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STAFF MEMORANDA

A LINEAR MODEL OF THE LONG-RUN NEUTRALITY OF MONEY

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One of the oldest verbal theories of economics is the quantity theory of money. Over the last two hundred years this general theory has been presented in a variety of forms. The common assumption of these alternative approaches is that a change in the quantity of money causes a proportional change in the level of prices and does not affect the level of real output in the long run. The theoretical arguments for this theory have been developed most clearly by Irving Fisher (2) and Milton Friedman (4).

An extension of the quantity theory of money assumes that a change in the rate of growth of money causes, in the long run, an equal change in the rate of inflation and does not have any permanent effect on real output or employment. In recent times this additional argument has been developed in the extensive literature on the lack of a permanent trade-off between the rate of inflation and the rate of unemployment. The pioneering work in this area was done by Milton Friedman (3) and Edmund Phelps (5).

The purpose of this paper is to translate these economic assumptions into mathematical constraints that then are imposed on a small macroeconomic model. Mathematically this model consists of linear ordinary difference equations, where one of the external forces is assumed to be a weighted average of a money variable. The first part of this paper reviews the basic assumptions of the quantity theory of money. The derivation of the constraints and the corresponding transformation of the data are presented in the appendixes. The second part of the paper examines some of the problems that arise in estimating and comparing these constrained models.

In its simplest form the quantity theory of money states that, other things being equal, doubling the quantity of money will cause of doubling of prices. This theory is an assumption about how an economic system works in our physical universe. It is not merely a simple if-then statement of logic. As Alexander Del Mar (1) was careful to specify over a hundred years ago, this latter interpretation does "violence to Nature, whose movements are performed only in time; an element, of which logic has usually taken but little account." Del Mar went on to state "whilst the volume of money might be increased or diminished instantly, the resulting movement of prices would only occur after an interval of time."

Irving Fisher (2) was very careful to make this distinction between the short-run and long-run effects of a change in the quantity of money. In his words:

> We have emphasized the fact that the strictly proportional effect on prices of an increase in M [the quantity of money] is only the *normal* or *ultimate* effect after transition periods are over. The proposition that prices vary with money holds true only in comparing two imaginary periods for each of which prices are stationary or are moving alike upward or downward and at the same rate.

He characterizes the dynamic relationship between money and the general level of prices with the following analogy:

> The peculiar effects during transition periods are analogous to the peculiar effects in starting or stopping a train of cars. Normally the caboose keeps exact pace with the locomotive, but when the train is starting or stopping this relationship is modified by the gradual transmission of effects through the intervening cars. Any special shock to one car is similarly transmitted to all the others and to the locomotive.

Although Fisher saw that a "sudden" change in the quantity of money initially would affect the volume of real output or trade, he considered this effect to be a temporary one. In terms of long run or "ultimate" effects, he assumed that "An inflation of the currency cannot increase the product of farms or factories, nor the speed of freight trains or ships. The stream of business depends on natural resources and technical conditions, not on the quantity of money."

Under a system of fiat money, like we have today, Fisher thought that a change in the quantity of money would not "appreciably affect the quantity of goods sold for money." He concluded that

> . . .the issue of paper money may affect the paper and printing trades, the employment of bank and government clerks, etc. In fact, there is no end to the minute changes in the Q's [measures of real output] which the changes mentioned, and others might bring about. But from a practical or statistical point of view they amount to nothing, for they could not add to nor subtract one tenth of 1 per cent from the general aggregate of trade.

Notice that the quantity theory of money focuses on the hypothesized effects of just one of a myriad of factors that determine the levels of prices and real outputs. On this matter Fisher wrote

> The importance and reality (sic) of this proposition are not dimished in the least by the fact that these othere causes do not historically remain quiescent and allow the effect on the p's [prices] of an increase in M [money] to be seen alone. The effects of M are blended with the effects of changes in the other factors in the equation of exchange just as the effects of gravity upon a falling body are blended with the effects of the resistance of the atmosphere.

In order to translate the preceeding arguments into a mathematical model, it is necessary to articulate some of the assumptions that are being made. This theory hypothesizes a dynamic relationship between a measure of the quantity of money (M) and measures of a price index (P) and the rate of producing real output or the level of real transactions (Q). We can define a measure of nominal output or transactions (Y) as the product of the price index and the measure of real output. Algebraically this definition can be written as

1) $Y = PQ$

Let y, p, q, and m represent the logarithms of Y, P. Q, and M, respectively. Equation 1 can be expressed in the following linear form

2) $y = p + q,$

where each of these variables are assumed to be functions of m, time (t), and whatever other variables one wishes to included in a model.

Given the accounting identity between y, p, and q, there are three possible ways to formulate a model. We can specify directly the dynamic linkage between m and y and p, between m and y and q, or between m and p and q, and then use the accounting identity (equation 2) to determine the corresponding values of q, p, or y, respectively. These three alternatives correspond to the three versions of the dynamic model that are estimated in this paper.

The basic form of each of these models consists of an ordinary difference equation, where the time step should be relatively "small" with respect to the speed of adjustment of the economy. Within the context of this mathematical framework, the current value of an economic variable is assumed to be a function of the lagged values of this variable plus a summation of other forces, one of which is a weighted average of the quantity of money.

Furthermore, these difference equations are assumed to be linear and have constant coefficients. Each version of the models consists of the accounting equation plus two of the following three equations that need to be estimated:

$$3)\quad y(t) = \sum_{i=1}^{N} a_y(i)y(t-i) + \sum_{j=o}^{M} b_y(j)m(t-j) + \Theta_y$$

$$4)\quad p(t) = \sum_{i=1}^{N} a_p(i)p(t-i) + \sum_{j=o}^{M} b_p(j)m(t-j) + \Theta_p$$

$$5)\quad q(t) = \sum_{i=1}^{N} a_q(i)q(t-i) + \sum_{j=o}^{M} b_q(j)m(t-j) + \Theta_q$$

where Θ_y, Θ_p and Θ_q can represent an intercept term plus whatever other variables one wishes to include. Notice that each of these difference equations is of the same order (N) and includes the same number (M) of lagged values of money.

Since this model is linear the incremental effect of change in the quantity of money is independent of the initial conditions of the model and of the effects of other variables that might be added to the model. Furthermore, the effects of an equal increase or decrease in the quantity of money are exactly equal in absolute magnitude, although they have opposite signs. Because of these inherent restrictions, the model is presented with the caveat that it is intended to predict the effects of relatively "small" changes in the quantity of money or its rate of growth.

Given the functional form of this model, the next step is to specify the mathematical constraints that correspond to the assumptions about the long-run neutrality of money. Recall that the quantity theory assumes that a change in the quantity of money causes a proportional change in the price index and level of nominal output, in the long run, and that a change in the rate of growth of money causes an equal change in the long-run rates of inflation and growth of nominal output. In order for these two conditions to exist, the coefficients on equations 3 and 4 must satisfy the following constraints:

$$6)\quad \sum_{i=1}^{N} a(i) + \sum_{j=o}^{M} b(j) = 1$$

$$7)\quad \sum_{i=1}^{N} ia(i) + \sum_{j=o}^{M} b(j) = 0$$

where a(i) represents $a_y(i)$ or $a_p(i)$ and b(j) represents $b_y(i)$ or $b_p(j)$. See Appendix A for the derivation of these constraints.

It should be pointed out that the second constraint (equation 7) differs from one that Dean Taylor (6) used in estimating Friedman's dynamic model of nominal output. This particular model is a second order difference equation (N=2) with one lag of the money variable (M=1). His constraint is

$$b(0) = 1 + (\lambda_1 e^{-\lambda_1} - \lambda_2 e^{-\lambda_2})/(\lambda_2 - \lambda_1)$$

where λ_1 and λ_2 are the roots of the characteristic polynomial of this difference equations. The fact that this constraint is mathematically incorrect can be shown by checking the derivation of equation 7 or by solving Taylor's model with his estimated coefficients. In his model a change in the quantity of money does not cause a proportional change in the level of nominal output, in the long run.

When the equation for the logarithm of real output (equation 5) is estimated, the following constraints will impose the assumptions of the quantity theory of money:

8) $$\sum_{i=1}^{N} a_q(i) + \sum_{j=o}^{M} b_q(j) = 0$$

9) $$\sum_{i=1}^{N} ia_q(i) + \sum_{j=o}^{M} jb_q(j) = 0$$

These constraints correspond to the assumptions that neither a change in the quantity of money nor a change in the rate of growth of money has a permanent effect of the level of real output.

In order to reduce the number of variables that have to be estimated, the lag weights in each of the equations of the model are assumed to be generated by a third-degree polynomial, e.g.

$$a(i) = \alpha_0 + \alpha_1 i + \alpha_2 i^2 + \alpha_3 i^3$$

$$b(j) = \beta_0 + \beta_1 j + \beta 2 j^2 + \beta_3 j^3$$

where a(i) represents $a_y(i)$, $a_p(i)$, or $a_q(i)$ and b(j) represents $b_y(j)$ or $b_p(j)$. The lag weights for the money variable in the equation for real output are assumed to be on a fourth-degree polynomial. Furthermore, the end-points are constrained to equal zero or

$$a(N+1) = 0$$

$$b(M+1) = 0.$$

The above constraints and the assumptions about the long-run neutrality of money can then be used to determine the corresponding transformation of the data. After the data has been transformed the model can be estimated by any appropriate statistical technique. See Appendixes B and C for the derivation of these transformations.

Given the preceeding method of imposing the assumptions about the long-run neutrality of money, the next step is to estimate a version of the model. This linear transfer equations model is general enough so that it can be fitted to any time series of a monetary aggregate, a price index, a measure of real economic activity, and the corresponding measure of nominal economic activity. The difference equations can be estimated with monthly or quarterly data, provided that the chosen time step is relatively small with respect to the speed of adjustment of the economy. This latter proviso is an inherent restriction of the mathematics of difference equations.

Notice the enormous number of combinations that can be tried for any given economy or country. Possible measures of nominal economic activity include retail sales, personal income, gross national product (GNP), and personal consumption expenditures. For a measure of real economic activity, one can try using industrial production, man-hours in private nonagricultural industry, or a deflated series of nominal economic activity. Some of the available price indices that could be estimated include the consumers' price index, wholesale price index, a wage rate series, and a price deflator for any nominal income or output series. In terms of a measure of monetary aggregate, one can try a series for the monetary base or a money supply series such as M-1, M-1+, M-2, M-3, . . . For each set of economic time series, it is necessary to specify a range in historical time for the purpose of estimation.

In order to provide a specific example of some of the problems that arise in estimating this model, I shall concentrate in the remainder of this paper on the dynamic linkage between M-1 and GNP. Because GNP data is available only on a quarterly basis, the time step of the model is a quarter of a year. The sample period for each model is from the first quarter of 1959 through the fourth quarter of 1976. Beginning in the first quarter of 1977, an eight quarter dynamic simulation of each model is run so as to provide measures of how well the models fit outside their period of estimation.

Each of the equations that are estimated directly in the models is fitted by minimizing the sum of squared errors with respect to the coefficients of the equation. The data for each model are first transformed into rates of growth by taking the first difference of the logarithms of the data. Next, these data have been transformed, according to the procedure that has been developed in this paper, so as to build in the long-run assumptions about the neutrality of money. Finally, an intercept term and some additional variables have been added to each equation and the least-squares estimates have been calculated.

The money variable is the rate of growth of M-1 between two successive quarters. The quarterly data for M-1 is equal to the average of the three months' seasonally adjusted data of each quarter. The nominal output variable is the rate of growth of GNP minus Federal Government purchases of goods and services. Federal purchases have been subtracted since one of the additional variables in each equation is a weighted average of its rates of growth. The price variable is the rate of change of the GNP deflator. For each of these quarterly series, seasonally adjusted data have been used.

The lag weights on the weighted average of the rates of growth of Federal Government purchases are generated by a third-degree polynomial. In order to simplify the model, the number of lagged values of Federal purchases is assumed to be equal to M, the number of lagged values of the money aggregate. The end-point is constrained to equal zero so that only three coefficients need to be estimated for this fiscal policy variable.

Two dummy variables have been included in each equation to provide estimates of the effects of wage and price controls during the Nixon administration and the effects of the quadraupling of crude oil prices in 1973 by OPEC. Those dummy variables are third-degree time polynomials with an end-point constraint. For example, the dummy variables for wage-price controls are generated by the following equation:

$$d_{wp}(t_{wp} + \tau) = \delta_0 + \delta_1\tau + \delta_2\tau^2 + \delta_3\tau^3, \ \tau=0,1....,\ NTWP$$

$$d_{wp}(t_{wp} + NTWP) = 0$$

where t_{wp} is the first quarter with controls and NTWP is the number of quarters these dummy variables are applied. The first quarter for the wage-price dummy variables is 1971-3, and the first quarter for the OPEC dummy variables is 1973-4. Each of these dummy variables requires that three additional coefficients be estimated in each equation.

Even after selecting the economic time series to be included, the sample period, and the method of estimation, there is a large number of possible models that can be tried. For example, what should be the order of the differences equations? How many lagged values of the monetary aggregate and Federal purchases should be used? How long should each of the dummy variables be applied? In terms of the parameters of the models, these issues are concerned about the appropriate values of N, M, NTWP, and NTOIL. NTOIL is the number of quarters the OPEC dummy variables are applied.

In order to reduce the number of possible models, I restricted the first series of regressions to three values for each of these four parameters. The corresponding set of 81, or 3^4, parameter combinations are summarized in the following table:

N	M	NTWP	NTOIL
3	8	8	8
4	12	10	10
5	16	12	12

For each of these parameter combinations, three equations were estimated directly -- one for the rate of growth of nominal GNP minus Federal Government purchases, the second for the rate of growth of the GNP deflector, and the third for the rate of growth of real GNP minus real Federal purchases.

These three estimated equations and the accounting identity for the rate of growth of nominal output determine the three versions of this model. Version I of the model uses the estimated equations for the rates of growth of nominal output and the price index and determines the corresponding values of the rate of growth of real output from the accounting identity. Version II of the model uses the estimated equations for the rates of growth of nominal and real output and determines the corresponding rate of growth of the deflator from the accounting identity. Version III of the model uses the estimated equations for the rates of growth of the price deflator and real output and determines the corresponding rate of growth of nominal output from the accounting identity. In total there were 243, or 81 times 3, equations and models estimated in the first set of regressions.

The next problem that logically arises is how to evaluate each of these alternative models and parameter combinations. My heuristic approach for selecting the "best" model developed along the following line. For each parameter combination I calculated two sets of summary statistics. The first set of statistics includes measures of fit within the sample period. The second set of statistics includes measures of fit within the forecast period.

In the first set of summary statistics I include fifteen numbers. Six of these numbers are the coefficients of determination and the root mean squared errors of the three equations that have been estimated directly. These statistics are labeled R^2 (Dy), R^2 (Dp), R^2 (Dq) and RMSE (Dy), RMSE (Dp), RMSE (Dq), respectively. Each of the three versions of this model uses two of the directly estimated equations and the accounting identity to calculate estimates of the "residual" variable. Therefore, there are also a coefficient of determination and a root mean squared error for each of the three possible residual variables. These six numbers are labeled R^2 (Dye), R^2 (Dpe), R^2 (Dqe) and RMSE (Dye), RMSE (Dpe), RMSE (Dqe), respectively.

The next summary statistics that are calculated are just the sums of the three root mean squared errors for each version of the models. These numbers are labeled $\overline{\text{RMSE}}$ (I), $\overline{\text{RMSE}}$ (II), and $\overline{\text{RMSE}}$ (III), and are defined by the following equations:

$$\overline{\text{RMSE}}\text{ (I)} = \text{RMSE (Dy)} + \text{RMSE (Dp)} + \text{RMSE (Dqe)}$$

$$\overline{\text{RMSE}}\text{ (II)} = \text{RMSE (Dy)} + \text{RMSE (Dpe)} + \text{RMSE (Dq)}$$

$$\overline{\text{RMSE}}\text{ (III)} = \text{RMSE (Dye)} + \text{RMSE (Dp)} + \text{RMSE (Dq)}.$$

These statistics provide a single measure of how well any model estimates the rates of growth of nominal output, the price deflator, and real output, within the sample period. While there are fifteen summary statistics in this first set for any parameter combinations, only seven of these numbers are related to any particular version of the model.

In the first set of regressions, where I tried 81 different parameter combinations, these summary statistics fell within the following ranges:

$.543 \geq R^2 \text{ (Dy)} \geq .375$ $\qquad$ $.423 \geq R^2 \text{ (Dye)} \geq .269$

$.608 \leq \text{RMSE (Dy)} \leq .702$ $\qquad$ $.683 \leq \text{RMSE (Dye)} \leq .760$

$.857 \geq R^2 \text{ (Dp)} \geq .799$ $\qquad$ $.764 \geq R^2 \text{ (Dpe)} \geq .513$

$.244 \leq \text{RMSE (Dp)} \leq .290$ $\qquad$ $.314 \leq \text{RMSE (Dpe)} \leq .452$

$.577 \geq R^2 \text{ (Dq)} \geq .388$ $\qquad$ $.656 \geq R^2 \text{ (Dqe)} \geq .436$

$.695 \leq \text{RMSE (Dq)} \leq .816$ $\qquad$ $.629 \leq \text{RMSE (Dqe)} \leq .783$

$1.501 \leq \overline{\text{RMSE}}\text{ (I)} \leq 1.774$

$1.631 \leq \overline{\text{RMSE}}\text{ (II)} \leq 1.900$

$1.648 \leq \overline{\text{RMSE}}\text{ (III)} \leq 1.856$

Notice that the best fits as measured by the size of the root mean squared errors, are provided by the inflation equation when it is estimated directly. This property is robust in the sense that the highest root mean squared error of this equation is lower than the smallest root mean squared error of any other equation.

By examining the summary statistics for each of the individual regressions, it is possible to spot several other patterns. For every parameter combination in this set of regressions, the root mean squared error for the rate of growth of nominal output is smaller when this variable is estimated directly instead of being treated as the residual variable. Furthermore, the root mean squared error for the rate of growth of real output is at least as small when this variable is treated as the residual variable instead of being estimated directly. These patterns can be summarized by the following inequalities:

$$\text{RMSE (Dy)} < \text{RMSE (Dye)}$$

$$\text{RMSE (Dp)} < \text{RMSE (Dpe)}$$

$$\text{RMSE (Dqe)} \leq \text{RMSE (Dq)}$$

when N = {3, 4, 5}, M = {8, 12, 16}, NTWP = {8, 10, 12}, and NTOIL = {8, 10, 12}. A direct implication of these findings is that the first version of the model always provides the smallest sum of the root mean squared errors. Like any empirical findings, these results are dependent upon the economic data that are used and the particular model that is estimated.

After selecting the first version of the model, where Dy and Dp are estimated directly, I compared the sums of the root mean squared errors for different values of NTWP and NTOIL. In every regression this sum was lowest when NTWP was equal to 10, instead of 8 or 12. In 78 of the 81 regressions the sum of the root mean squared errors was lowest when NTOIL was equal to 10, instead of 8 or 12. In the three other regressions, the model had a slightly better fit when NTOIL was equal to 12. Given these patterns, I concluded that ten quarters, or two and a half years, was an appropriate length of time to apply the dummy variables for Nixon's wage and price controls and for the formation of the OPEC cartel.

By deciding to set the value of NTWP and NTOIL to equal 10, I had narrowed down my search to nine parameter combinations in the first set of regressions. The sums of the root mean squared errors of these nine regressions are presented in the following table:

TABLE 1: $\overline{\text{RMSE}}$ (I) IN SAMPLE PERIOD

N/M	8	12	16
3	1.605	1.521	1.524
4	1.619	1.508	1.504
5	1.572	1.506	1.501

Judging only by this criterion, the best fitting model within the sample period would be the one with fifth-order difference equations that are functions of a 16 quarter, or 4 year, average of the rates of growth of M-1 and Federal Government purchases.

In order to compare these models under alternative criteria, I next examined the set of summary statistics for the forecast period. This set includes the three root mean squared errors in the forecast period and the sum of these errors. These statistics, for the first version of the model, are RMSE ($D\hat{y}$), RMSE ($D\hat{p}$), RMSE ($D\hat{q}$), and RMSE ($\hat{I}$), respectively. The sums of the root mean squared errors of these nine forecasts are presented in the following table.

TABLE 2: $\overline{\text{RMSE}}$ ($\hat{I}$) IN FORECAST PERIOD

N/M	8	12	16
3	2.423	2.492	2.465
4	2.415	2.487	2.453
5	2.471	2.542	2.514

Notice that, among these nine models, some of the parameter combinations that provide the lowest sums of the root mean squared errors within the sample period have the highest sums in the forecast period. In fact, the correlation coefficient between these two sets of root mean squared errors is negative (-.78)!

Another set of summary statistics includes the accumulative errors over the eight quarter forecast. For example, between the first quarter of 1977 and the fourth quarter of 1978, the GNP deflator increased by 13.860 percent. Over this time period, a model that predicts this deflator would have increased by 14.988 percent would have an accumulative eight quarter error of -1.128 percentage points. Table 3 and 4 display the accumulative forecast errors for the rates of growth of nominal output and the price deflator during the two year dynamic simulation.

TABLE 3: ACCUMULATIVE ERROR FOR $D\hat{y}$

N/M	8	12	16
3	2.844	3.626	4.131
4	2.774	3.617	4.046
5	3.134	3.750	4.299

TABLE 4: ACCUMULATIVE ERROR FOR $D\hat{p}$

N/M	8	12	16
3	-.334	-1.128	-.811
4	-.366	-1.035	-.632
5	-.261	-1.148	-.716

By using the accounting identify, the accumulative error for the rate of growth of real output can be determined by subtracting the accumulative error for inflation from the accumulative error for the rates of growth of nominal output.

An examination of the coefficients of the GNP deflator regressions reveals an interesting pattern. Five of these estimated difference equations are unstable. In these models an increase in the rate of growth of money causes a temporary increase in the rate of inflation. After a relatively long period of time, the rate of inflation begins to decrease. Eventually these models enter a period of self-sustaining hyperdeflation, following the initial <u>increase</u> in the rate of growth of money. Since the equations for the rate of growth of nominal output are stable, this version of the model predicts that real output eventually will be growing at an ever increasing rate! Needless to say, these unstable models are inconsistent with the assumptions about the long-run neutrality of money.

The "mechanics" of this peculiar instability problem can be seen easily by examining the sums of the coefficients for the lagged values of inflation. Table 5 displays these sums for the nine regressions.

TABLE 5: SUM OF $a_p(i)$'s, i = 1.2,..., N

N/M	8	12	16
3	1.088	.976	.858
4	1.077	1.053	.928
5	1.054	1.101	.967

With one trivial exception, the coefficients for the lagged values of inflation in these regressions are always positive numbers that are less than one. Therefore, whenever their sum is greater than one, the model is unstable. Recall that one of the constraints that is imposed upon these models is that the sum of the coefficients for lagged values of inflation plus the sum of the coefficients for the money variables are equal to one (equation 6). Whenever the model is unstable because the sum of the $a_p(i)$ coefficients is greater than one, the sum of the $b_p(j)$ coefficients must be negative. This is why these unstable models predict an eventual hyperdeflation following an increase in the rate of growth of money.

Even when the inflation equations are stable, the sums of the coefficients for the lagged values of inflation are close to one. These models display high degrees of "inflationary momentum". By this expression, economists mean that an economy takes a relatively long period of time to adjust to a one period disturbance in the rate of inflation. One of the perplexing questions raised by these regressions is why does the inflation equation predict a relatively slow speed of adjustment when the directly estimated equations for the rates of growth of nominal and real output predict a relatively fast speed of adjustment.

In order to see how the three versions of this model fit for alternative values of M, I ran a second set of regressions and calculated some of the dynamic impact multipliers. In this set of regression I estimated third-order difference equations and varied M between 10 and 16. By setting N equal to 3, the end-point and third-degree polynominal constraints on the coefficients of lagged values of the dependent variables are nonbinding. Figure 1 plots some of the summary statistics for these regressions.

The three graphs in the first column plot the root mean squared errors within the sample period. The different values of M are arranged along the horizontal axes. There are two distinct patterns in these graphs. Each version of the model fits about the same within the sample period regardless of the value of M. In other words, the plots of the root mean squared errors are very flat. The second pattern is that the first version of the model consistently provides the best fit within the sample period. Recall that the relevant root mean squared errors for this version of the model are RMSA (Dy), RMSE (Dp), and RMSE (Dqe).

The three graphs in the middle column plot the root mean squared errors in the forecast period. Notice that these errors always are higher in the forecast period than they are in the sample period. Furthermore, none of the three versions of the model consistently provides the best fit within the forecast period.

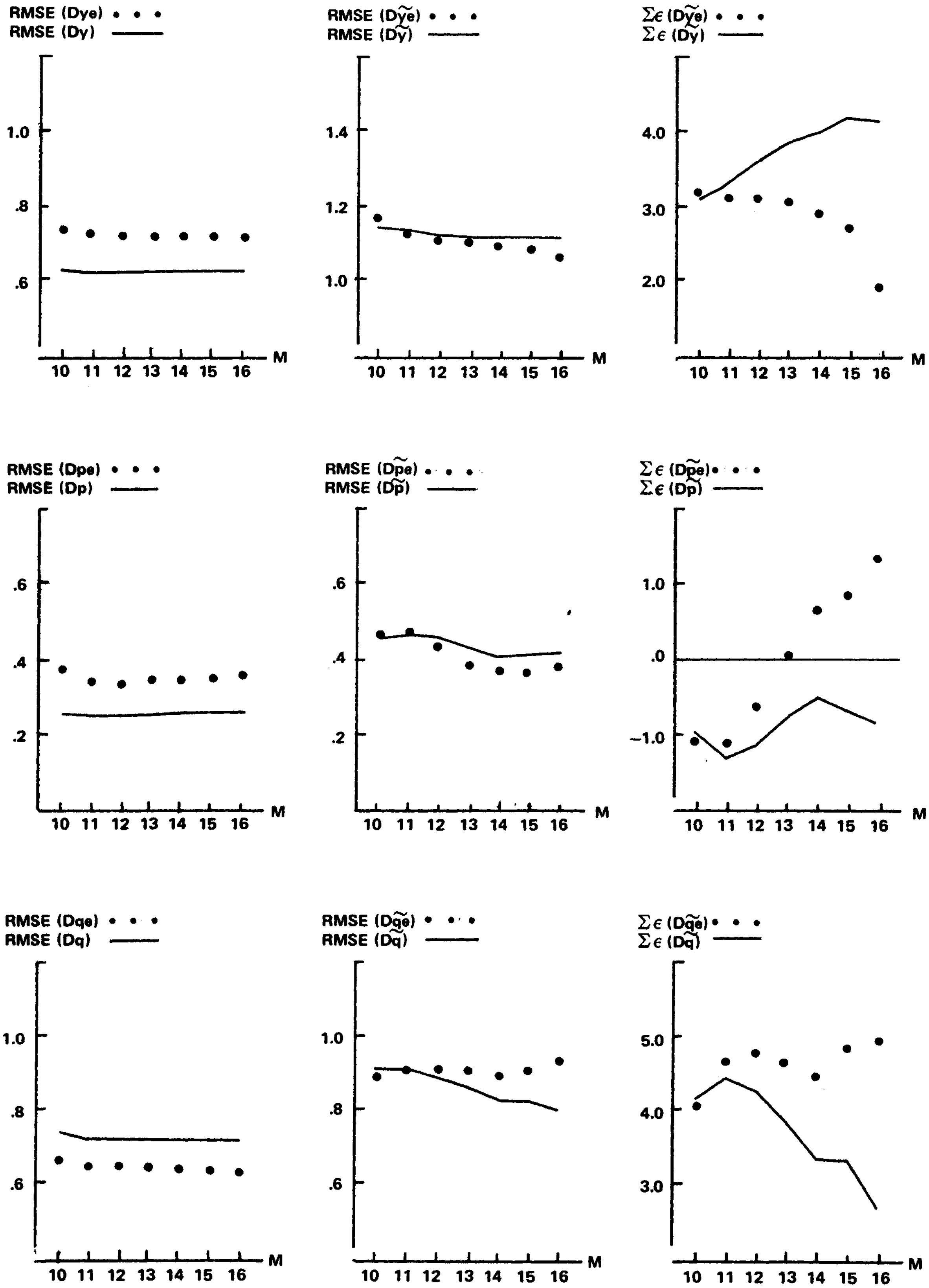

Figure 1: Summary Statistics

The graphs in the third column plot the accumulative errors for each variable in the two year forecast period. Notice that most of the accumulative errors for the GNP deflator estimates are less than one percentage point. These errors are very small when compared with the 13.86 percent increase in the GNP deflator during 1977 and 1978. On the other hand, the accumulative errors for the real GNP estimates are substantial. For the different models they range between 3 and 5 percentage points, compared with a 9.85 percent increase in real GNP between the first quarter of 1977 and the fourth quarter of 1978.

After weighing how well these models fit within the sample and the forecast periods, I subjectively selected 14 to be the value of M. This model uses weighted averages of the current rates of growth of money and of Federal government purchases and their lagged values for the previous three and a half years. The dynamic impact multipliers for a one percentage point change in the rate of growth of money are plotted in Figure 2. The three graphs correspond to the three versions of the model where N, M, NTWP, and NTOIL are equal to 3, 14, 10, and 10, respectively.

Given these parameter combinations, the equation that is estimated directly for the rate of growth of nominal income predicts that Dy will increase rapidly, overshoot, and then converge within five percent of its long-run value by eight quarters. The inflation equation that is estimated directly predicts that Dp will pass through its new equilibrium value in two years, overshoot by about fifteen percent and then very gradually converge onto its long-run value. Therefore, the implicit prediction of the first version of the model is that real output will increase for two years and then gradually return to its equilibrium

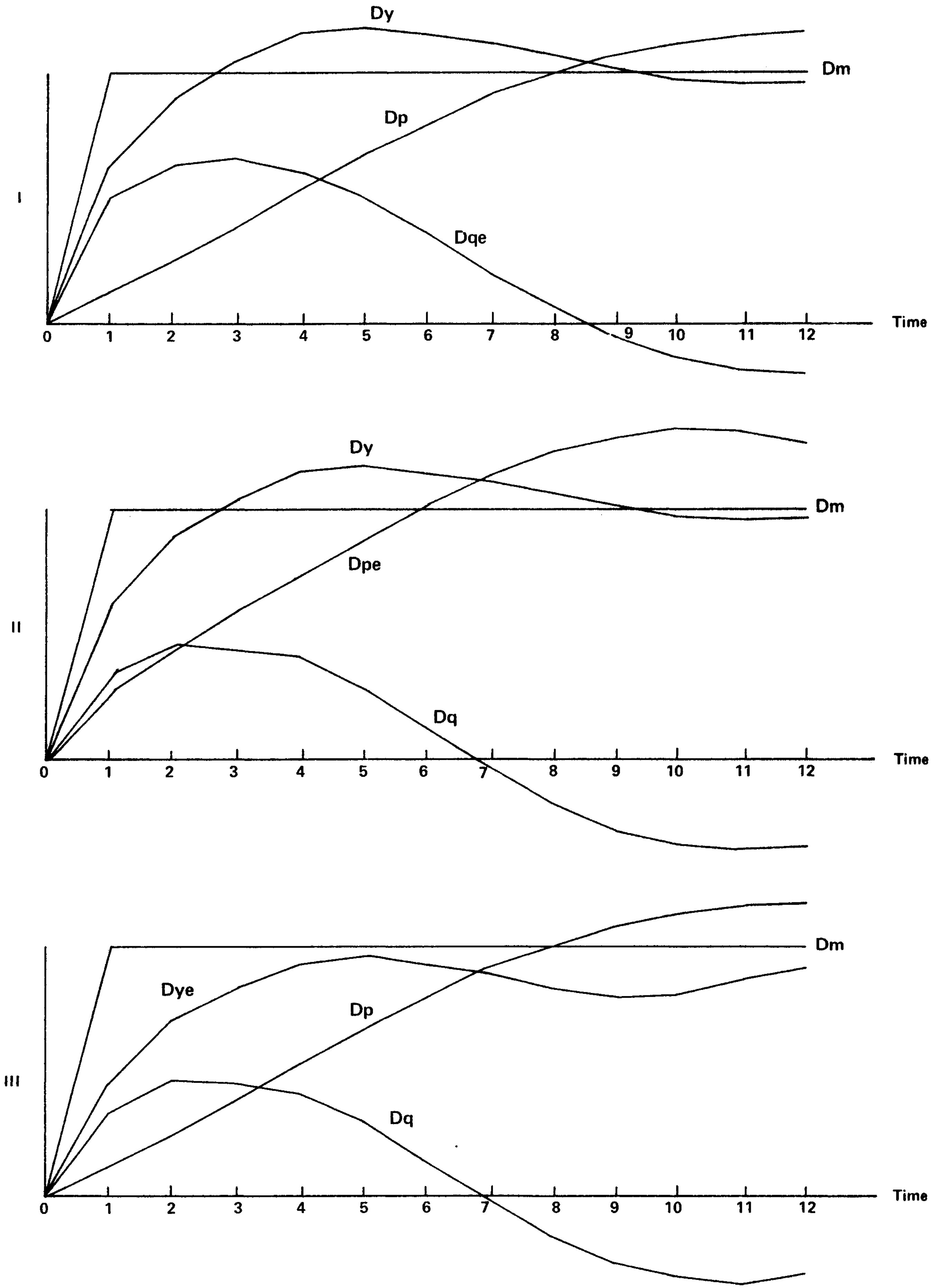

Figure 2: Dynamic Impact Multipliers for a Change in the Rate of Growth of Money

value. By summing the first eight values of Dqe, one obtains 0.886. This is the maximum percentage increase in output that temporarily is caused by a one percentage point increase in the rate of growth of money.

However, when the rate of growth of real output is estimated directly, the regression predicts the Dq will be positive for only six quarters or one and a half years. According to this regression, the maximum percentage increase in real output is only 0.518 following a one percentage increase in the rate of growth of money. In the second version of the model, where inflation is the residual variable, the model predicts that Dpe will increase by its equilibrium value in six quarters, overshoot, and then converge to within five percent of its equilibrium value by four years.

The coefficients that are estimated directly for this particular set of data and combination of parameters are listed in Table 6. Except for the intercept terms, these are some of the coefficients of the various polynomial generating functions of the model. The corresponding t-statistics are listed in parentheses. The extremely low t-statistics for β_3 in the equations for Dy and Dp and for β_4 in the equation of Dq indicate that lower order polynomials can be used to generate the lag weights of the monetary aggregates. In none of these equations is a weighted average of Federal government purchases of goods and services statistically significant at any conventional level. On the other hand, the dummy variables, especially the ones for the quadrupling of crude oil prices by OPEC, are statistically significant in the sample period. These patterns are found in all of the equations in the second set of regressions.

TABLE 6: CONSTRAINED ESTIMATES OF THE LONG-RUN NEUTRALITY OF MONEY
N=3, M=14, NTWP=10, NTOIL=10, (t-statistics in parentheses)

Dependent Variable	Intercept	Coefficients of the Polynomial Constraints: Lagged Dependent Variables α_0	α_2	α_3	Monetary Aggregate β_0	β_3	Federal Government Purchases γ_0	γ_1	γ_2
Dy	1.2537	-0.4231	--	-0.0236	0.6342	-0.0008	-0.0138	-0.0036	0.0004
	(3.60)	(-1.50)		(-0.08)	(1.74)	(-0.003)	(-0.02)	(-0.002)	(0.0003)
Dp	-0.0555	1.7193	--	-0.1628	0.1220	-0.000008	0.0256	-0.0151	0.0021
	(-0.46)	(1.26)		(-1.22)	(1.36)	(-0.00008)	(0.09)	(-0.02)	(0.004)
Dq	0.9656	0.4608	-0.9966	0.5626	0.3372	$\beta_4=3\text{x}10^{-6}$	-0.0296	0.0075	-0.0008
	(1.81)	(0.14)	(-0.13)	(0.13)	(2.41)	(0.00002)	(-0.04)	(0.004)	(-0.001)

Dependent Variable	R^2	Coefficients of the Polynomial Dummy Variables: Wage-Price Controls δ_0	δ_2	δ_3	OPEC Cartel δ_0	δ_2	δ_3
Dy	.518	-1.5294	-0.1323	0.0059	0.4249	0.4247	-0.0029
		(-2.03)	(-0.91)	(0.64)	(0.61)	(2.98)	(3.27)
Dp	.837	-0.5160	-0.0855	-0.0067	0.2483	-0.1516	0.0010
		(-1.73)	(-1.45)	(-1.78)	(0.86)	(-2.55)	(2.71)
Dq	.537	-1.1428	-0.2863	0.0156	-0.1730	0.4885	-0.0311
		(-1.11)	(-1.70)	(1.45)	(-0.21)	(2.95)	(-3.02)

In conclusion, let us review the path followed in this paper. The main purpose of this paper is to translate the assumptions about the long-run neutrality of money into mathematical constraints that then can be imposed when estimating a small macroeconomic model. For a clear verbal presentation of this theory, I used Irving Fisher's description of the quantity theory of money. Next, I specified that the mathematical form of the models to be estimated would consist of linear ordinary difference equations. For this type of model, the assumptions about the long-run neutrality of money correspond to certain linear restrictions on the coefficients of each equation. In the appendixes, these restrictions were derived and the appropriate transformations of the data were presented.

The second part of this paper uses M-1 and GNP quarterly data to estimate three versions of the model under a variety of parameter combinations. The different parameters tried include the order of the difference equations, the number of lagged values of the monetary aggregate, and the length of time to apply dummy variables for Nixon's wage-price controls and the quadrupling of crude oil prices by OPEC.

In order to compare these alternative models after they have been estimated, I calculated two sets of summary statistics. These two sets include measures of fit within the sample period and within a two-year forecast period. This latter set of statistics was used in selecting the "best" model, since many parameter combinations provide about the same degree of fit within the sample period. On the other hand, some of the estimated models that did not fit relatively well within the sample period did predict relatively well within the forecast period.

Furthermore, some of the better fitting models within both of these periods contain an unstable inflation equation, where an increase in the rate of growth of money ultimately causes a self-generating hypderdeflation. After subjectively weighing these summary statistics and stability conditions, I chose a single set of parameters and calculated the dynamic impact multipliers associated with a change in the rate of money.

The major contribution of this paper is that it provides economists with a simple method of imposing the assumptions about the long-run neutrality of money within a system of linear difference equations. Furthermore, these models can be estimated with a large variety of economic time series for money, prices, nominal output, and real output. Additional explanatory variables may be added, and these alternative models may be estimated and compared by any method one chooses to use. The final test for any of these models is how well do they predict the future vis-a-vis other existing models.

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APPENDIX A
CONSTRAINTS FOR A NOMINAL SERIES

Assume the following linear transfer function that relates the logarithm of a nominal output or price series y(t) to its lagged values and to the current and lagged values of the logarithm of a monetary aggregate m(t):

$$1)\quad y(t) = \sum_{i=1}^{N} a(i)y(t-i) + \sum_{j=o}^{M} b(j)m(t-j).$$

The assumption that a change in the quantity of money causes a proportional change in the level of the nominal variable, in the long run, implies that

$$2)\quad y(t) + \Delta = \sum_{i=1}^{N} a(i)(y(t-i) + \Delta) + \sum_{j=o}^{M} b(j)(m(t-j) + \Delta).$$

Subtract equation 1 from equation 2 and divide by Δ to obtain the first constraint:

$$3)\quad 1 = \sum_{i=1}^{N} a(i) + \sum_{j=o}^{M} b(j)$$

Without loss of generality we can normalize the long-run values of y(t) and m(t) so that they equal one. The assumption that a change in the rate of growth of money (δ) causes an equal change in the long-run rate of inflation implies that the values of y(t-i) and m(t-j) are equal to $1-i\delta$ and $1-j\delta$, respectively. In terms of our linear transfer function this means that

$$4)\quad 1 = \sum_{i=1}^{N} a(i)(1-i\delta) + \sum_{j=o}^{M} b(j)(1-j\delta)$$

Subtract equation 2 from equation 4 and divide by δ to obtain the second constraint:

$$5)\quad 0 = \sum_{i=1}^{N} ia(i) + \sum_{j=o}^{M} jb(j)$$

Notice that this constraint means that at least one of the coefficients of the transfer function must be negative.

It is interesting to notice the different roles that these two constraints play when the model is expressed as a rate of change equation. Let D represent a difference operator that is defined such that

$$Dx(t) \equiv x(t) - x(t-1).$$

The transfer function in terms of rates of changes can be written as

$$Dy(t) = \sum_{i=1}^{N} a(i)Dy(t-i) + \sum_{j=o}^{M} b(j)Dm(t-j)$$

In this model the constraint that the sum of the coefficients equals one (equation 3) imposes the assumption that a change in the rate of growth of money causes, in the long run, an equal change in the rate of inflation. The additional constraint that the weighted sum of coefficients equals zero (equation 5) means that a change in the quantity of money will cause a proportional change in the level of the nominal variable.

APPENDIX B

The dynamic relationship between the logarithms of a money variable m(t) and a nominal variable y(t) is assumed to be represented by the following linear transfer function, where the lag weights are generated by third-degree polynomials.

1) $$y(t) = \sum_{i=1}^{N} a(i)y(t-i) + \sum_{j=o}^{M} b(j)m(t-j)$$

2) $$a(i) = \alpha_0 + \alpha_1 i + \alpha_2 i^2 + \alpha_3 i^3$$

3) $$b(j) = \beta_0 + \beta_1 j + \beta_2 j^2 + \beta_3 j^3$$

There are eight basic coefficients in this model - α_0, α_1, α_2, α_3, β_0, β_1, β_2, and β_3. Four of these coefficients can be determined by constraining the end points -- a(N+1) and b(M+1) -- to equal zero and by making the two assumptions about the long-run neutrality of money (see Appendix A). These constraints are summarized by the following four equations:

4) $$a(N+1) = 0$$

5) $$b(M+1) = 0$$

6) $$\sum_{i=1}^{N} a(i) + \sum_{j=o}^{M} b(j) = 1$$

7) $$\sum_{i=1}^{N} ia(i) + \sum_{j=o}^{M} jb(j) = 0$$

These equations can be substituted into the model to determine a linear transformation of the data that will impose the four constraints. The appendix will outline the transformation that can be used when α_2, α_3, β_0, and β_3 are estimated directly. The remaining coefficients can then be determined from equations 2-6.

Use equations 2-4 to evaluate the two end-point constraints and rearrange to obtain the following expressions for α_0 and β_1:

8) $\alpha_0 = -(N+1)\alpha_1 - (N+1)^2\alpha_2 - (N+1)^3\alpha_3$

9) $\beta_1 = -\beta_0/(M+1) - (M+1)\beta_2 - (M+1)^2\beta_3$

Next expand the two economic constraints by substituting in the polynomial generating function to obtain

10) $N\alpha_0 + \alpha_1\Sigma i + \alpha_2\Sigma i^2 + \alpha_3\Sigma i^3 + (M+1)\beta_0 + \beta_1\Sigma j$

$+ \beta_2\Sigma j^2 + \beta_3\Sigma j^3 = 1$

11) $\alpha_0\Sigma i + \alpha_1\Sigma i^2 + \alpha_2\Sigma i^3 + \alpha_3\Sigma i^4 + \beta_0\Sigma j + \beta_1\Sigma j^2$

$+ \beta_2\Sigma j^3 + \beta_3\Sigma j^4 = 0$

The ranges of summation have been omitted with the understanding that the terms with i's are summed from 1 to N and terms with j's are summed from 0 to M.

Use equations 8 and 9 to factor out α_0 and β_1 in equations 10 and 11 and rearrange into matrix format.

$$\begin{bmatrix} \Sigma i - N(N+1) & \Sigma j^2 - (M+1)\Sigma j \\ \\ \Sigma i^2 - (N+1)\Sigma i & \Sigma j^3 - (M+1)\Sigma j^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \\ \beta_2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & N(N+1)^2-\Sigma i^2 & N(N+1)^3-\Sigma i^3 & \Sigma j/(M+1)-(M+1) & (M+1)^2\Sigma j-\Sigma j^3 \\ 0 & (N+1)^2\Sigma i-\Sigma i^3 & (N+1)^3\Sigma i-\Sigma i^4 & \Sigma j^2/(M+1) - \Sigma j & (M+1)^2\Sigma j-\Sigma j^4 \end{bmatrix} \begin{bmatrix} 1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \\ \beta_3 \end{bmatrix}$$

or

$$Au = Bv,$$

where $u = (\alpha_1 \;\; \beta_2)'$ and $v = (1 \; \alpha_2 \; \alpha_3 \; \beta_0 \; \beta_3)'$.

The next several steps require a fair amount of tedius algebra that can be described briefly as follows. Recall (or look up in any mathematical handbook) that the sums of powers os positive integers are given by the following equations:

$$\Sigma k = K(K+1)/2$$

$$\Sigma k^2 = K(K+1)(2K+1)/6$$

$$\Sigma k^3 = K^2(K+1)^2/4$$

$$\Sigma k^4 = K(K+1)(2K+1)(3K^2+3K-1)/30,$$

where the summations are from 0 or 1 to K. Use these equations to evaluate the summation terms in matrix A and invert this matrix to get

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} -M(M+1)^2(M+2)/12 & -M(M+1)(M+2)/6 \\ \\ -N(N+1)(N+2)/6 & -N(N+1)/2 \end{bmatrix}$$

where

$$|A| = M(M+1)(M+2)N(N+1)(3M-2N-1)/72.$$

In order to use this transformation it is necessary that 3M-2N-1 does not equal zero. This special case can be solved by selecting a different set of four coefficients that are to be estimated directly and by calculating the corresponding transformation of the data.

Evaluate the summation terms in matrix B and then premultiply by A^{-1} to obtain the following equations:

$$12)\quad \alpha_1 = c_0 + c_1\alpha_2 + c_2\alpha_3 + c_3\beta_0 + c_4\beta_3$$

$$13)\quad \beta_2 = d_0 + d_1\alpha_2 + d_2\alpha_3 + d_3\beta_0 + d_4\beta_3$$

where

$$c_0 = \frac{-6(M+1)}{N(N+1)(3M-2N-1)}$$

$$c_1 = \frac{-M(4N+5)+3N^2+5N+1}{3M-2N-1}$$

$$c_2 = \frac{(3N+4)(-15M(N+1)+12N^2+17N+1)}{10(3M-2N-1)}$$

$$c_3 = \frac{(M+2)(M+3)}{N(N+1)(3M-2N-1)}$$

$$c_4 = \frac{(M-1)M(M+1)(M+2)(M+3)}{10N(N+1)(3M-2N-1)}$$

$$d_0 = \frac{12(N+2)}{M(M+1)(M+2)(3M-2N-1)}$$

$$d_1 = \frac{-(N-1)N(N+1)(N+2)}{M(M+1)(M+2)(3M-2N-1)}$$

$$d_2 = \frac{-3(N-1)N(N+1)(N+2)(3N+4)}{5M(M+1)(M+2)(3M-2N-1)}$$

$$d_3 = \frac{6(M-N-2)}{M(M+1)(3M-2N-1)}$$

$$d_4 = \frac{3(5N(M+1)-2(4M^2+3M-2))}{5(3M-2N-1)}.$$

The accuracy of the preceding steps can be checked by first assuming arbitrary values for N, M, α_2, α_3, β_0, and $\beta 1$. By using equations 8, 9, 12, and 13 it is then possible to determine the corresponding values of α_0, α_1, β_1, and β_2. These coefficients' values can then be plugged into equations 10 and 11 to see if these equations sum to 1 and 0, respectively.

Equations 8, 9, 12, and 13 can be substituted into polynomial constants to yield

14) $a(i) = \phi_0(i) + \phi_1(i)\alpha_2 + \phi_2(i)\alpha_3 + \phi_3(i)\beta_0 + \phi_4(i)\beta_3$

15) $b(j) = \Psi_0(j) + \Psi_1(j)\alpha_2 + \Psi_2(j)\alpha_3 + \Psi_3(j)\beta_0 + \Psi_4(j)\beta_3$

where

$$\phi_0(i) = -(N+1)c_0 + c_0 i$$

$$\phi_1(i) = -(N+1)(c_1+N+1) + c_1 i + i^2$$

$$\phi_2(i) = -(N+1)(c_2+(N+1)^2) + c_2 i + i^3$$

$$\phi_3(i) = -(N+1)c_3 + c_3 i$$

$$\phi_4(i) = -(N+1)c_4 + c_4 i$$

$$\Psi_0(j) = -(M+1)d_0 j + d_0 j^2$$

$$\Psi_1(j) = -(M+1)d_1 j + d_1 j^2$$

$$\Psi_2(j) = -(M+1)d_2 j + d_2 j^2$$

$$\Psi_3(j) = 1-((M+1)d_3 + 1/(M+1))j + d_3 j^2$$

$$\Psi_4(j) = -(M+1)(d_4+M+1)j + d_4 j^2 + j^3$$

Equations 13 and 14 can then be substituted into the original transfer function to determine the corresponding linear transformation of the data. The constrained model is given by

$$z(t) = \alpha_2 x_1(t) + \alpha_3 x_2(t) + \beta_0 x_3(t) + \beta_3 x_4(t)$$

where

$$z(t) = y(t) - \Sigma\phi_0(i)y(t-i) - \Sigma\Psi_0(j)m(t-j)$$

$$x_k(t) = \Sigma\phi_k(i)y(t-i) + \Sigma\Psi_k(j)m(t-j), \qquad k = 1,2,3,4$$

After transforming the data, this model can be estimated by any standard regression technique. The estimated coefficients will determine how quickly the nominal variable responds to a change in the quantity of money. By calculating the dynamic input multipliers of this model, it is possible to check the accuracy of this transformation by seeing if the long-run multipliers correspond to the two economic constraints.

APPENDIX C
CONSTRAINTS FOR A REAL SERIES

Assume that the current level of the logarithm of real output q(t) is a function of its lagged values plus a weighted average of the logarithms of a money variable m(t).

$$1)\quad q(t) = f(q(t-i)) + \sum_{j=0}^{M} b(j)m(t-j)$$

The assumptions that the level of the money variable and its rate of growth do not have a long-run impact of the level of real output imply that

$$2)\quad \Sigma b(j) = 0$$

$$3)\quad \Sigma j b(j) = 0$$

where terms with j's are summed from 0 to M. When the lag coefficients are generated by a polynomial, these two constraints can be used to eliminate two of the coefficients that must be estimated.

Suppose the lag weights are determined by the following fourth-degree polynomial

$$4)\quad b(j) = \beta_0 + \beta_1 j + \beta_2 j^2 + \beta_3 j^3 + \beta_4 j^4$$

where the end point b(M+1) is constrained to equal zero.

$$5)\quad b(M+1) = 0$$

Equations 2, 3 and 5 can be used to factor out three of the coefficients. Suppose β_1, β_2, and β_3 are factored out by substituting these equations into equation 4. In matrix format we can derive the following system of equations:

$$\begin{bmatrix} M+1 & (M+1)^2 & (M+1)^3 \\ \Sigma j & \Sigma j^2 & \Sigma j^3 \\ \Sigma j^2 & \Sigma j^3 & \Sigma j^4 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} -1 & -(M+1)^4 \\ -(M+1) & -\Sigma j^4 \\ -\Sigma j & -\Sigma j^5 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_4 \end{bmatrix}$$

or

$$Au = Bv$$

where $u = (\beta_1 \quad \beta_2 \quad \beta_3)'$ and $v = (\beta_0 \quad \beta_4)'$. The summation terms can be evaluated by using the equations for sum of powers of positive integers (see Appendix B). The equation for the sum of fifth powers is

$$\Sigma k^5 = K^2(K+1)^2(2K^2+2K-1)/12,$$

where k is summed from 0 to K.

Invert matrix A and then premultiply matrix B by A^{-1} to obtain the following equations

$$\beta_1 = \phi_1\beta_0 + \Psi_1\beta_4$$

$$\beta_2 = \phi_2\beta_0 + \Psi_2\beta_4$$

$$\beta_3 = \phi_3\beta_0 + \Psi_3\beta_4$$

where

$$\phi_1 = -(3M+1)(3M+2)/((M-1)M(M+1))$$

$$\phi_2 = 6(3M+2)/((M-1)M(M+1))$$

$$\phi_3 = -10/((M-1)M(M+1))$$

$$\Psi_1 = -(M+1)(M^2+2M+2)/5$$

$$\Psi_2 = (6M^2+12M+7)/5$$

$$\Psi_3 = -2(M+1)$$

These constraints can then be used to transform the money data into two sets that can be used to estimate the model.

$$\beta_0 x_1(t) + \beta_4 x_2(t) = \Sigma b(j)m(t-j)$$

where

$$x_1(t) = \Sigma m(t-j)(1 + \phi_1 j + \phi_2 j^2 + \phi_3 j^3)$$

$$x_2(t) = \Sigma m(t-j)(\Psi_1 j + \Psi_2 j^2 + \Psi_3 j^3 + j^4)$$

The same transformation can be used when the model is estimated using the first difference or rates of growth of the real input and money variables. Notice that it is independent of the function that includes the lagged values of the real output variable.