term structure. This duration is referred to as the Fisher-Weil duration [13].

However, single factor models can describe the consequences of only a restricted set of term structure processes. Evidence from empirical term structure studies suggests that changes in actual term structures are more complex and require a number of factors to describe accurately. For example, the term structure of U.S. Treasury securities on August 27, 1987 is fitted well by a polynomial of degree 3:

$$r_t = 7.453 + 0.852\ 1nt + 0.016(1nt)^2 - 0.023(1nt)^3 \qquad (6A)$$

If on other dates the estimated coefficients, including the constant term, differ, the term structure may be driven by as many as four independent sources of interest rates movements. The evidence suggests that this may be the case. For example, term structure on November 18, 1987 is fitted well by the following third degree polynomial equation:

$$r_t = 7.342 + 0.984\ 1nt + 0.128(1nt)^2 + 0.009(1nt)^3 \qquad (6B)$$

Multiple factor duration return generating models may be developed in a number of ways. One method makes assumptions about both the number of factors and the nature of the stochastic process relating them. If, for example, the four coefficients in Equation 6A represent four independent factors, the changes in the coefficients through time are predetermined by the nature of the stochastic process assumed. Another method makes assumptions about the functional form of the term structures, assumes a number of factors equal to or less than the number of terms in the estimated functional form (e.g., in Equations 6A and 6B) and derives durations from changes in the estimated coefficients for each factor in the different periods. Whether the number of factors assumed in either approach is sufficiently accurate is an empirical question to be determined by cost-benefit analysis similar to that for the appropriate number of terms to use in the Taylor expansion. An insufficient number of factors increases stochastic process risk.

Using the first approach, Bierwag [5, 6], for example, has developed a two-factor model based on a stochastic process that describes linear combinations of two independent interest rates (one a short-term rate and the other a long-term rate) and derives two durations:

$$D1 = \sum_{t=1}^{n} tF_t P(O,t)/A \qquad (7)$$

$$D2 = \sum_{t=1}^{n} t^2 F_t P(O,t)/A$$

where A = value of the portfolio.

Note that the second duration is denominated in units of time squared. Because this term resembles the second term in the Taylor series expansion (Equation 3C), it and other similar duration measures are often confused for the second term in the Taylor expansion and given the same interpretation. But these second factors are not measures of convexity and the second-order terms derived from Taylor expansions are not useful descriptions of the stochastic processes driving term structures.

This can be easily demonstrated. For example, if the relationship between the two independent interest rates in the Bierwag model is specified in logarithmic rather than in linear form, the second duration measure will look as follows:

$$D2 = \sum_{t=1}^{n} t \ln t F_t P(O,t)/A \qquad (8)$$

The coefficient of the second factor is no longer a squared term and cannot be confused with the second term in a Taylor expansion.

Another approach to deriving multi-factor duration models was used by Carleton, Chambers, and McEnally [11]. They fitted term structures for Treasury securities to polynomials of various higher-order degrees. The equations resemble Equation 5, but are not in log form. They assumed that the motion in the term structures through time could be captured adequately from changes in the coefficients of only the first two terms of these equations (the constant and slope terms) and derived two duration factors by differentiating the final accumulation value with respect to the assumed changes in interest rates. Because they assumed the term structure to be described by a polynomial in terms to maturity, their higher-order terms resemble the higher-order terms from the Taylor expansion. But again, the resemblance is mostly coincidental. A different functional specification of the term structure could yield quite different duration measures and eliminate the similarity.

Multiple-factor duration models have also been developed by Brennan-Schwartz [10] and Nelson-Schaefer [19], among others. As in the two models described above, the second factor terms in these models need not be higher-order terms resembling those in the Taylor expansion.

IV

Duration models are useful tools for security analysis only if they are fully understood and correctly specified. Multi-term models for approximating instantaneous price elasticities must not be confused with multi-factor models for estimating bond returns, even if they may look alike. Thus, the existence of squared terms in multi-term duration equations need not indicate the presence of multiple factors and multiple factor models need not contain squared terms.

TABLE 1

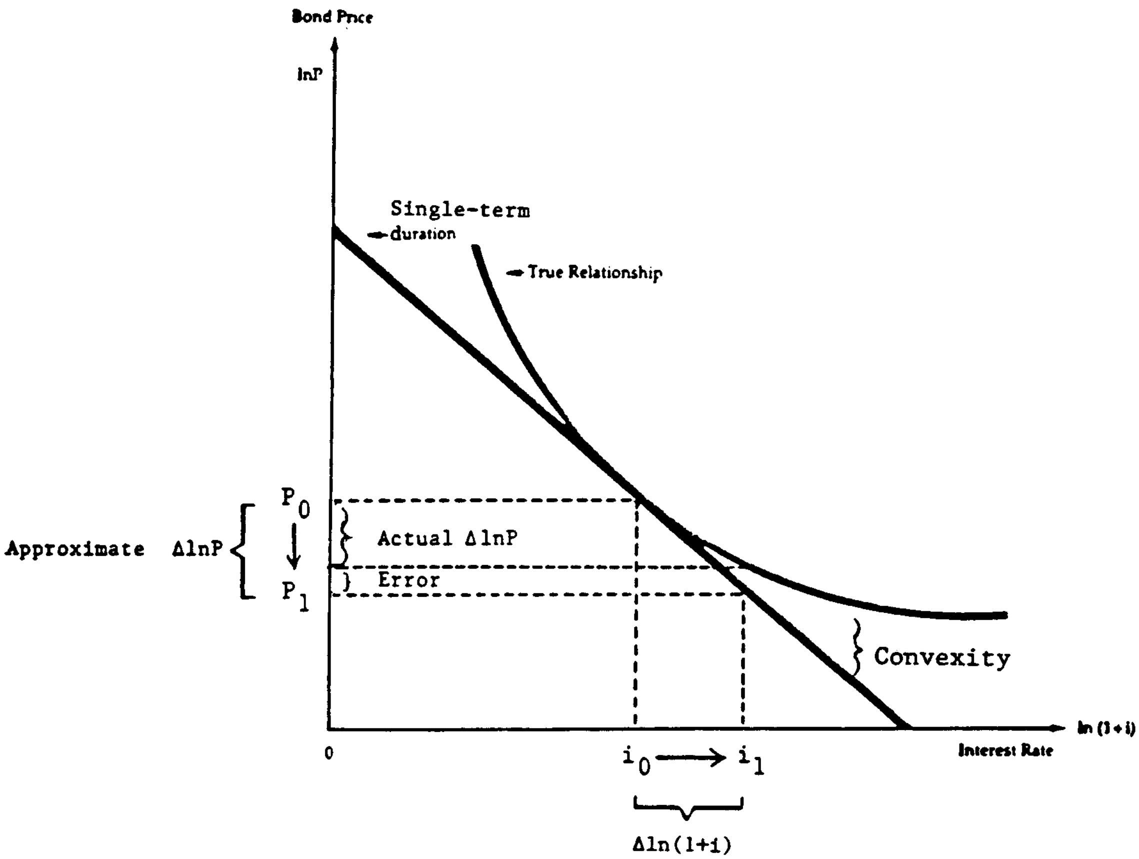
Estimated and Actual Decreases in Instantaneous Changes in Bond Prices For Different Bonds and Yields to Maturities When Interest Rates Increase 50 Basis Points

			Price Change		
			Estimated[a]		
Coupon Rate	Term to Maturity	Modified Duration	1st Term	1st/2nd Terms	Actual
(Percent)	(Years)		(Percent)		
YTM = 4%					
4	3	2.776	-1.388	-1.374	-1.374
	10	8.110	-4.055	-3.955	-3.956
	30	17.292	-8.646	-8.121	-8.144
16	3	2.556	-1.278	-1.266	-1.266
	10	6.566	-3.283	-3.209	-3.210
	30	13.816	-6.908	-6.549	-6.563
YTM = 12%					
4	3	2.564	-1.282	-1.271	-1.271
	10	6.934	-3.467	-3.389	-3.390
	30	9.228	-4.614	-4.416	-4.423
16	3	2.342	-1.171	-1.161	-1.161
	10	5.364	-2.682	-2.628	-2.629
	30	7.898	-3.949	-3.806	-3.811

[a]From Taylor series expansion using modified duration, annual coupon payments, and annual compounding.

FIGURE 1

Single-Term Duration and Convexity



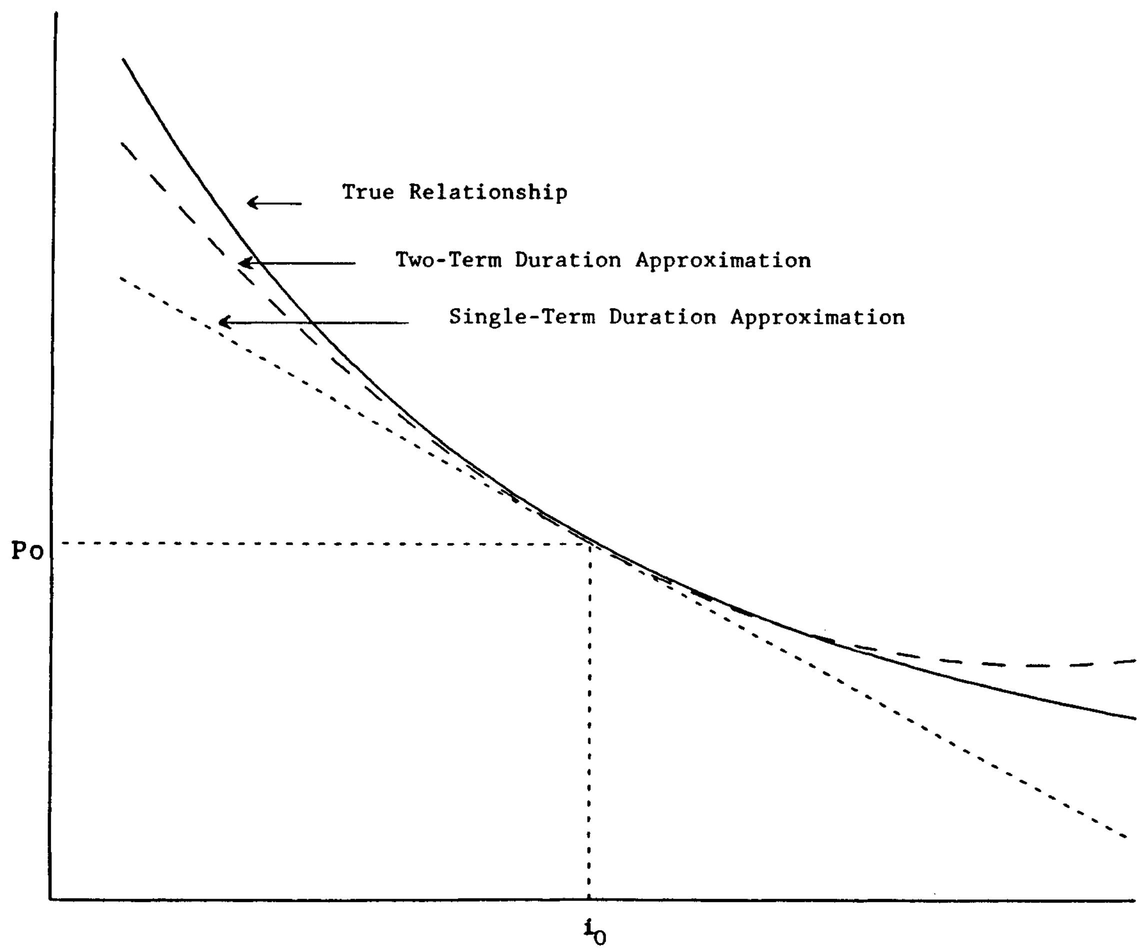
where:

$$\Delta \ln P = \frac{\Delta P}{P}$$

$$\Delta \ln(1+i) = \frac{\Delta i}{(1+i)}$$

FIGURE 2

Single- and Two-Term Duration Approximations
for Instantaneous Price Changes



REFERENCES

Babcock, Guilford C., "Duration As a Link Between Yield and Value", *Journal of Portfolio Management* (Summer 1984), pp. 58-65.

Babcock, Guilford C., "Erratum: Duration As a Link Between Yield and Value", *Journal of Portfolio Management* (Fall 1984), pp. 97-98.

Bierwag, G. O., "Measures of Duration", *Economic Inquiry* (October 1978), pp. 497-507.

Bierwag, G. O., *Duration Analysis: Managing Interest Rate Risk* (Cambridge, MA.: Ballinger, 1987).

Bierwag G. O., "Bond Returns, Discrete Stochastic Processes, and Duration", *Journal of Financial Research* (Fall 1987), pp. 191-209.

Bierwag, G. O., George G. Kaufman, and Cynthia M. Latta, "Bond Portfolio Immunization: Tests of Maturity, One- and Two-Factor Duration Matching Strategies", *Financial Review* (May 1987), pp. 203-219.

Bierwag, G. O., George G. Kaufman, Cynthia M. Latta, and Gordon S. Roberts, "Usefulness of Duration in Bond Portfolio Management: Response to Critics", *Journal of Portfolio Management* (Winter 1987), pp. 48-52.

Bierwag, G. O., George G. Kaufman, and Alden Toevs, "Recent Developments in Bond Portfolio Immunization Strategies", in George G. Kaufman, G. O. Bierwag, and Alden Toevs, eds., *Innovations in Bond Portfolio Management: Duration Analysis and Immunization* (Greenwich, CT.: JAI Press, 1983), pp. 105-157.

Bierwag, G. O., George G. Kaufman, and Alden Toevs, "Duration: Its Development and Use in Bond Portfolio Management", *Financial Analysts Journal* (July/August 1983), pp. 113-123.

Brennan, Michael J. and Edwardo S. Schwartz, "Duration, Bond Pricing, and Portfolio Management", in George G. Kaufman, G. O. Bierwag and Alden Toevs, eds., *Innovations in Bond Portfolio Management: Duration Analysis and Immunization* (Greenwich, CT.: JAI Press, 1983), pp. 3-36.

Chambers, Donald, Willard Carleton, and Richard McEnally, "Immunizing Default-Free Bond Portfolios With a Duration Vector", *Journal of Financial and Quantitative Analysis* (forthcoming).

Diller, Stanley, *Parametric Analysis of Fixed Income Securities* (New York: Goldman Sachs, June 1984).

Fisher, Lawrence and Roman L. Weil, "Coping With the Risk of Market Rate Fluctuations: Returns to Bondholders from Naive and Optimal Strategies", *Journal of Business* (October 1971), pp. 408-431.

Goodman, Laurie S. and N. R. Vijayaraghavan, "Generalized Duration Hedging With Futures Contracts", *Review of Futures Markets* (vol. 6, no. 1, 1986), pp. 96-108.

Gultekin, N. Bulent and Richard J. Rogalski, "Alternative Duration Specifications and the Measurement of Basis Risk: Empirical Tests", *Journal of Business* (April 1984), pp. 243-262.

Hopewell, Michael H. and George G. Kaufman, "Bond Price Volatility and Term to Maturity: A Generalized Respecification", *American Economic Review* (September 1973), pp. 749-753.

Khang, Chulsoon, "Bond Immunization When Short-Term Rates Fluctuate More Than Long-Term Rates", *Journal of Financial and Quantitative Analysis* (December 1979), pp. 1085-1095.

Klotz, Richard G., *Convexity of Fixed-Income Securities* (New York: Salomon Brothers, October 1985).

Nelson, Jeffrey and Stephen Schaefer, "The Dynamics of the Term Structure and Alternative Portfolio Immunization Strategies", in George G. Kaufman, G. O. Bierwag and Alden Toevs, eds., *Innovations in Bond Portfolio Management: Duration Analysis and Immunization* (Greenwich, CT.: JAI Press, 1983), pp. 61-101.

Toevs, Alden L., "The Error Incurred By Using Duration as a Measure of Bond Portfolio Volatility", *Research Report* (Federal Research Foundation of Canada, 1982).

FOOTNOTES

[1] In the academic literature, convexity is sometimes used to describe the nonequilibrium characteristics of some duration measures, such as Macaulay, that generate greater returns for equal duration bonds the higher the coupon.

[2] For some long-term deep discount bonds, decreases in maturity may be associated with increases in duration so that the maturity relationship may not always hold.

DURATIONS OF NONDEFAULT-FREE SECURITIES

G. O. Bierwag and George G. Kaufman*

I

Duration is a measure of the elasticity of the market price of an asset (liability)—the present value of an expected stream of cash payments (outflows)—with respect to the market discount rate. At the time a bond is purchased, only the stream of promised future payments is known with certainty. A number of factors may cause the amount and time pattern of the cash flows actually realized to differ from those promised. These include changes in interest rates, losses from default, and exercises of call, put, or other options. To be useful and accurate in designing and implementing bond investment strategies for managing interest rate risk, measures of duration, be they single- or multiple-factor, must allow for these uncertainties. Otherwise ex-ante duration may not equal ex-post duration, i.e., the investor will experience stochastic process risk. This adjustment for uncertainty does not require specifying the unexpected changes in cash flows precisely, but specifying only the stochastic processes governing these unexpected changes.

Most previous constructs of duration have been restricted to default- and option-free bonds.[11] These bonds are subject to loss only from unexpected changes in default-free interest rates. More recently, in large measure motivated by the sharp jump in prepayments on residential mortgages and thereby also on mortgage-backed securities when interest rates declined sharply, attention has been devoted to the problem of computing durations for bonds having call options [17, 21, 23]. However, because prepayments are in large measure related to interest rates, the unexpected changes in cash flows on these bonds also result solely from unexpected changes in interest rates. Measures of duration have not been developed for debt securities for which promised cash flows may change because of unexpected default by the issuer, although, in practice, duration-based strategies are most frequently applied to such portfolios, including portfolios of high yield or "junk" bonds [13]. This paper will show that there are biases introduced into the computation of duration for nondefault-free bonds when the time

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pattern of losses from default is not explicitly taken into account, examine the problems associated with constructing durations for such bonds, and develop simple single-factor measures of duration for option-free bonds subject to default risk for hypothetical stylized stochastic processes governing the time pattern of default losses.

II

Bonds subject to default risk trade at higher interest rates than comparable default-free bonds in order to compensate investors for expected losses resulting from reduced and delayed promised payments. The interest rates on bonds subject to default must be sufficiently high so that the expected returns are equal to those on comparable default-free bonds.[1] The difference between the yield to maturity on a bond subject to default and the yield on an otherwise comparable default-free bond is referred to as the default yield premium. The determinants of default yield premiums have been studied extensively in the literature [2, 3, 14, 16, 18, 19, 20, 25]. A few studies have computed durations for non-default-free bonds based on the expected loss from default and the corresponding default yield premiums [1, 12, 15]. However, for the same expected losses from default that are impounded in the default yield premium, the timing of the reductions in payments can vary greatly depending upon the unique circumstances of the particular issuer in default. The outcomes of any two bankruptcy proceedings are rarely the same. Thus, although the dollar amount of the default may be expected, the actual timing of the default may be unexpected.

Each different time pattern of cash payments translates into a different value of duration for a bond for a given stochastic process of default-free interest rates. Although the precise timing of the expected losses is random, it may be described statistically by a stochastic process. To obtain an accurate measure of duration, the stochastic process governing the timing of the losses from default for a given expected present value of future payments must be incorporated in the computation of duration for these bonds as well as the stochastic process governing interest rates.

The importance of including both stochastic processes in the computation of duration may be demonstrated as follows. Assume, for the sake of simplicity, that the stochastic process driving default-free interest rates is additive and consistent with the one-factor Macaulay duration measure so that interest rates on all terms to maturity are the same and the yield curve is flat.[2] Macaulay durations for par bonds with coupon rates and yields to maturity both equal to 10 percent are shown in column (b) of Table 1 for progressively longer terms to maturity. Now let the expected loss from default on the promised payments be such that the expected return is only 9 percent. The resulting default yield premium is 1 percent. This return is consistent with a number of payment patterns. The duration of any of these bonds will depend on the pattern of the expected cash payments particular to that bond.

In the two extremes, we can assume that the default occurs either (1) on the first coupon(s), but that sufficient payments are made on the later coupons to maintain the risk-adjusted interest rate at 9 percent, or (2) on the last payments in a magnitude necessary to reduce the return from 10 to 9 percent. The Macaulay durations for these assumptions are shown in columns (c) and (d) of Table 1, respectively. (The derivation of these default-time-adjusted durations are shown in Appendix A). As can be seen, the durations for the two patterns differ. The early default pattern generates consistently longer durations and the later default pattern consistently shorter durations. The differences in durations are shown in columns (e) and (f).

There is obviously a hypothetical time pattern of default that is consistent with the initial unadjusted duration values shown in column (b) so that no adjustment need be made in the duration computation. One such pattern reduces the cash payments in each period by an amount such that the present value of the after-default cash flows on the bond discounted by the risk-adjusted 9 percent interest rate is equal to the present value of the initial before-default cash flows discounted by the 10 percent yield to maturity. (This default pattern is derived in Appendix B.) But there is no reason to believe that this pattern is the appropriate one for every bond. Thus, the unadjusted duration does not necessarily apply to all bonds subject to default and an investor needs to specify an expected pattern of defaults for each bond. Of course, the particular pattern specified, just as the expected amount of default, may not be realized. As a result, the investor assumes stochastic process risk for default as well as for interest rates [10].

The unadjusted, maximum (early default) and minimum (late default) one-factor Macaulay durations for par bonds with 10 percent market yields to maturity when the default yield premium is 2 percent and the expected interest return declines to 8 percent are shown in Table 2. As can be seen from columns (e) and (f), the magnitude of the potential biases from not taking the stochastic process of default into account explicitly is larger than when the default risk premium is only 1 percent (Table 1). It follows that the greater the default yield premium, the greater the need to adjust the bond's duration for the expected timing of default losses.

A more realistic pattern of expected default may be one in which all scheduled coupon and principal payments are delayed K years after default, but eventually are paid in full by the end of the new maturity. This pattern is shown in Table 3 for 10 percent coupon par bonds with progressively longer terms to maturity having expected after-default returns of 9 percent. The number of years the payments will be delayed if default occurs immediately to satisfy these conditions is shown in column (c). Thus, for example, for a bond with 20 years to maturity, the payments will be delayed exactly one year, that is, the maturity is extended to 21 years. All default-adjusted durations exceed their corresponding unadjusted durations, and the size of the adjustment increases with term to maturity. The adjustment also exceeds the length of the delay in the payments.

It follows from the above numerical examples that adjusting for stochastic default risk as well as stochastic term structure risk, modifies the computation of the relevant duration. The new measures of duration depend on the stochastic process of default assumed. For example, if the assumed process produced defaults only on the last payments and interest rates are the same for all maturities and change only additively (the Macaulay assumptions) the single-factor measure of default-adjusted duration (D_A) is given by:

$$D_A = \sum_{1}^{K-1} tS_t(1+r)^{-t} + K(1-\alpha)S_K(1+r)^{-K}/P \qquad (1)$$

where:

S_t = the scheduled cash payment at time t
K = periods to the first payment that suffers loss from default
α = the proportion of the K^{th} payment lost from default
P = bond price
r = default risk-adjusted yield to maturity
N = term to maturity

In this formulation, partial or total default occurs on the K^{th} payment and total default on all of the remaining $N-(K+1)$ payments in such a magnitude as is consistent with the default risk premium included in the initial market yield to maturity. The duration of this equation is shown in Appendix A. Note that, in contrast with unadjusted duration, default adjusted duration requires the specification of both a description of the time pattern of the expected reduced cash flows and the default risk-adjusted discount rate.

On the other hand, if the same stochastic process of interest rates holds but if all scheduled payments are delayed for K^* years, then the correct single-factor measure of duration is:

$$D_A = K^* + \sum_{1}^{N} tS_t(1+r)^{-t}/P \qquad (2)$$

where K^* is determined so as to be consistent with a given default-risk premium included in the market yield to maturity. The derivation of this duration measure is contained in Appendix C. Other stochastic processes of default for the same stochastic process of interest rates may result in different single-factor measures of default-adjusted duration. More complex stochastic processes would produce more complex differences in the relevant durations.

III

The implications for portfolio management of failing to take the stochastic process of default into account in computing durations may be demonstrated easily. Again for simplicity, assume that the relationship between the expected risk-adjusted return on a security or portfolio of securities and duration is linear and may be expressed as [4, 5, 7, 11]:

$$r = r_o + \frac{PL - D}{PL}(r_1 - r_o) \tag{3}$$

where:

r = expected risk-adjusted return
r_o = initial risk-adjusted yield to maturity
r_1 = expected risk-adjusted yield to maturity immediately after purchase
D = Macaulay duration
PL = investor's planning period.

The bond's interest rate risk is proportional to the value of $(PL\text{-}D)/PL$. To immunize the return on a bond against unexpected changes in risk-adjusted interest rates, an investor needs to choose a bond with a duration equal to the expected planning period. This reduces the term $(PL\text{-}D)/PL$ to zero. Assume that the investor's planning period was 8½ years and that the universe of available bonds are those shown in Table 1. If the investor computed simple unadjusted Macaulay durations, he or she would select the 17 year maturity bond, whose duration is 8½ years. But if, as discussed above, the stochastic process of default were not one that reduced every promised payment by the present value of the expected default, the actual value of $(PL\text{-}D)/PL$ would not be zero and the investor would not be immunized. Instead, the investor would unknowingly be assuming interest rate risk.

If all the expected defaults occur on the last payments, the investor wishing to immunize should have selected the twenty-one year bond which has an appropriate duration of 8½ years; if all the expected defaults occur on the early payments, the investor should have chosen the 14 year maturity bond, which has an approximately 8½ year duration for this default pattern. Because any degree of interest rate risk is measured by the term $(PL\text{-}D)/PL$, similar errors from neglecting the time pattern of losses from default occur for investors who prefer to assume specific nonzero risk-adjusted interest rate risk exposures on bonds subject to default risk rather than to immunize.

IV

This paper has demonstrated that for securities subject to default, durations, whether single- or multiple-term, computed on the basis of market rates of interest need to be adjusted to reflect the expected timing of the expected losses from default. This requires both estimation of the expected loss from default so as to obtain the risk-adjusted interest rate and a chronological description of the reduced cash flows. Although including the stochastic processes generating both interest rates and the time pattern of losses from default complicates the derivation and computation of duration, it is necessary to do so in order to minimize total stochastic process risk and to manage portfolios of nondefault-free bonds intelligently. Because relatively little is currently known about the timing of reductions in payments for bond issues after default, much additional research on the bankruptcy process is required to attain this objective. A primary purpose of this paper is to stimulate such efforts.

Table 1

Macaulay Durations for Alternative Default and Cash Flow Scenarios for Par Bonds with Yields to Maturity of 10 Percent and Expected Returns of 9 Percent

(a) Maturity	(b) Unadjusted Duration D	(c) Maximum Duration D_U	(d) Minimum Duration D_L	(e) $D_U - D$	(f) $D_L - D$
		(years)			
1	.976	.976	.976	0	0
2	1.862	1.878	1.861	0.016	-0.001
3	2.665	2.713	2.661	0.048	-0.004
4	3.393	3.484	3.385	0.091	-0.008
5	4.054	4.196	4.038	0.142	-0.016
6	4.653	4.855	4.627	0.202	-0.026
7	5.197	5.461	5.157	0.264	-0.040
8	5.690	6.018	5.633	0.328	-0.057
9	6.137	6.533	6.060	0.396	-0.077
10	6.543	7.009	6.441	0.466	-0.102
11	6.911	7.449	6.780	0.538	-0.131
12	7.244	7.855	7.082	0.611	-0.162
13	7.547	8.230	7.349	0.683	-0.198
14	7.822	8.576	7.584	0.754	-0.238
15	8.071	8.897	7.790	0.826	-0.281
16	8.296	9.192	7.969	0.896	-0.327
17	8.501	9.465	8.124	0.964	-0.377
18	8.687	9.717	8.256	1.030	-0.431
19	8.856	9.950	8.368	1.094	-0.488
20	9.009	10.165	8.461	1.156	-0.548
21	9.147	10.363	8.537	1.216	-0.610
22	9.273	10.545	8.597	1.272	-0.676
23	9.387	10.713	8.642	1.326	-0.745
24	9.491	10.875	8.675	1.384	-0.816
25	9.584	11.010	8.695	1.426	-0.889

D_U - Duration when default expected to occur on earliest payments.
D_L - Duration when default expected to occur on last payments.

Table 2

Macaulay Durations for Alternative Default and Cash Flow Scenarios for Par Bonds with Yields to Maturity of 10 Percent and Expected Returns of 8 Percent

(a) Maturity	(b) Unadjusted Duration D	(c) Maximum Duration D_U	(d) Minimum Duration D_L	(e) $D_U - D$	(f) $D_L - D$
		(years)			
1	.976	.976	.976	0	0
2	1.862	1.896	1.859	0.034	-0.003
3	2.665	2.758	2.657	0.093	-0.008
4	3.393	3.559	3.376	0.166	-0.017
5	4.054	4.313	4.022	0.259	-0.032
6	4.653	5.024	4.600	0.371	-0.053
7	5.197	5.681	5.116	0.484	-0.081
8	5.690	6.299	5.574	0.609	-0.116
9	6.137	6.880	5.979	0.743	-0.158
10	6.543	7.427	6.334	0.884	-0.209
11	6.911	7.936	6.643	1.025	-0.268
12	7.244	8.411	6.911	1.167	-0.333
13	7.547	8.858	7.139	1.311	-0.408
14	7.822	9.278	7.331	1.456	-0.491
15	8.071	9.673	7.490	1.602	-0.581
16	8.296	10.044	7.618	1.748	-0.678
17	8.501	10.389	7.717	1.888	-0.784
18	8.687	10.711	7.790	2.024	-0.897
19	8.856	11.013	7.839	2.157	-1.017
20	9.009	11.297	7.865	2.288	-1.144
21	9.147	11.563	7.871	2.416	-1.276
22	9.273	11.813	7.871	2.540	-1.402
23	9.387	12.047	7.871	2.660	-1.516
24	9.491	12.267	7.871	2.776	-1.620
25	9.584	12.472	7.871	2.888	-1.713

Table 3

Macaulay Durations for Default Patterns On Par Bonds with Yields to Maturity of 10 Percent In Which All Payments Are Delayed by K Years To Generate Expected Returns of 9 Percent

(a) Maturity	(b) Unadjusted Duration D	(c) Delay In Payments K	(d) D_K	(e) $D_K - D$
		(years)		
1	.976	.106	1.082	.106
2	1.862	.202	2.065	.203
3	2.665	.289	2.959	.294
4	3.393	.369	3.773	.380
5	4.054	.441	4.516	.462
6	4.653	.506	5.193	.540
7	5.197	.566	5.813	.616
8	5.690	.621	6.380	.690
9	6.137	.670	6.899	.762
10	6.543	.716	7.374	.831
11	6.911	.757	7.810	.899
12	7.244	.795	8.209	.965
13	7.547	.829	8.576	1.029
14	7.822	.861	8.913	1.091
15	8.071	.889	9.222	1.151
16	8.296	.916	9.507	1.211
17	8.501	.940	9.768	1.267
18	8.687	.962	10.008	1.321
19	8.856	.982	10.230	1.374
20	9.009	1.000	10.433	1.424
21	9.147	1.017	10.620	1.473
22	9.273	1.032	10.793	1.520
23	9.387	1.046	10.951	1.564
24	9.491	1.059	11.097	1.606
25	9.584	1.070	11.232	1.648

D_K - Duration when default expected to cause all payments to be postponed K years.

Appendix A

Default-Adjusted Durations for Default Patterns on the Earliest and Latest Payments for a Given Default Risk Premium

If the expected default patterns occur on the early (late) payment dates, then the duration of the entire income stream is larger (smaller). This result is shown to be true for a given default risk premium included in the market yield to maturity. The price of a bond is given as

$$P = \sum_{1}^{N} S_t(1 + r_t^*)^{-t} = \sum_{1}^{N} (S_t - d_t)(1 + r)^{-t} \tag{A.1}$$

where S_t is the promised cash flow at date t, d_t is the expected default at date t, r^* is the market yield to maturity, r is the risk-adjusted yield to maturity, and N is the maturity. The Macaulay duration is

$$D = \sum_{1}^{N} tS_t(1 + r^*)^{-t}/P. \tag{A.2}$$

The duration adjusted for the risk of the expected default is

$$D_A = \sum_{1}^{N} t(S_t - d_t)(1 + r)^{-t}/P. \tag{A.3}$$

To find the maximal adjusted duration, we let

$$d_t = \begin{cases} S_t, \; t = 1,2, \dots , K-1 \\ \alpha S_K, \; t = K \\ 0, \; t = K+1, \dots , N \end{cases} \tag{A.4}$$

for some $\alpha (0 \leq \alpha \leq 1)$ and K such that (A.1) holds for the given values of r^*, r, and N. In this way, all of the expected defaults occur on the earliest possible dates consistent with the pricing equations. Substitution of (A.4) into (A.1) allows us to define

$$F(\alpha, K) = (1 - \alpha)S_K(1 + r)^{-K} + \sum_{K+1}^{N} S_t(1 + r)^{-t}, \; K + 1 < N. \tag{A.5}$$

Here, $F(\alpha, K)$ is the value of the income stream implied by the expected defaults in (a.4). If (α, K) is chosen correctly, then $F(\alpha, K) = P$. The function $F(\alpha, K)$ is decreasing a function of K and α. When $K = 1$ and $\alpha = 0$, $F(0,1) > P$, and when $K = N$ and $\alpha = 1$, $F(1,N) = 0$. Therefore, there is some value, (α^*, K^*), at which $F(\alpha^*, K^*) = P$. To find these values, we may proceed iteratively by letting $\alpha = 0$, and increasing K, integer by integer, until we have some K^* at which

$$F(0,K^* - 1) \leq P \leq F(0,K^*). \tag{A.6}$$

Noting that $F(1,K^* - 1) = F(0,K^*)$, it follows that there is some α^* at which $F(\alpha^*, K^* - 1) = P$. That is,

$$(1 - \alpha^*)S_{K^*-1}(1 + r)^{-K^*+1} + F(0,K^*) = P, \tag{A.7}$$

or

$$(1 - \alpha^*) = \frac{P - F(0,K^*)}{S_{K^*-1}(1 + r)^{1-K^*}} \tag{A.8}$$

Using (α^*,K^*) to compute the excepted defaults in (A.4) and substituting these defaults into (A.3) gives us the maximal duration consistent with the pricing equation, (A.1).

To find the minimal adjusted duration, we let

$$d_t = \begin{cases} 0, & t = 1,2, \dots , K - 1 \\ \alpha S_K, & t = K \\ S_t, & t = K + 1, \dots , N \end{cases} \tag{A.9}$$

for some $\alpha (0 \leq \alpha \leq 1)$ and K such that (A.1) holds for the given values of r^*, r, and N. In this way, all of the expected defaults occur in the latest possible dates consistent with the pricing equations. Substitution of (A.9) into (A.1) allows us to define

$$G(\alpha,K) = \sum_{1}^{K-1} S_t(1 + r)^{-t} + (1 - \alpha)S_K(1 + r)^{-K} \tag{A.10}$$

Here, $G(\alpha,K)$ is a monotonic increasing function of K and a monotonic decreasing function of α. We can observe that $G(0,N) > P > G(1,1) = 0$. Therefore, there are values (α^*,K^*) at which $G(\alpha^*,K^*) = P$. Assuming $\alpha = 1$, we may proceed monotonically until we have found a value of K^* such that

$$G(1,K^*) \leq P \leq G(1,K^* + 1), \tag{A.11}$$

so that for some α^*, $G(\alpha^*,K^*) = P$. That is, (A.12) $$(1-\alpha^*) = \frac{P - G(1, K^*)}{S_{K^*}(1+r)^{-K}}$$

Using (α^*, K^*) to compute the expected defaults in (A.9) and substituting these defaults into (A.3) gives us the minimal duration consistent with pricing equation (A.1)

The procedures for finding (α^*, K^*) in the case of coupon bonds allow for explicit expressions for $F(\alpha, K)$ and $G(\alpha, K)$. For example, if we (a) allow for semi-annual discounting, (b) allow for semi-annual coupon payments, and (c) measure maturity in six-month periods, then

(A.13) $$F(\alpha,K) = (1 + r/2)^{-(K-1)}p(c,r,N - K + 1) - \alpha(c/2)F(1 + r/2)^{-K}$$

where $(c/2)$ is the semi-annual coupon rate, F is the face value of the bond, and $p(c,r,N\text{-}K+1)$ is the price of a bond with annual coupon rate c, annual yield to maturity r, and maturity $N - K + 1$. Similarly,

(A.14) $$G(\alpha,K) = A(c,r,K) - \alpha F(1 + r/2)^{-K}(c/2)$$

where $A(c,r,K)$ is the value of an annuity consisting of $(c/2)F$ dollars every six months for K six month periods.

Appendix B

Default-Adjusted Durations for Default Patterns Consistent with Unadjusted Durations

It is clear from Table 1 and 2 that there are expected default patterns in which the adjusted duration is identical to the unadjusted duration. Some of these special cases are of interest.

The price of the security in those cases where the promised payments are subject to default can be expressed as

$$P = \sum_{1}^{N} S_t(1 + r^*)^{-t} = \sum_{1}^{N} (S_t - d_t)(1 + r)^{-t} \tag{B.1}$$

Where S_t is the promised payment at date t, d_t is the expected default, r^* is the market yield to maturity, r is the default-risk-adjusted yield to maturity, and (r^*-r) is the default risk premium included in the market yield to maturity. The unadjusted duration is

$$D = \sum_{1}^{N} tS_t(1 + r^*)^{-t}/P \tag{B.2}$$

and the adjusted duration is

$$D_A = \sum_{1}^{N} t(S_t - d_t)(1 + r)^{-t}/P. \tag{B.3}$$

An obvious default pattern—$d_1, d_2, \ldots, d_N$—in which $D = D_A$ is one in which

$$\frac{S_t - d_t}{(1 + r)^t} = \frac{S_t}{(1 + r^*)^t} t = 1,2, \ldots, N. \tag{B.4}$$

Here, the present values of the payments after the expected defaults are equal to the present values of the payments before the expected defaults. If the income stream, $S_1, S_2, \ldots, S_N$, can be broken into its separate components and be bought and sold separately, then equation (B.4) gives two alternative ways for evaluating each of the cash flows in equilibrium for the given expected default pattern.

If (1) the term structures—r^* and r—are flat, (2) the term structures can only shift in an additive manner, and (3) cash flow components can be separated and sold separately, then no other expected default pattern is consistent with an equilibrium in which equation (B.4) holds. On the other hand, if the components of the income stream cannot be separated and sold separately, then a variety of expected default patterns are possible because then equation (A.4) need not be required to hold in an equilibrium for those expected defaults.

It is clear that stochastic processes over the default patterns and interest rates and the degree to which income stream components can be bought and sold separately are relevant features of an equilibrium model of the term structure and of the corresponding relationship between the adjusted and unadjusted durations.

Appendix C

Default-Adjusted Durations for Default Patterns in Which All Scheduled Payments are Delayed

The value of the promised income of a bond can be expressed as

$$\text{(C.1)} \qquad P = \sum_{1}^{N} S_t(1 + r^*)^{-t},$$

where S_t is the promised cash flow at time t, N is the maturity, and r^* is the market yield to maturity on the bond. If the expected default pattern is one in which all payments are delayed by K years but on which there is some interest accumulation, the expected cash flows, after default, become

$$\text{(C.2)} \qquad \overline{S}_t = 0,\ t = 1,2, \dots, K$$

$$\overline{S}_t = (1 + \gamma r^*)^K S_{t-K},\ t = K + 1, \dots, N + K$$

where γr^*, $0 \leq \gamma \leq 1$, is the interest rate on delayed payments. If r is the risk-adjusted discount rate, then

$$\text{(C.3)} \qquad P = \sum_{K+1}^{N+K} \overline{S}_t(1 + r)^{-t},$$

where r^*-r is the default risk premium included in the market yield to maturity r^*. Substitution of (C.2) into (C.3) gives

$$\text{(C.4)} \qquad P = \sum_{K+1}^{N+K} (1 + \gamma r^*)^K (1 + r)^{-t} S_{t-K}$$

The duration of this income stream, after the expected defaults, may be computed as

$$\text{(C.5)} \qquad D_K = \sum_{K+1}^{N+K} tS_{t-K}(1 + r)^{-t}(1 + \gamma r^*)^K / P$$

$$= \sum_{K+1}^{N+1} tS_{t-K}(1 + r)^{-t} / \sum_{K+1}^{N+1} S_{t-K}(1 + r)^{-t}$$

$$= \sum_{1}^{N} (t + K)S_t(1 + r)^{-t} / \sum_{1}^{N} S_t(1 + r)^{-t}$$

We can thus write the adjusted duration as

(C.6) $$D_K = K + \sum_{1}^{N} tS_t(1 + r)^{-t} / \sum_{1}^{N} S_t(1 + r)^{-t}$$

The result in Table 3 are derived by assuming $r^* = .10$, $r = .09$, and the cash flows are such that the bond sells at par, ($P = 100$). The Macaulay duration of this bond is

(C.7) $$D = \sum_{1}^{N} tS_t(1 + r^*)^{-t} / \sum_{1}^{N} S_t(1 + r^*)^{-t}.$$

Given that duration is a decreasing function of the rate of interest and that $r^* > r$, it follows that

(C.8) $$D < D_K$$

or that

(C.9) $$D_K > D + K,$$

which is clearly shown to be the case in comparing columns (e) and (c) of Table 3. In words, the default-adjusted duration increases by more than the delay in payment.

It is also apparent that the expected default-adjusted duration is not affected by γ or the extent to which interest is paid on delayed payments.

REFERENCES

Alexander, Gordon J. and Bruce G. Resnick, "Using Linear and Goal Programming to Immunize Bond Portfolios", *Journal of Banking and Finance* (March 1985), pp. 35-54.

Altman, Edward I. and Scott A. Nammacher, *Investing in Junk Bonds* (New York: John Wiley, 1987).

Atkinson, Thomas R., *Trends in Corporate Bond Quality* (New York: National Bureau of Economic Research, 1967).

Babcock, Guilford C., "Duration As a Link Between Yield and Value", *Journal of Portfolio Management* (Summer 1984), pp. 58-65.

Babcock, Guilford C., "Erratum: Duration As a Link Between Yield and Value", *Journal of Portfolio Management* (Fall 1984), pp. 97-98.

Beedles, William L., "On the Use of Certainty Equivalent Factors as Risk Proxies", *Journal of Financial Research* (Winter 1978), pp. 15-21.

Bierwag, G. O., *Duration Analysis: Managing Interest Rate Risk* (Cambridge, MA.: Ballinger, 1987).

Bierwag, G. O., George G. Kaufman, and Cynthia M. Latta, "Bond Portfolio Immunization: Tests of Maturity, One- and Two-Factor Duration Matching Strategies", *Financial Review* (May 1987), pp. 203-220.

Bierwag, G. O., George G. Kaufman, Cynthia M. Latta, and Gordon S. Roberts, "Usefulness of Duration in Bond Portfolio Management: Response to Critics", *Journal of Portfolio Management* (Winter 1987), pp. 48-52.

Bierwag, G.O., George G. Kaufman, and Alden Toevs, "Bond Portfolio Immunization and Stochastic Process Risk", *Journal of Bank Research* (Winter 1983), pp. 282-291.

Bierwag, G. O., George G. Kaufman, and Alden Toevs, "Duration: Its Development and Use in Bond Portfolio Management", *Financial Analysts Journal* (July/August 1983), pp. 15-35.

Chance, Don M., "A Note on Default Risk and the Duration of Zero Coupon Bonds", Working Paper, Virginia Polytechnic Institute and State University (November 1987).

Feldstein, Sylvan G., Peter E. Christensen, and Frank Fabozzi, "Bond Portfolio Immunization", in F. Fabozzi, ed., *Readings in Investment Management* (Homewood, IL.: Richard Irwin, Inc., 1983), pp. 189-206.

Fons, Jerome S., "The Default Premium and Corporate Bond Experience", *Journal of Finance* (March 1987), pp. 81-97.

Fons, Jerome S., "Single-Factor Certainty-Equivalence Duration for Non-Default Free Bonds", *Working Paper,* Federal Reserve Bank of Cleveland (July 1987).

Fraine, Harold G., and Robert H. Mills, "Effect of Default and Credit Deterioration on Yields of Corporate Bonds", *Journal of Finance* (September 1961), pp. 423-434.

Garman, Mark B., "The Duration of Option Portfolios", *Journal of Financial Economics* (June 1985), pp. 309-315.

Hempel, George H., *The Postwar Quality of State and Local Debt* (New York: National Bureau of Economic Research, 1971).

Hickman, Braddock W., *Corporate Bond Quality and Investor Experience* (New York: National Bureau of Economic Research, 1958).

Johnson, Ramon E., "Term Structures of Corporate Bond Yields as a Function of Risk of Default", *Journal of Finance* (May 1967), pp. 313-345.

Kopprasch, Robert W. et al., *Effective Duration of Callable Bonds* (New York: Salomon Brothers, April 1987).

McEnally, Richard W. and Calvin Broadman, "Aspects of Corporate Bond Portfolio Diversification", *Journal of Financial Research* (Spring 1979), pp. 27-35.

Toevs, Alden L., "Interest Rate Risk and Uncertain Lives", *Journal of Portfolio Management* (Spring 1985), pp. 45-56.

Warner, Jerold B., "Bankruptcy Costs: Some Evidence", *Journal of Finance* (May 1977), pp. 337-347.

Warner, Jerold B., "Bankruptcy, Absolute Priority, and the Pricing of Risky Debt Claims", *Journal of Financial Economics* (May 1977), pp. 239-276.

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1. The price (P) of bond subject to default risk may be expressed as:

$$P = \Sigma S_t(1 + r^*)^{-t} = \Sigma(S_t - d_t)(1 + r)^{-t}$$

Where: S_t = promised cash flows
d_t = expected loss from default
r = risk-adjusted interest rate
r^* = market interest rate
t = time to payment

2. Although more complex and possibly realistic measures of duration have been developed, the single-factor Macaulay measure performs reasonably well in empirical tests [7, 8, 9].