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THE PERSISTENCE OF AGGREGATE EMPLOYMENT AND THE NEUTRALITY OF MONEY

Thomas J. Sargent

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THE PERSISTENCE OF AGGREGATE EMPLOYMENT AND THE NEUTRALITY OF MONEY

By

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Introduction

Recently it has widely been claimed[1] that equilibrium models cannot account for the observed high serial correlation of aggregates such as employment and real GNP.[2] If valid, that claim would constitute a decisive objection against equilibrium theories of the business cycle, since any successful theory should be capable of explaining the serial correlation properties of these aggregates. Indeed, the serial correlations of aggregates such as real GNP and employment, together with their cross correlations, are the defining characteristics of the "business cycle." In fact, however, it turns out to be straightforward to generate equilibrium models that do imply that output and employment are serially correlated. Such models are readily formed by combining dynamic demand and supply curves for factors of production, thereby obtaining dynamic stochastic versions of the "classical" macroeconomic model. In the context of these equilibrium models, there is no problem in explaining why aggregate employment and output are serially correlated, but there is a problem in explaining the price-output and money-output correlations that have persisted over many business cycles. It was toward explaining these price-output and money-output correlations, while largely abstracting from the task of explaining observed patterns of serial correlation, that Lucas's initial work on aggregate supply was directed.

The theory of aggregate supply that was advanced by Lucas [1973] provided no endogenous explanation for why aggregate-demand-induced movements in output and employment might persist, i.e. might be serially correlated. True, the theory accounts for such movements through the presence of the term λy_{t-1} in the aggregate supply schedule, but it does not provide an explanation for the presence of that term.[3] The main purpose of this essay is to describe an endogenous theory of persistence in employment, all the elements of which are to be found in the works of Robert Lucas [1967, 1973] and Leonard Rapping [1969]. In effect, the construction here provides a rationalization for the inclusion of terms like λy_{t-1} on the right side of the Phillips curves estimated by Lucas [1973].

Lucas [1975] has already proposed a model of aggregate-demand-induced persistence in output which relies on a different mechanism than that proposed here. The key features of Lucas's model are, first, the hypothesis that nominal aggregate demand is unobservable, an hypothesis which permits agents' forecasting errors to be serially correlated even under the hypothesis that their expectations are rational; and, second, the presence of capital, which gives rise to a variety of accelerator mechanism. By way of contrast, the mechanism described here does not resort to any device, like Lucas's hypothesis that aggregate demand is unobservable, to render rational agents' forecasting errors serially correlated. In the present model, agents' forecast errors are themselves serially uncorrelated, but nevertheless give rise to serially correlated movements in employment. The mechanism that produces this situation depends on the interaction of dynamic supply and demand schedules for employment. Equilibria in this model have the characteristic that agents' forecast errors induce movements in the real wage which in turn cause persistent movements in employment as agents reallocate their employment over time.

As may already be evident even from this very brief description, the model is really a version of Lucas and Rapping's, one with rational expectations imposed. As it happens, when one imposes rational expectations on Lucas and Rapping's model with all trades occurring in a single market, as Lucas and Rapping assumed, one obtains a "real" theory of the business cycle in the sense that stochastic processes for employment and real wages are predetermined with respect to stochastic processes for the money supply and price level. One purpose of this paper is to highlight the "real" character of their model as well as the role of Lucas's disparate-information model of the Phillips curve in restoring to the system some interaction between the "real" and purely nominal parts.

An essential assumption of the present model is that there are costs associated with adjusting the labor force quickly. Some of the phenomena that this widely-made assumption is designed to explain are summarized by Walter Oi [1962] and Sherwin Rosen [1968]. Prominent among these phenomena are the cyclical behavior of average hours and of employment across different skill classes, employment in lower skills being more variable over the cycle. I have followed Holt, Modigliani, Muth and Simon [1960] in using quadratic functions to model the firm's costs. This step has been taken in order to deliver linear decision rules, without which the calculations in section 3 would become even more intractable.

In explaining serially correlated employment, this theory does assign an important role to real wage movements, a role to which many macroeconomists might object. Following the Dunlop-Tarshis-Keynes controversy, it has widely been believed that the interactions between employment and real wages that exist in the aggregate data are very weak. However, a recent empirical study by Neftci [1976] indicates that rather substantial real wage-employment interactions are to be found in the aggregate data. In any event, the presence of such interactions is an important ingredient in the present model. Partly, this is a consequence of my having simplified things drastically by suppressing the real interest rate from the labor supply schedule, a simplification that permits me to ignore capital accumulation. However, this simplification is purchased at the cost of suppressing another mechanism generating persistence, which emerges from the mutual interaction of capital, real interest rates, and labor supply. This richer mechanism is briefly described in a concluding section. As the reader will see, however, even the computations involved with the simplified model are tedious enough.

In section 2 a simplified, aggregative version of the model is described in which information discrepancies are suppressed. Section 3 then introduces information discrepancies and describes the equilibrium in a particular market. Section 4 then describes the behavior of the economy-wide aggregates in the model. Section 5 discusses alternative mechanisms for generating persistence of employment in response to shocks in aggregate demand.

2. An Aggregative Model

I begin by describing a simple version of a model in which all trades occur in a single market. Labor supply is governed by the schedule

$$(1) \qquad n_t = \sum_{j=0}^{\infty} h_j E[(w_{t+j} - p_{t+j}) | \Omega_t] + v(L)u_{1t}$$

where n_t is the level of employment, w_t is the natural logarithm of the money wage, p_t is the log of the price level, u_{1t} is a serially uncorrelated random process with mean zero and variance σ_1^2, and $v(L) = \sum_{j=0}^{\infty} v_j L^j$, $\sum v_j^2 < \infty$, where L is the lag operator. Here $Ex|\Omega_t$ is the linear least squares projection of x on an information set Ω_t, which I will assume includes at least current and lagged values of w_t, p_t and n_t. Equation (1) is an infinite horizon version of Lucas and Rapping's supply schedule of employment with the real interest rate arguments suppressed. Suppressing the real interest rate[4] in (1) simplifies things by permitting building a smaller model, one that ignores capital accumulation.[5] Lucas and Rapping argue that it is plausible that in (1) h_j is positive for small j but negative for large j. This pattern indicates a larger response to changes in the real wage that are viewed as temporary than to those that are viewed as permanent. If $\sum_{j=0}^{\infty} h_j = 0$, the labor supply schedule is vertical with respect to what are perceived as permanent movements in the real wage.

I posit that firms' demand schedule for employment is

$$(2) \qquad n_t - \lambda_1 n_{t-1} = -\lambda_1 \sum_{i=0}^{\infty} \left(\frac{1}{\lambda_2}\right)^i E_t\left(\frac{1}{d}(w_{t+i} - p_{t+i}) + u_{2t+i}\right)$$

where $E_t\{\cdot\} \equiv E[\{\cdot\}|\Omega_t]$, where u_{2t} is a serially uncorrelated random process with mean zero and variance σ_2^2 ; d is a nonnegative parameter that measures adjustment costs; and $\lambda_2>1>\lambda_1>0$. The parameters λ_1 and λ_2 and these restrictions on them will be discussed below. I assume that $Eu_{1t}u_{2s} = 0$ for $t\neq s$. Equation (2) is an approximation to the demand schedule for employment of a firm that maximizes expected present value in the face of quadratic costs of adjusting its labor force. Thus, let W_t and P_t be the levels of the firm's wage and price and time t. Suppose the firm's output is $a_0n_t + a_1n_t^2$, $a_0 > 0$, $a_1 \leq 0$. The firm bears costs of adjusting its labor force of $P_t \frac{d}{2}(n_t - n_{t-1})^2$ where d>0. The firm maximizes its expected real present value

$$v_t = \tilde{E}_t \sum_{j=0}^{\infty} \beta_{t,j}\{\frac{P_{t+j}}{P_t}(a_0n_{t+j} + a_1n_{t+j}^2) - \frac{W_{t+j}}{P_t} n_{t+j}$$

$$- \frac{P_{t+j}}{P_t}\frac{d}{2}(n_{t+j} - n_{t+j-1})^2\}$$

where $\{\beta_{t,j}\}_{j=0}^{\infty}$ is a sequence of nominal discount factors with $\beta_{t,0}=1$, $\tilde{E}$ is the mathematical expectation operator, and $\tilde{E}_t\{\cdot\} \equiv \tilde{E}[\{\cdot\}|\Omega_t]$. The marginal condition associated with a maximum of v_t with respect to n_t is

$$(a_0 + 2a_1n_t) - \frac{W_t}{P_t} - dn_t + dn_{t-1}$$

$$+ \tilde{E}_t\{\frac{P_{t+1}}{P_t}\beta_{t,1}dn_{t+1} - \frac{P_{t+1}}{P_t}\beta_{t,1}dn_t\} = 0$$

Rearranging gives

$$d\beta_{t,1}\tilde{E}_t \frac{P_{t+1}}{P_t} n_{t+1} + (2a_1-d-d\beta_{t,1}\tilde{E}_t \frac{P_{t+1}}{P_t})n_t + dn_{t-1} = \frac{W_t}{P_t} - a_0$$

Now impose the approximations

$$\beta_{t,j}\tilde{E}_t\frac{P_{t+j}}{P_t} \simeq b^j \qquad\qquad j = 0,1,\ldots;\ \text{all } t$$

$$\beta_{t,1}\tilde{E}_t\frac{P_{t+1}}{P_t}\, n_{t+1} \simeq bE_t n_{t+1}$$

where b is a constant real discount factor, approximations which amount to assuming a constant real rate of interest. Substituting these approximations into the preceding equation gives

$$db\tilde{E}_t n_{t+1} + (2a_1 - d(1+b))n_t + dn_{t-1} = \frac{W_t}{P_t} - a_0 \quad .$$

I now assume that units are chosen to make W_t/P_t equal unity on average. Then use the approximation[6]

$$\frac{W_t}{P_t} \simeq 1 + \log\frac{W_t}{P_t} = 1 + w_t - p_t \ ,$$

which comes from the first two terms of a Taylor's series expansion of $\exp(\log \frac{W}{P})$ about $\frac{W}{P} = 1$. Substituting this approximation in the above equation, replacing mathematical expectations with linear least squares projections, adding a random term du_{2t}, and dropping the constant gives

$$(3) \qquad bdE_t n_{t+1} + (2a_1 - d(1+b))n_t + dn_{t-1} = (w_t - p_t) + du_{2t}$$

The stochastic difference equation (3) is thus a linear approximation to the "Euler equation" that emerges from the firm's present value maximization problem.

The demand schedule or decision rule (2) is the unique solution to (3) that satisfies the pertinent boundary condition (transversality condition). To indicate how the demand schedule is derived from (3), let me write (3) in the form

$$bE_t n_{t+1} + \phi n_t + n_{t-1} = \frac{1}{d}(w_t - p_t) + u_{2t} \tag{4}$$

where $\phi = (\frac{2a_1}{d} - (1+b))$. Since $a \leq 0$, $d>0$, $\phi<-(1+b)<0$. It is instructive to solve (4) for $E_t n_{t+1}$. Project both sides of (4) on Ω_{t-1} to get

$$bE_{t-1}n_{t+1} + \phi E_{t-1}n_t + E_{t-1}n_{t-1} = \frac{1}{d}E_{t-1}(w_t - p_t) + E_{t-1}u_{2t} \quad ,$$

or

$$\{bB^{-1} + \phi + B\}E_{t-1}n_t = \frac{1}{d}E_{t-1}(w_t - p_t) + E_{t-1}u_{2t}$$

where $B^{-j}E_{t-1}n_t \equiv E_{t-1}n_{t+j}$ for all integer j. Operating on both sides of the preceding equation with B gives[7]

$$(b + \phi B + B^2)E_{t-1}n_t = \frac{1}{d}\, E_{t-1}(w_{t-1} - p_{t-1}) + E_{t-1}u_{2t-1}$$

or

(5) $$b(1-\lambda_1 B)(1-\lambda_2 B)E_{t-1}n_t = \frac{1}{d}\, E_{t-1}(w_{t-1} - p_{t-1}) + E_{t-1}u_{2t-1}$$

where λ_1 and λ_2 are the roots of the characteristic polynomial

$$\lambda^2 + \frac{\phi}{b}\lambda + \frac{1}{b} = 0$$

That is, we want to find λ_1 and λ_2 such that

$$b(1 + \frac{\phi}{b}B + \frac{1}{b}B^2) = b(1-\lambda_1 B)(1-\lambda_2 B) \quad .$$

Equating powers of B gives

$$-\frac{\phi}{b} = (\lambda_1 + \lambda_2)$$

$$\frac{1}{b} = \lambda_1\lambda_2$$

or $$\frac{1}{\lambda_1 b} = \lambda_2$$

Thus we have that λ_1 must satisfy

$$-\frac{\phi}{b} = (\lambda_1 + \frac{1}{\lambda_1 b}) \qquad \text{or} \qquad -\phi = (\lambda_1 b + \frac{1}{\lambda_1})$$

Figure 1 depicts the determination of λ_1 and λ_2. The function $(\lambda b + \frac{1}{\lambda})$ attains a minimum at $\lambda = \sqrt{1/b}$, attaining a value of $2\sqrt{b}$ there. For $0<b\leq 1$, we have that

$$(1+b) \geq 2\sqrt{b} \quad ,$$

with equality at b=1, so that even with a=0 (in which case $\phi = -(1+b)$), we have $-\phi \geq 2\sqrt{b}$. This assures that with a<0, the solutions for λ_1 and λ_2 as depicted in Figure 1 are real.

From the figure, it can be seen that with b close enough to unity and ϕ large enough in absolute value, one root will be less than unity and another greater than unity, a situation I will assume to prevail. Without loss of generality, I choose λ_1 to be the root that is less than unity. Notice that for a<0, as the adjustment cost parameter goes to zero, the parameter $-\phi$ goes to $+\infty$, driving λ_1 and $\frac{1}{\lambda_2}$ both toward zero. As will be seen, this produces a situation in which larger costs of adjustment lead to more sluggish adjustments over time of employment in response to the "signals" that the firm receives.

Solving (5) we have

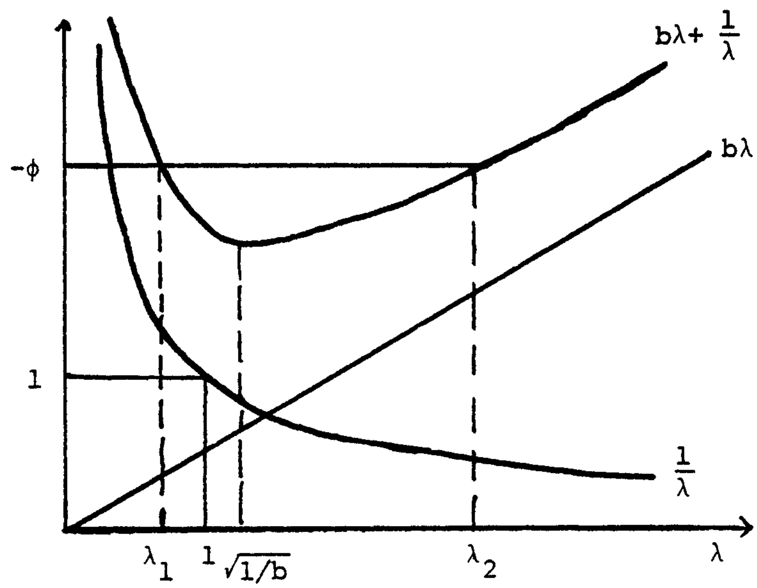
$$E_{t-1}n_t = \left(\frac{1}{1-\lambda_1 B}\right)\left(\frac{1}{1-\lambda_2 B}\right)\frac{1}{b}\left\{\frac{1}{d} E_{t-1}(w_{t-1}-p_{t-1}) + E_{t-1}u_{2t}\right\} \tag{6}$$

Notice that

$$\frac{1}{(1-\lambda_1 B)(1-\lambda_2 B)} = \frac{\lambda_1}{\lambda_1-\lambda_2}\cdot\frac{1}{1-\lambda_1 B} - \frac{\lambda_2}{\lambda_1-\lambda_2}\,\frac{1}{1-\lambda_2 B}$$

Also notice that formally

$$\frac{1}{1-\lambda_2 B} = \frac{-\frac{1}{\lambda_2 B}}{1-\frac{1}{\lambda_2}B^{-1}}$$



$b\lambda + \frac{1}{\lambda}$
$b\lambda$
$-\phi$
1
$\frac{1}{\lambda}$
λ_1
1
$\sqrt{1/b}$
λ_2
λ

Figure 1

If $|\lambda_2|>1$, the "backwards" expansion

$$\frac{1}{1-\lambda_2 B} = \sum_{i=0}^{\infty} \lambda_2^i B^i$$

has coefficients λ_2^i that are not square summable, i.e. $\sum_{i=0}^{\infty} \lambda_2^{2i}$ does not converge. However, the "forward" expansion

$$\frac{1}{1-\lambda_2 B} = \frac{-\left(\frac{1}{\lambda_2}\right) B^{-1}}{1-\left(\frac{1}{\lambda_2}\right) B^{-1}} = -\frac{1}{\lambda_2} B^{-1} \sum_{i=0}^{\infty} \left(\frac{1}{\lambda_2}\right)^i B^{-i}$$

has coefficients $\left(\frac{1}{\lambda_2}\right)^i$ that are square summable. Since by way of imposing the trans versality condition[8] we shall need to insist that all lag distributions are square-summable, it is appropriate to take

$$\frac{1}{(1-\lambda_1 B)(1-\lambda_2 B)} = \frac{\lambda_1}{\lambda_1-\lambda_2} \frac{1}{1-\lambda_1 B} + \frac{\lambda_2}{\lambda_1-\lambda_2} \cdot \frac{\frac{1}{\lambda_2} B^{-1}}{1-\frac{1}{\lambda_2} B^{-1}}$$

$$= \frac{\lambda_1}{\lambda_1-\lambda_2} \frac{1}{1-\lambda_1 B} + \frac{B^{-1}}{\lambda_1-\lambda_2} \cdot \frac{1}{1-\frac{1}{\lambda_2} B^{-1}}$$

Operating on both sides of the above equation with $(1-\lambda_1 B)$ gives

$$\frac{(1-\lambda_1 B)}{(1-\lambda_1 B)(1-\lambda_2 B)} = \frac{\lambda_1}{\lambda_1-\lambda_2} + \frac{(1-\lambda_1 B)\, B^{-1}}{(\lambda_1-\lambda_2)(1-\frac{1}{\lambda_2} B^{-1})}$$

Operating on both sides of (5) with the above equation then gives

$$(6) \qquad (1-\lambda_1 B)E_{t-1}n_t = \frac{1}{b}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right) E_{t-1}\left(\frac{1}{d}(w_{t-1}-p_{t-1}) + u_{2t-1}\right)$$

$$+ \frac{1}{b}\left(\frac{1}{\lambda_1-\lambda_2}\right)\left(\frac{1}{1-\frac{1}{\lambda_2}B^{-1}}\right) E_{t-1}\ (1-\lambda_1 L)\left(\frac{1}{d}(w_t-p_t) + u_{2t}\right)$$

In deriving the above from (5), I have used the property

$$(1-\lambda_1 B)E_{t-1}\chi_t = E_{t-1}\chi_t - \lambda_1\chi_{t-1} = E_{t-1}(\chi_t-\lambda_1\chi_{t-1}) = E_{t-1}(1-\lambda_1 L)\chi_t \quad .$$

The solution (6) for $E_{t-1}n_t$ suggests that the solution for n_t is given by

$$(7) \qquad n_t - \lambda_1 n_{t-1} = \frac{1}{b}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)\left(\frac{1}{d}(w_{t-1}-p_{t-1}) + u_{2t-1}\right)$$

$$+ \frac{1}{b}\left(\frac{1}{\lambda_1-\lambda_2}\right)\left(\frac{1}{1-\frac{1}{\lambda_2}B^{-1}}\right)E_t(1-\lambda_1 L)\left(\frac{1}{d}(w_t-p_t) + u_{2t}\right) \quad .$$

That (7) is a solution can be verified directly by substituting (7) into (4). Equation (7) is equivalent with the demand schedule (2). Equation (7) gives the solution for $E_{t-1}n_t$ as (square summable) sums of past w_t-p_t's and u_{2t}'s, and expected future values of w_t-p_t. The decay parameters λ_1 and λ_2 are functions of the structural parameters b, d, and a by virtue of their being roots of the characteristic polynomial $\lambda^2 + \frac{\phi}{b}\lambda + \frac{1}{b} = 0$. Equation (7) indicates how the firm responds to the signals it receives in the form of current and past real wages and forecasts of future real wages. As we have seen, decreases in the adjustment cost parameter d cause λ_1 and λ_2^{-1} to decrease, thereby speeding up the firm's response to "signals" via (7). This is the sense in which larger adjustment costs lead to more sluggish adjustments over time of employment in this model.

It is convenient to represent the model in the form

$$(1) \qquad n_t = \sum_{j=0}^{\infty} h_j E_t(w_{t+j} - p_{t+j}) + v(L)u_{1t}$$

$$(4) \qquad bE_t n_{t+1} + \phi n_t + n_{t-1} = \frac{1}{d}(w_t - p_t) + u_{2t}$$

This pair of stochastic difference equations by itself determines a bivariate stochastic process for n_t and $w_t - p_t$. A solution of the system of stochastic difference equations (1) and (4) is a jointly covariance stationary bivariate stochastic process for $(n_t, w_t - p_t)$ which satisfies (1) and (4), where $E_t n_{t+1}$ and $E_t(w_{t+j} - p_{t+j})$ are linear least squares projections calculated with respect to the solution stochastic process. It should be noted that including the "Euler equation" (4) rather than the demand schedule (2) in this representation of the model means that the representation fails to include the restrictions imposed by the transversality condition which was used to derive (2) from (4). The information in the transversality condition will be imposed on the solution to (1) and (4) by requiring that the solution stochastic process be covariance stationary, or equivalently, that the polynomials in the lag operator appearing in (8) be square-summable. I shall use a version of Muth's method to solve (1) and (4). The solution will have moving average representation

$$(8) \qquad n_t = \alpha(L)u_{2t} + \beta(L)u_{1t}$$

$$(w_t - p_t) = \gamma(L)u_{2t} + \delta(L)u_{1t}$$

where $\alpha(L)$, $\beta(L)$, $\gamma(L)$ and $\delta(L)$ are each square summable polynomials in the lag operator that are one-sided on non-negative powers of L, e.g. $\alpha(L) = \sum_{j=0}^{\infty} \alpha_j L^j$ with $\sum_{j=0}^{\infty} \alpha_j^2 < \infty$. The object is to find a 4-tuple $\alpha(L)$, $\beta(L)$, $\gamma(L)$, and $\delta(L)$ that satisfies (1) and (4). First use (8) in conjunction with the classic Wiener-Kolmogorov prediction formulas to calculate

$$(9) \qquad E_t n_{t+1} = \left(\frac{\alpha(L)}{L}\right)_+ u_{2t} + \left(\frac{\beta(L)}{L}\right)_+ u_{1t}$$

$$E_t(w_{t+j} - p_{t+j}) = \left(\frac{\gamma(L)}{L^j}\right)_+ u_{2t} + \left(\frac{\delta(L)}{L^j}\right)_+ u_{1t}$$

where $\left(\sum_{j=\infty}^{\infty} \phi_j L^j\right)_+ \equiv \sum_{j=0}^{\infty} \phi_j L^j$.

Substituting (9) into (1) and (4) guarantees that expectations are rational (i.e., are linear least squares forecasts made using the solution moving average representation):

$$n_t = \sum_{j=0}^{\infty} h_j \left(\frac{\gamma(L)}{L^j}\right)_+ u_{2t} + \sum_{j=0}^{\infty} h_j \left(\frac{\delta(L)}{L^j}\right)_+ u_{1t} + v(L)u_{1t}$$

$$b\left(\frac{\alpha(L)}{L}\right)_+ u_{2t} + b\left(\frac{\beta(L)}{L}\right)_+ u_{1t} + \phi\alpha(L)u_{2t} + \phi\beta(L)u_{1t} + \alpha(L)\cdot Lu_{2t}$$

$$+ \beta(L)Lu_{1t} = \frac{1}{d}(w_t - p_t) + u_{2t}$$

These equations can be rewritten

(10) $$n_t = [h(L^{-1})\gamma(L)]_+ \; u_{2t} + \{[h(L^{-1})\delta(L)]_+ + v(L)\} \; u_{1t}$$

$$\frac{1}{d}(w_t - p_t) = \{b\left[\frac{\alpha(L)}{L}\right]_+ + \phi\alpha(L) + \alpha(L)L - I\} \; u_{2t}$$

$$+ \{b\left[\frac{\beta(L)}{L}\right]_+ + \phi\beta(L) + \beta(L)L\} \; u_{1t}$$

Since (8) and (10) are of the same form, we have the restrictions

(11) $$\alpha(L) = [h(L^{-1})\gamma(L)]_+$$

$$\beta(L) = [h(L^{-1})\delta(L)]_+ + v(L)$$

$$\gamma(L) = d\{b\left[\frac{\alpha(L)}{L}\right]_+ + \phi\alpha(L) + \alpha(L)\cdot L - I\}$$

$$\delta(L) = d\{b\left[\frac{\beta(L)}{L}\right]_+ + \phi\beta(L) + \beta(L)\cdot L\}$$

Equations (11) are a set of restrictions across α, β, γ, δ, restrictions that depend on the structural parameters $h(L)$, $v(L)$, b, d, and ϕ. Let $\ell_2(0,\infty)$ be the space of one-sided square summable sequences $(\chi_0, \chi_1, \ldots)$ with $\sum_{j=0}^{\infty} |\chi_j|^2 < \infty$. Then notice that for fixed $h(L)$, $v(L)$, b, d and ϕ, where v and h are square summable, equation (11) defines a mapping of the space $\ell_2 x \ell_2 x \ell_2 x \ell_2$ into itself. A natural way to find a solution is to iterate on the mapping defined by (11) until the sequences α, β, γ, and δ converge. It is possible to restrict the parameters $h(L)$, $v(L)$, b, d, and ϕ so that (11) defines a contraction, in which case the contraction mapping theorem[9] guarantees that a unique solution to (11) exists and that it can always be found by iterating on (11). However, there is no reason in general to expect the parameters to obey the requirements needed to make (11) a contraction. Obviously, solutions to (11) can exist even when (11) fails to be a contraction.

By iterating on (11), I have calculated the equilibrium α, β, γ and δ for the following parameter values:

$$a = -1. \quad , \quad b = .95 \; , \; d = 10, \quad \phi = -2.15 \; , \quad v(L) = I$$

$$\begin{aligned}100\cdot h(L) = 1.0 &+ .75L + .5L^2 + .25L^3 + 0L^4 - .2L^5 - .3L^6 - .4L^7 \\ &- .4L^8 - .3L^9 - .2L^{10} - .2L^{11} - .1L^{12} - .1L^{13}\end{aligned}$$

For these parameter values, $\lambda_1 = .65$ and $\lambda_2^{-1} = .62$. The values of α, β, γ and δ are recorded in Table 1. For d=20, and all other parameter values remaining the same as above, α, β, γ, and δ are recorded in Table 2.

The solutions for $\beta(L)$ indicate that serially uncorrelated shocks to labor supply u_{1t} lead to positively serially correlated movements in employment. Further, the serial correlation becomes stronger, ceteris paribus, the higher is the adjustment cost parameter d. The solutions for $\delta(L)$ indicate that a positive labor supply shock u_{1t} causes real wages to fall temporarily but to rise in subsequent periods. It is this behavior of the real wage in response to labor supply shocks that causes current employment to display positive responses to past values of those shocks.

To study analytically a special case of (1) and (4), consider the system

$$n_t = h_0(w_t - p_t) + u_{1t}$$

$$bE_t n_{t+1} + \phi n_t + n_{t-1} = \frac{1}{d}(w_t - p_t) + u_{2t} \; .$$

Substituting the first equation into the second and rearranging gives

$$bE_t n_{t+1} + \phi' n_t + n_{t-1} = u'_{1t} + u_{2t}$$

Table 1

$d = 10\ (\lambda_1 = .65,\ \lambda_2^{-1} = .62,\ \phi = -2.15)$

j	α_j	β_j	γ_j	δ_j
0	-.0876	.8759	-8.1896	-18.1039
1	-.0077	.0767	-.7174	7.1737
2	-.0007	.0067	-.0628	.6284
3	-.0001	.0006	0.0055	.0550

Table 2

$d = 20\ (\lambda_1 = .75,\ \lambda_2^{-1} = .71,\ \phi = -2.05)$

j	α_j	β_j	γ_j	δ_j
0	-.1581	.7903	-13.9942	-30.0289
1	-.0250	.1249	-2.2119	11.0596
2	-.0039	.0197	-.3496	1.7481
3	-.0006	.0031	-.0553	.2763

where $\phi' = (\phi - \frac{1}{dh_0})$ and $u_{1t}' = \frac{-1}{dh_0} u_{1t}$. Paralleling the solution (7), we have

$$n_t - \lambda_1 n_{t-1} = \frac{1}{b}\left(\frac{\lambda_1}{\lambda_1-\lambda_2}\right)(u_{1t-1}' + u_{2t-1})$$

$$+ \frac{1}{b}\left(\frac{1}{\lambda_1-\lambda_2}\right)\left(\frac{1}{1-\frac{1}{\lambda_2}B^{-1}}\right)E_t(1-\lambda L)(u_{1t} + u_{2t})$$

where λ_1 and λ_2 are the roots of the characteristic polynomial $\lambda^2 + \frac{\phi'}{b} + \frac{1}{b} = 0$ with $\lambda_1 < 1$ and $\lambda_2 > 1$. Since u_{1t} and u_{2t} are serially uncorrelated, we have

$$n_t - \lambda_1 n_{t-1} = -\lambda_1(u_{1t}' + u_{2t}) \quad .$$

Thus, n_t is a first-order Markov process with moving average representation

$$n_t = \frac{-\lambda_1}{1-\lambda_1 L}\left(\frac{-1}{dh_0} u_{1t} + u_{2t}\right)$$

The solution for $(w_t - p_t)$ is then given by

$$(w_t - p_t) = \frac{1}{h_0}\left(\frac{-(1-\lambda_1)+\lambda_1 L}{1-\lambda_1 L}\right)u_{1t} - \frac{1}{h_0}\left(\frac{\lambda_1}{1-\lambda_1 L}\right)u_{2t}$$

Inspection of the solution for $(w_t - p_t)$ indicates that $\delta(L)$ will have a negative coefficient on L° and positive coefficients on higher powers of L, matching the pattern exhibited in Tables 1 and 2.

As it happens, the solution to (1) and (4) always has the first-order Markov property when v(L) = I . This can be seen by noticing that if v(L) = I, then in (1) and (4) n_{t-1} is the only "state" variable that is predetermined at time t. It is then natural to guess that a solution to (1) and (4) will have the property that

$$E_{t-1}n_t = \lambda n_{t-1}$$

$$E_{t-1}(w_t - p_t) = \mu n_{t-1}$$

These imply by the Markov property that

$$E_{t-1}n_{t+j} = \lambda^{j+1}n_{t-1}$$

$$E_{t-1}(w_{t+j} - p_{t+j}) = \mu^{j+1}n_{t-1}$$

Project both sides of (1) and (4) on information dated t-1 and earlier, and substitute the preceding equations to obtain the restrictions

$$\lambda = \sum_{j=0}^{\infty} h_j \mu^{j+1}$$

$$b\lambda^2 + \phi\lambda + 1 = \frac{1}{d} \cdot \mu$$

These are two equations that can be solved for λ and μ as functions of b, ϕ, d and h(L). Suppose that the h_j's are zero for $j \geq m+1$. Then substitute the first equation into the second to obtain

$$b\left(\sum_{j=0}^{m} h_j \mu^{j+1}\right)^2 + \phi\left(\sum_{j=0}^{m} h_j \mu^{j+1}\right) + 1 = \frac{1}{d}\,\mu$$

which is a 2(m+1) order equation in μ, which in general has 2(m+1) roots. In practice, many of the roots will be complex and can be disregarded as they have no physical interpretation. However, notice that if there are any real roots to the above equation (and there need not be any), then there will be an <u>even</u> number of such roots. In calculating roots for sample parameter values, I have encountered on occasions multiple solutions, each of which corresponds to a solution that is a stationary stochastic process. An example of this is reported in Table 4, which reports solutions for μ and λ for certain parameter values. There are four real roots, with three solutions for λ being less than unity. It is straightforward[10] to establish that the solution values of (μ,λ) correspond to a stationary vector stochastic process if and only if $|\lambda|<1$. Thus, in this case we have three solutions that correspond to stationary stochastic processes for $(w_t - p_t, n_t)$. Table 3 gives an example where there are only two real solutions. The solution with $\lambda>1$ can be discarded as it violates the "transversality condition," so that in this particular example there is a unique admissible solution. Notice that the solutions in Tables 3 and 4 are associated with substantial serial correlation in employment. The possibility that there are multiple equilibrium stochastic processes for (n_t, w_t-p_t) raises a number of interesting possibilities and questions, which I won't pursue in this paper.

From the preceding calculations, it is evident that the model is capable of generating serially correlated movements in employment in response to serially uncorrelated shocks. The existence of serially correlated movements in employment is no paradox from the viewpoint of equilibrium models of the class represented by (1) and (4). However, the model formed by (1) and (4) is inconsistent with the notion that movements in "aggregate demand" or "money" are sources of the shocks impinging on employment. That is, the model can't account for positive price-employment or money-employment correlations that originate from the action of aggregate demand on employment. To show this formally,

Table 3

$b = .95,\ \phi = -2.0,\ d^{-1} = .03$

$h(L) = .8 + .8L + OL^2 - .8L^3 - .8L^4$

Real roots:

$(\mu,\lambda) = (-1.288,\ .931)$

$(\mu,\lambda) = (-1.329,\ 1.169)$

Table 4

$b = .95,\ \phi = -2.0,\ d^{-1} = .005$

$h(L) = .8 + .8L + .8L^2 + OL^3 - .8L^4 - .8L^5 - .8L^6$

Real roots:

$(\mu,\lambda) = (.594,\ .811)$

$(\mu,\lambda) = (.861,\ .808)$

$(\mu,\lambda) = (-1.200,\ 1.274)$

$(\mu,\lambda) = (-1.153,\ .831)$

notice that the two equations (1) and (4) are sufficient completely to determine the bivariate stochastic process for $(n_t, w_t - p_t)$. From this fact it follows that the model is a "real" model of business fluctuations in the following very strong sense: the vector stochastic process for $(n_t, w_t - p_t)$ can be determined analytically before the stochastic process for the price level p_t has been determined. In this model monetary policy is powerless to affect employment. Indeed, even unexpected movements in money are asserted to have no effects on n_t or $w_t - p_t$. The strong real property is a consequence of two features of the system. First, there is the specification of the structural equations (1) and (2) with their exclusion of "money illusion." Second, there is the imposition of rational expectations, which enters by way of our having assumed that agents' expectations are linear least squares forecasts formed with reference to the probability distribution generated by the solution to (1) and (4). The strong real property will not in general prevail where agents' expectations are posited to be arbitrary _ad hoc_ functions of, say, own past values.

The tendency of economists to regard real models of this type as incomplete or wrong is due to the fact that positive output-price and output-money correlations have characterized business cycles for a long time, coupled with a widespread belief that those correlations reflect a reality in which both the price level and aggregate output and employment are responding sympathetically to a common causal force such as "aggregate demand" or "money." Most economists have desired to explain the positive price-output correlations with a model in which the price level and aggregate output and employment are simultaneously determined; this _desideratum_ is to be contrasted with the preceding real model in which a complete theory of employment and the real wage is obtained without any theory of the price level having been advanced.

3. Disparate Information Sets

It was to explain demand-induced price-output correlations while retaining the hypothesis of rational expectations that Lucas resorted to a setup with dispersed markets and information discrepancies. That setup can be adapted to generate demand-induced price-employment correlations in the present model while delivering the added dividend that there emerges an endogenous theory of the persistence of effects of aggregate demand shocks.

Firms and households are dispersed over a continuum of markets. Both firms and workers are forever stuck in the market where they happen to be, a simplification that rules out interesting search problems. The price in a particular market at time t obeys

$$p_t(z_t) = p_t + z_t \tag{12}$$

where p_t is the logarithm of the economy-wide average price level, and z_t is a serially independent random process with mean zero and probability density $g(z)$. The average price p_t follows a stochastic process to be described below. I assume that $Ep_t z_s = 0$ for all t and s, so that the price in a particular market is the sum of the average price level and a market specific relative demand component z_t which is orthogonal to the average price level.

The state of a particular market relative to the average economy-wide state at time t is entirely determined by the sequence of relative demand shocks $\{z_t, z_{t-1}, z_{t-2}, \ldots\}$ that it has received up to the present time. That is, all markets are identical in their behavioral parameters and face environments differentiated only by the different sequences

$(z_t, z_{t-1}, \ldots)$ that they face. This fact permits us to index individual markets by the sequences $\{z_t, z_{t-1}, \ldots\}$ which they have faced: if two markets have drawn identical z sequences over the past, they will have behaved in identical fashion. Notice that from the earlier assumption that z_t is serially independent, it follows that the sequence $(z_t, z_{t-1}, \ldots)$ has joint density $g(z_t)g(z_{t-1})\ldots$. Let $\bar{z}_t$ denote the sequence $\{z_t, z_{t-1}, \ldots\}$.

In this model, it will turn out that at time t employment and the money wage in market $\bar{z}_t$ will depend on $\bar{z}_t$ as well as the other state variables that influence all markets. I denote the equilibrium wage and employment in market $\bar{z}_t$ as $w_t(\bar{z}_t)$ and $n_t(\bar{z}_t)$ respectively. I denote the economy-wide averages of $w_t(\bar{z}_t)$ as w_t and n_t, so that

$$w_t = \int w_t(\bar{z}_t)g(z_t)g(z_{t-1})\ldots dz_t dz_{t-1}\ldots$$

$$n_t = \int n_t(\bar{z}_t)g(z_t)g(z_{t-1})\ldots dz_t dz_{t-1}\ldots$$

I assume that at time t agents in market $\bar{z}_t$ have an information set $I_t(\bar{z}_t)$ consisting of $\{\theta_{t-1}, w_t(\bar{z}_t), p_t(z_t)\}$ where[11]

$$\theta_{t-1} = \{w_{t-1}, w_{t-2}, \ldots, p_{t-1}, p_{t-2}, \ldots, n_{t-1}, n_{t-2}, \ldots$$
$$w_{t-1}(\bar{z}_t), w_{t-2}(\bar{z}_t), \ldots, p_{t-1}(\bar{z}_t), p_{t-2}(\bar{z}_t), \ldots\}$$

Agents in market $\bar{z}_t$ thus have information about the <u>lagged</u> values of the economy-wide aggregates n, p, and w, and know current <u>and</u> lagged values of the market-specific variables $w_t(\bar{z}_t)$ and $p_t(\bar{z}_t)$.

I assume that in market $\bar{z}_t$ agents behave according to

$$(13) \qquad n_t(\bar{z}_t) = \sum_{j=0}^{\infty} h_j \{Ew_{t+j}(\bar{z}_t) | I_t(\bar{z}_t) - Ep_{t+j} | I_t(\bar{z}_t)\} + u_{1t}$$

$$(14) \qquad bEn_{t+1}(\bar{z}_t)|I_t(\bar{z}_t) + \Phi n_t(\bar{z}_t) + n_{t-1}(\bar{z}_t) = \frac{1}{d}\{w_t(\bar{z}_t) - p_t(\bar{z}_t)\} + u_{2t}$$

Equation (13) is the counterpart of the labor demand schedule while equation (14) is the counterpart of the labor supply schedule of section 2. In the labor supply schedule I have deflated the (expected) money wage in market $\bar{z}_t$ by the expected average economy-wide price level, to reflect that an employee cares about his prospective wage measured not in terms of own-market goods but in terms of an economy-wide average bundle of goods. The terms u_{1t} and u_{2t} are economy-wide shocks to labor supply and demand, which I assume are serially uncorrelated random processes with further properties to be spelled out below.

To complete the model in the simplest way, I assume that p_t is governed by the exogenous process

$$(15) \qquad p_t = w(L)u_{3t}$$

where $w(L)$ is a square-summable polynomial in the lag operator that is one-sided on the present and past. I assume that u_{1t}, u_{2t}, u_{3t}, z_t are serially uncorrelated random processes with means of zero. I further assume that $Ez_t u_{js} = 0$ for all t, j and s. I assume that $Eu_{js}u_{kt} = 0$ for all j and k, and all $t \neq s$. Thus, the u_{jt}'s are orthogonal at all leads and lags, but possibly correlated contemporaneously. Let the contemporaneous covariance matrix of $u_t = (u_{1t}, u_{2t}, u_{3t})'$ be $\sum = Eu_t u_t'$. To simplify things a bit, I will assume that $\sum$ is diagonal, though dropping this assumption would require modifying only equations (22), (29) and (32) below.

Combining (12) and (15) we also have

$$(16) \qquad p_t(z_t) = w(L)u_{3t} + z_t$$

A solution of the stochastic difference equations (13), (14) and (16) is a stochastic process that has a moving average representation

$$(17) \qquad n_t(\bar{z}_t) = a(L)u_{1t} + b(L)u_{2t} + c(L)u_{3t} + d(L)z_t$$

$$(18) \qquad w_t(\bar{z}_t) = e(L)u_{1t} + f(L)u_{2t} + g(L)u_{3t} + k(L)z_t$$

where all polynomials in the lag operator are square summable and one-sided on the present and past. We can use the same technique exhibited in section 2 to characterize the restrictions that equations (13), (14), and (16) impose on the polynomials in L that appear in (17) and (18). The reader not interested in these tedious calculations can turn without loss of continuity to the end of this section and find a statement of the restrictions.

From (16) and (18) it is straightforward to calculate

$$(19) \qquad p_t(z_t) - Ep_t|\theta_{t-1} = w_0u_{3t} + z_t$$

$$(20) \qquad w_t(\bar{z}_t) - Ew_t(\bar{z}_t)|\theta_{t-1} = e_0u_{1t} + f_0u_{2t} + g_0u_{3t} + k_0z_t$$

Application of the Wiener-Kolmogorov formula[12] to (17) gives

$$(21) \qquad En_{t+1}(\bar{z}_t)|\theta_{t-1} = \left[\frac{a(L)}{L^2}\right]_+ u_{1t-1} + \left[\frac{b(L)}{L^2}\right]_+ u_{2t-1} + \left[\frac{c(L)}{L^2}\right]_+ u_{3t-1}$$

$$+ \left[\frac{d(L)}{L^2}\right]_+ z_{t-1}$$

From the recursive projection (Kalman filter) formula,[13] we have

$$En_{t+1}(\bar{z}_t)\,|\,I_t(\bar{z}_t) = En_{t+1}(\bar{z}_t)\,|\,\theta_{t-1}$$

$$+ E[(a_1u_{1t} + b_1u_{2t} + c_1u_{3t} + d_1z_t)\,|\,(p_t(z_t) - Ep_t|\theta_{t-1}),$$

$$(w_t(\bar{z}_t) - Ew_t(\bar{z}_t)|\theta_{t-1})]$$

Let us write the second projection on the right side as

$$\phi_1(p_t(z)-Ep_t|\theta_{t-1}) + \phi_2(w_t(\bar{z}_t) - Ew_t(\bar{z}_t)|\theta_{t-1})$$

where ϕ_1 and ϕ_2 are determined by the least squares normal equations[14]

$$\text{(22)}\qquad \begin{pmatrix} w_0^{\,2}Eu_3^{\,2} + Ez^2, & w_0g_0Eu_3^{\,2} + k_0Ez^2 \\ w_0g_0Eu_3^{\,2} + k_0Ez^2, & c_0^{\,2}Eu_1^{\,2} + f_0^{\,2}Eu_2^{\,2} + g_0^{\,2}Eu_3^{\,2} + k_0^{\,2}Ez^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$= \begin{pmatrix} w_0c_1Eu_3^{\,2} + d_1Ez^2 \\ e_0a_1Eu_1^{\,2} + f_0b_1Eu_2^{\,2} + g_0c_1Eu_3^{\,2} + k_0d_1Ez^2 \end{pmatrix}$$

Thus, we have

$$\text{(23)}\qquad En_{t+1}(\bar{z}_t)\,|\,I_t(\bar{z}_t) = En_{t+1}(\bar{z}_t)\,|\,\theta_{t-1} + \phi_1(p_t(z_t) - Ep_t|\theta_{t-1})$$

$$+ \phi_2(w_t(\bar{z}_t) - Ew_t(\bar{z}_t)|\theta_{t-1})$$

Substituting from (21), (23), (17), (19) and (20) into (14) gives

$$\text{(24)}\qquad b\left(\frac{a(L)}{L^2}\right)_+ u_{1t-1} + b\left(\frac{b(L)}{L^2}\right)_+ u_{2t-1} + b\left(\frac{c(L)}{L^2}\right)_+ u_{3t-1} + b\left(\frac{d(L)}{L^2}\right)_+ z_{t-1}$$

$$+ b\phi_1\{w_0u_{3t}+z_t\} + b\phi_2\{e_0u_{1t} + f_0u_{2t} + g_0u_{3t} + k_0z_t\}$$

$$+ \{\phi + L\}\{a(L)u_{1t} + b(L)u_{2t} + c(L)u_{3t} + d(L)z_t\} =$$

$$\frac{1}{d}\{w_t(\bar{z}_t) - w(L)u_{3t} - z_t\} + u_{2t}$$

Rearranging, we have

$$\text{(25)} \qquad \{b\left[\frac{a(L)}{L^2}\right]_+ L + b\phi_2 e_0 I + (\phi+L)a(L)\}\ u_{1t}$$

$$+ \{b\left[\frac{b(L)}{L^2}\right]_+ L + b\phi_2 f_0 I + (\phi+L)b(L) - I\}\ u_{2t}$$

$$+ \{b\left[\frac{c(L)}{L^2}\right]_+ L + b\phi_1 w_0 I + b\phi_2 g_0 I + (\phi+L)c(L) + \frac{1}{d}\, w(L)\}\ u_{3t}$$

$$+ \{b\left[\frac{d(L)}{L^2}\right]_+ L + b\phi_1 I + b\phi_2 k_0 I + (\phi+L)d(L) + \frac{1}{d}\cdot I\}\ z_t$$

$$= \frac{1}{d}\, w_t(\bar{z}_t)$$

Comparing (25) with (18), we have the restrictions

$$\text{(26)} \qquad e(L) = d\{b\left[\frac{a(L)}{L^2}\right]_+ + b\phi_2 e_0 I + (\phi+L)a(L)\}$$

$$f(L) = d\{b\left[\frac{b(L)}{L^2}\right]_+ + b\phi_2 f_0 I + (\phi+L)b(L) - I\}$$

$$g(L) = d\{b\left[\frac{c(L)}{L^2}\right]_+ + (b\phi_1 + b\phi_2 g_0)I + (\phi+L)c(L) + \frac{1}{d}\, w(L)\}$$

$$k(L) = d\{b\left[\frac{d(L)}{L^2}\right]_+ + (b\phi_1 + b\phi_2 k_0)I + (\phi+L)d)L) + \frac{1}{d}\, I\}$$

These form a set of four restrictions across the polynomials in L in the solution (17) and (18). Notice that the restrictions (26) involve the least squares regression parameters (ϕ_1, ϕ_2) which are determined by the normal equations (22) and are therefore functions of all the parameters that appear in (22). In particular, ϕ_1 and ϕ_2 themselves depend on the first two coefficients in the polynomials that appear (17) and (18). In effect, equations (26) are the restrictions that the labor demand schedule imposes on the solution (17)-(18).

I now perform analogous calculations in order to deduce the restrictions imposed by the labor supply schedule. From (16) we have

(27) $$Ep_{t+j}(\bar{z}_t)\,|\,\theta_{t-1} = Ep_{t+j}\,|\,\theta_{t-1} = \left[\frac{w(L)}{L^{j+1}}\right]_+ u_{3t-1} \quad , \quad j\geq 0 \quad .$$

We also have from the recursive projection formula

(28) $$Ep_{t+j}\,|\,I_t(\bar{z}_t) = Ep_{t+j}\,|\,\theta_{t-1} + \phi_{3j}(p_t(z) - Ep_t\,|\,\theta_{t-1})$$

$$+ \phi_{4j}(w_t(\bar{z}_t) - Ew_t(\bar{z}_t)\,|\,\theta_{t-1}) \qquad j=0,1,2,\ldots$$

where ϕ_{3j} and ϕ_{4j} are least squares regression coefficients. In particular, we have

$$Ep_{t+j}\,|\,\theta_t - Ep_{t+j}\,|\,\theta_{t-1} = w_j u_{3t}$$

Then from the recursive projection formula, ϕ_{3j} and ϕ_{4j} satisfy the least squares normal equations

(29) $$M \cdot \begin{bmatrix} \phi_{3j} \\ \phi_{4j} \end{bmatrix} = \begin{bmatrix} w_0 w_j Eu_3^{\,2} \\ w_0 g_j Eu_3^{\,2} \end{bmatrix} \qquad j=0,1,2,\ldots$$

where M is the same (2x2) moment matrix on the left side of equation (22).

From (18) we have

$$(30)\qquad Ew_{t+j}(\bar{z}_t)|\theta_{t-1} = \left[\frac{e(L)}{L^{j+1}}\right]_+ u_{1t-1} + \left[\frac{f(L)}{L^{j+1}}\right]_+ u_{2t-1} + \left[\frac{g(L)}{L^{j+1}}\right]_+ u_{3t-1}$$

$$+ \left[\frac{k(L)}{L^{j+1}}\right]_+ z_{t-1}$$

It is straightforward to calculate that

$$Ew_{t+j}(\bar{z}_t)|\theta_t - Ew_{t+j}(\bar{z}_t)|\theta_{t-1} = e_j u_{1t} + f_j u_{2t} + g_j u_{3t} + k_j z_t \quad .$$

Then from the recursive projection formula we have

$$(31)\qquad Ew_{t+j}(\bar{z}_t)|I_t(\bar{z}_t) = Ew_{t+j}(\bar{z}_t)|\theta_{t-1} + \pi_{1j}(p_t(z_t) - Ep_t|\theta_{t-1})$$

$$+ \pi_{2j}(w_t(\bar{z}_t) - Ew_t(\bar{z}_t)|\theta_{t-1}) \qquad j=1,2,\ldots$$

where π_{1j} and π_{2j} satisfy the least squares normal equations

$$(32)\qquad M\begin{bmatrix}\pi_{1j}\\ \pi_{2j}\end{bmatrix} = \begin{bmatrix} w_0 e_j Eu_1^{\ 2} + k_j Ez^2 \\ e_0 p_j Eu_1^{\ 2} + f_0 f_j Eu_2^{\ 2} + g_0 g_j Eu_3^{\ 2} + k_0 k_j Ez^2\end{bmatrix} \qquad j=1,2,\ldots$$

where M is again the (2x2) matrix on the left side of (22).

Substituting (28) and (31) into (13) gives

$$n_t(\bar{z}_t) = h_0[w_t(\bar{z}_t) - Ep_t|\theta_{t-1}] + \sum_{j=1}^{\infty} h_j\{Ew_{t+j}(\bar{z}_t)|\theta_{t-1} - Ep_{t+j}|\theta_{t-1}\}$$

$$+ \Gamma_1\{p_t(z) - Ep_t|\theta_{t-1}\} + \Gamma_2\{w_t(\bar{z}_t) - Ew_t(\bar{z}_t)|\theta_{t-1}\} + u_{1t}$$

where $\qquad \Gamma_1 = -h_0\theta_{30} + \sum_{j=1}^{\infty} h_j(\pi_{1j} - \theta_{3j})$

$$(33)\qquad \Gamma_2 = -h_0\theta_{40} + \sum_{j=1}^{\infty} h_j(\pi_{2j} - \theta_{4j})$$

By substituting (27), (30), (16), and (18) into the above equation and rearranging, we can derive the following restrictions:

$$\text{(34)} \qquad a(L) = \{h_0 e(L) + \sum_{1}^{\infty} h_j \left[\frac{e(L)}{L^{j+1}}\right]_+ L + \Gamma_2 e_0 I + I\}$$

$$b(L) = \{h_0 f(L) + \sum_{1}^{\infty} h_j \left[\frac{f(L)}{L^{j+1}}\right]_+ L + \Gamma_2 f_0 I\}$$

$$c(L) = \{h_0 g(L) - h_0 \left[\frac{w(L)}{L}\right]_+ L + \sum_{1}^{\infty} h_j \{\left[\frac{g(L)}{L^{j+1}}\right]_+ - \left[\frac{w(L)}{L^{j+1}}\right]_+\} L + \Gamma_1 w_0 I + \Gamma_2 g_0 I\}$$

$$d(L) = \{h_0 k(L) + \sum_{1}^{\infty} h_j \left[\frac{k(L)}{L^{j+1}}\right]_+ L + \Gamma_1 I + \Gamma_2 k_0 I\}$$

The restrictions (26) and (34) where Γ_1, Γ_2, ϕ_1 and ϕ_2 are given by (22), (29), (32), and (33) fully characterize the solution to the stochastic difference equations (13) and (14).

Collecting some results, we have

$$\text{(26)} \qquad e(L) = d\{b\left[\frac{a(L)}{L^2}\right]_+ + b\phi_2 e_0 I + (\phi+L)a(L)\}$$

$$f(L) = d\{b\left[\frac{b(L)}{L^2}\right]_+ + b\phi_2 f_0 I + (\phi+L)b(L) - I\}$$

$$g(L) = d\{b\left[\frac{c(L)}{L^2}\right]_+ + (b\phi_1 w_0 + b\phi_2 g_0)I + (\phi+L)c(L) + \frac{1}{d} w(L)\}$$

$$k(L) = d\{b\left[\frac{d(L)}{L^2}\right]_+ + (b\phi_1 + b\phi_2 k_0)I + (\phi+L)d(L) + \frac{1}{d} I\}$$

(34)
$$a(L) = \{h_0 e(L) + \sum_{j=1}^{\infty} h_j \left[\frac{e(L)}{L^{j+1}}\right]_+ L + \Gamma_2 e_0 I + I\}$$

$$b(L) = \{h_0 f(L) + \sum_{1}^{\infty} h_j \left[\frac{f(L)}{L^{j+1}}\right]_+ L + \Gamma_2 f_0 I\}$$

$$c(L) = \{h_0 g(L) + h_0 \left[\frac{w(L)}{L}\right]_+ L + \sum_{1}^{\infty} h_j \left[\frac{g(L)-w(L)}{L^{j+1}}\right]_+ L + (\Gamma_1 w_0 + \Gamma_2 g_0) I\}$$

$$d(L) = \{h_0 k(L) + \sum_{1}^{\infty} h_j \left[\frac{k(L)}{L^{j+1}}\right]_+ L + (\Gamma_1 + \Gamma_2 k_0) I\}$$

(33)
$$\Gamma_1 = -h_0 \phi_{30} + \sum_{1}^{\infty} h_j (\pi_{1j} - \phi_{3j})$$

$$\Gamma_2 = -h_0 \phi_{40} + \sum_{1}^{\infty} h_j (\pi_{2j} - \phi_{4j})$$

(22)
$$M \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} w_0 c_1 Eu_3^2 + d_1 Ez^2 \\ e_0 a_1 Eu_1^2 + f_0 b_1 Eu_2^2 + g_0 c_1 Eu_3^2 + k_0 d_1 Ez^2 \end{bmatrix}$$

(29)
$$M \begin{bmatrix} \phi_{3j} \\ \phi_{4j} \end{bmatrix} = w_j Eu_3^2 \begin{bmatrix} w_0 \\ g_0 \end{bmatrix}$$

(32)
$$M \begin{bmatrix} \pi_{1j} \\ \pi_{2j} \end{bmatrix} = \begin{bmatrix} w_0 e_j Eu_1^2 + k_j Ez^2 \\ e_0 e_j Eu_1^2 + f_0 f_j Eu_2^2 + g_0 g_j Eu_3^2 + k_0 k_j Ez^2 \end{bmatrix}$$

(35)
$$M = \begin{bmatrix} w_0^2 Eu_3^2 + Ez^2 , & w_0 g_0 Eu_3^2 + k_0 Ez^2 \\ w_0 g_0 Eu_3^2 + k_0 Ez^2 , & e_0^2 Eu_1^2 + f_0^2 Eu_2^2 + g_0^2 Eu_3^2 + k_0^2 Ez^2 \end{bmatrix}$$

Equations (26), (34), (33), (22), (29), (32), and (35) embody the restriction that the model imposes on the bivariate stochastic process (17)-(18) for n_t and w_t.

The following algorithm provides a possible method of computing an equilibrium stochastic process for n_t, w_t. Let D(L) be the (2x4) matrix of polynomials in the lag operator

$$D(L) = \begin{bmatrix} a(L) & b(L) & c(L) & d(L) \\ & & & \\ e(L) & f(L) & g(L) & k(L) \end{bmatrix}$$

The object is to compute a D(L) that satisfies the preceding restrictions. The parameters of the model are ϕ, b, h(L), w(L), Ez^2, and $\sum = Eu_t u_t'$. A possible algorithm begins with a guess at D(L) and the parameters Γ_1, Γ_2, ϕ_1 and ϕ_2. A new estimate of D(L) can be calculated from (26) and (34). Then equations (33), (22), (29), (32), and (35) are solved for new estimates of Γ_1, Γ_2, ϕ_1, ϕ_2. These together with the estimate of D(L) are used to calculate a new D(L) via (26) and (34). The proposal is to continue iterating.

What complicates these calculations relative to those reported in section 2 is the dependence of the parameters ϕ_1, ϕ_2, Γ_1 and Γ_2 on the equilibrium value of D(L), as well as on Ez^2 and $\sum = Eu_t u_t'$. For arbitrarily fixed values of ϕ_1, ϕ_2, Γ_1, and Γ_2, a D(L) that satisfies (26)-(34) can be computedusing the same iterative scheme as used in section 2.

4. Behavior of Economy-Wide Averages

By averaging (17) and (18) across markets (i.e., integrating both sides with respect to $g(z_t)g(z_{t-1})\ \ldots\ dz_t dz_{t-1})$ we obtain

(17') $$n_t = a(L)u_{1t} + b(L)u_{2t} + c(L)u_{3t}$$

(18') $$w_t = e(L)u_{1t} + f(L)u_{2t} + g(L)u_{3t}$$

Subtracting (15) from (18') gives

(36) $$(w_t - p_t) = e(L)u_{1t} + f(L)u_{2t} + (g(L) - w(L))u_{3t}$$

Equations (17'), 18') and (36) form a moving average representation for the economy-wide averages n_t, w_t, w_t-p_t. Here a(L), b(L), c(L), e(L), f(L), and g(L) are the same polynomials in L that appear in the solutions (17) and (18) for the individual-market solutions.

Notice that, in contrast to our results in section 2, the solutions (17') and (36) for n_t and w_t-p_t involve polynomials in the price level innovation u_{3t} as well as the economy-wide labor supply and labor demand shocks u_{1t} and u_{2t}. This fact establishes that the strong real property of the section 2 model does not obtain here, a direct result of the information discrepancies in the present setup. That is, the price level shock u_{3t} in general appears in the solutions for n_t and w_t-p_t, in contrast to the model of section 2 in which stochastic processes for n_t and w_t-p_t were completely predetermined with respect to the stochastic process for p_t.

To pursue further the sense in which the current model fails to be a "real" model, it is useful to exhibit the economy-wide "structural equations" that (17') and (36) in effect solve. Averaging both sides of equation (24) over markets (i.e., integrating both sides with respect to $g(z_t)g(z_{t-1})\ldots dz_t dz_{t-1}\ldots$) gives

$$(37) \qquad b\left(\frac{a(L)}{L^2}\right)_+ u_{1t} + b\left(\frac{b(L)}{L^2}\right)_+ u_{2t-1} + b\left(\frac{c(L)}{L^2}\right)_+ u_{3t-1} + b\phi_1\{w_0 u_{3t}\}$$

$$+ b\phi_2\{e_0 u_{1t} + f_0 u_{2t} + g_0 u_{3t}\} + \{\phi + L\}\{a(L)u_{1t} + b(L)u_{2t} + c(L)u_{3t}\}$$

$$= \frac{1}{d}\{w_t - w(L)u_{3t}\} + u_{2t} \quad .$$

Let Ω_{t-1} be the information set consisting of the economy-wide aggregates $\{p_{t-1}, p_{t-2}, \ldots, w_{t-1}, w_{t-2}, \ldots, n_{t-1}, n_{t-2}, \ldots\}$. Then notice that

$$w_0 u_{3t} = p_t - Ep_t|\Omega_{t-1}$$

$$e_0 u_{1t} + f_0 u_{2t} + g_0 u_{3t} = w_t - Ew_t|\Omega_{t-1}$$

$$En_{t+1}|\Omega_{t-1} = \left(\frac{a(L)}{L^2}\right)_+ u_{1t-1} + \left(\frac{b(L)}{L^2}\right)_+ u_{2t-1} + \left(\frac{c(L)}{L^2}\right)_+ u_{3t-1}$$

Consequently, equation (37) becomes

$$(38) \qquad bEn_{t+1}|\Omega_{t-1} + \Phi n_t + n_{t-1} = \frac{1}{d}\{w_t - p_t\} + u_{2t}$$

$$- b\phi_1\{p_t - Ep_t|\Omega_{t-1}\} - b\phi_2\{w_t - Ew_t|\Omega_{t-1}\}$$

Equation (38) is in the nature of an aggregate demand schedule for employment.

Similarly, by averaging the labor supply schedule across markets we obtain

$$(39) \qquad n_t = h_0[w_t - Ep_t|\Omega_{t-1}] + \sum_{j=1}^{\infty} h_j\{Ew_{t+j}|\Omega_{t-1} - Ep_{t+j}|\Omega_{t-1}\}$$

$$+ \Gamma_1\{p_t - Ep_t|\Omega_{t-1}\} + \Gamma_2\{w_t - Ew_t|\Omega_{t-1}\} + u_{1t}$$

Equation (39) is in the nature of an aggregate supply schedule for labor. Equations (38) and (39) are two stochastic difference equations in the three endogenous stochastic processes w_t, p_t, and n_t. In general, there is no way of writing the system in terms of the real variables (w_t-p_t) and n_t alone. In other words, the system does not possess the strong real property possessed by the model of section 2. In order to determine stochastic processes for n_t and w_t-p_t, we must add to (38) and (39) an equation or system of equations that permits us to determine the stochastic process for p_t as well. The model could be completed by adding either a final form for p_t or a system of structural relations (e.g. portfolio balance schedules) that permits determination of p_t (and any additional endogenous variables appearing in the added structural equations). In any event, the strong real character of the model in section 2 has been broken by the information discrepancies introduced in this section. These modifications potentially allow scope for price-output or price-employment correlations that stem from unexpected movements in aggregate demand.

While the strong real character of the model is destroyed by the modifications made in this section, the model remains quite "classical" in the sense that systematic countercyclical monetary policy is impossible. To show this, project both sides of (38) and (39) against Ω_{t-1} to get

$$(40) \qquad En_t|\Omega_{t-1} = \sum_{j=0}^{\infty} h_j E(w_{t+j} - p_{t+j})|\Omega_{t-1}$$

$$(41) \qquad bEn_{t+1}|\Omega_{t-1} + \phi En_t|\Omega_{t-1} + n_{t-1} = \frac{1}{d} E(w_t - p_t)|\Omega_{t-1} \quad .$$

These are two difference equations that are capable by themselves of determining the sequences $En_{t+j}|\Omega_{t-1}$ and $E(w_{t+j} - p_{t+j})|\Omega_{t-1}$, $j=0,1,2,\ldots$. In this sense, equations (38) and (39) determine the systematic or predictable (from the viewpoint of information in Ω_{t-1}) parts of the real wage and employment sequences, independently of how the model is completed to determine the price level. This means that the predictable parts of n_t and w_t-p_t cannot be influenced by the choice of the coefficients by which, say, the money supply is made a linear function of the information in Ω_{t-1}. In other words, this model possesses the same "neutrality" property as the model studied by Sargent and Wallace [1975]: one linear deterministic feedback rule setting money as a function of information in Ω_{t-1} is equivalent with any other such rule so far as concerns the predictable part of employment.[15] Further, the model has the characteristic that as a block, (w-p) and n are econometrically exogenous with respect to processes measuring nominal magnitudes such as the absolute price level and the money supply. This can be seen by noting that (40) and (41) by themselves are capable of determining $En_{t+j}|\Omega_{t-1}$ and $E(w_{t+j} - p_{t+j})|\Omega_{t-1}$ as functions of past values of n and (w-p) alone. The block econometric exogeneity of "real" variables with respect to "nominal" magnitudes was asserted earlier (Sargent [1976]) to be a distinguishing characteristic of "classical" macroeconometric models. The models of section 5 also possess this characteristic.

5. Other Mechanisms Generating Persistence

The role of the real rate of interest in determining labor supply was suppressed in the preceding models in order to simplify the calculations. However, including the real interest rate as a determinant of labor supply leads to models with additional mechanisms for generating persistent movement in employment in response to serially uncorrelated surprises. For example, consider the aggregative model

$$n_t = \sum_{j=0}^{\infty} h_j E_t(w_{t+j} - p_{t+j}) + \sum_{j=0}^{\infty} v_j E_t \rho_{t+j} + u_{1t} \tag{43}$$

$$bd_1 E_t n_{t+1} + (2a_1 - d_1(1+b))n_t + d_1 n_{t-1} = (w_t - p_t) + f_1 k_t + d_1 u_{2t} \tag{44}$$

$$bd_2 E_t k_{t+1} + (2a_2 - d_2(1+b))k_t + d_2 k_{t-1} = \rho_t + f_2 n_t + d_2 u_{3t} \tag{45}$$

$$k_{t+1} - k_t = e_1 k_t + e_2 n_t + e_3 z_t + \sum_{j=1}^{\infty} g_j E_t(w_{t+j} - p_{t+j}) + \sum_{j=1}^{\infty} m_j E_t \rho_{t+j} + u_{4t} \tag{46}$$

$$a_1, a_2 < 0 \;,\; f_1, f_2 < 0,\; d_1, d_2 > 0,\quad 0 < b < 1$$

Here u_{jt} , $j=1,\ldots,4$ are stochastic disturbance processes; k_t is the log of the capital stock while ρ_t is the one-period real rate of interest; z_t is a vector of exogenous variables including tax rates, government expenditures, and other measures of fiscal policy. Equation (43) is the labor supply schedule of Lucas and Rapping [1969]. Equations (44) and (45) are log-linear approximations to the Euler equations that emerge from a firm's present value maximation problem where varying both capital and labor requires bearing quadratic adjustment costs; d_1 and d_2 are adjustment cost parameters, while a_1, a_2, f_1 and f_2 are production function parameters. Equation (46) is a supply curve of capital that is a log-linear approximation to the equation that emerges from combining Lucas and Rapping's consumption function with the national income identity and a production function and solving for national investment. In (46), the terms in w-p and ρ come from Lucas and Rapping's consumption schedule; the terms in k_t and n_t arise from the aggregate production function, while the term in z_t arises from the national income identity and the consumption schedule. With respect to the process governing capital accumulation, the model formed by (43)-(46) resembles Tobin's "Dynamic Aggregative Model."

The model (43)-(46) is a complete model that determines the vector stochastic process n_t, k_t, w_t-p_t, and ρ_t. Consequently the model has the strong real property that stochastic processes for these four "real" variables are predetermined with respect to the stochastic process for the aggregate price level p_t. By resorting to information discrepancies, this strong real property can be broken, exactly as it was in moving from the simple model of section 2 to the model of section 3. In this way, the model (43)-(46) can be amended to account for demand-induced price-output correlations.[16] Further, the model contains a richer mechanism than does our simple model of section 3 for inducing serially correlated responses to serially uncorrelated surprises in the price level. In particular, it will turn out that surprise movements in the price level induce movements in both real wages and the real interst rate that cause workers to reallocate their labor over time via (43).

Footnotes

[1] By equilibrium model I mean a model in which all markets clear each period, and in which agents' decision rules are optimal given their information.

[2] For example, see Poole [1976] and especially the "general discussion" following that paper.

[3] Demand-induced persistence in unemployment is similarly accounted for but not explained in Sargent [1976].

[4] In their empirical work, Lucas and Rapping also suppressed the real interest rate argument.

[5] In section 5 below, I will describe a model that includes the real interest rate in (1). That model incorporates a mechanism for generating persistent aggregate-demand-induced output movements that works both through the real wage and the real rate of interest.

[6] Linear approximations are used for the same reason they were used by Lucas [1975]: to make the subsequent computations manageable.

[7] It is necessary to distinguish two operators, B and L. The operator B is defined by

$$B^{-1}[Ex_{t+j}|\Omega_{t-1}] = Ex_{t+j+1}|\Omega_{t-1} \quad ,$$

i.e., application of B^{-1} shifts forward by one period the date on the variables whose conditional forecast is being computed, but leaves the information set unaltered. The lag operator L is defined by

$$L^j x_t = x_{t-j} \quad .$$

In particular, notice that this definition implies that

$$L^{-1}(Ex_{t+j}|\Omega_{t-1}) = Ex_{t+j+1}|\Omega_t \quad ,$$

so that application of L^{-1} shifts both the random variable x <u>and</u> the information set Ω forward by one period.

[8] The difference equation (5) has many solutions, all but one of which imply that it will be expected that eventually employment will increase at increasing absolute rates. Along such paths, the cost-of-adjustment terms $-\frac{d}{2}(n_t - n_{t-1})^2$ come to dominate the present value expression, implying that such paths correspond to minima rather than maxima of expected present value. Imposing square-summability on all lag distributions in effect rules out such paths. For a discussion of the transversality condition in a closely related context, see Lucas [1967].

[9] For a discussion of the contraction mapping theorem, see Apostol.

[10] This is proved by writing the system as $z_t = Az_{t-1} + a_t$ where $z_t = [n_t, n_{t-1}, (w_t - p_t), (w_{t-1} - p_{t-1})]'$, $a_t = [a_{1t}, 0, a_{2t}, 0]'$, and

$$A = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \mu & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Here (a_{1t}, a_{2t}) are the errors in predicting n and w-p, respectively from observations on past n's and (w-p)'s. The process z_t is stationary if and only if the eigenvalue of A with maximum absolute value is bounded in absolute value by unity. It is readily verified that λ is the eigenvalue of A with maximum absolute value.

[11] I have taken the liberty of denoting p_t as a function of the sequence $\bar{z}_t$ as $p_t(\bar{z}_t)$. The price at time t-1 in market facing sequence $\bar{z}_t$ is denoted $p_{t-1}(\bar{z}_t)$ and of course by (12) equals $p_{t-1} + z_{t-1}$.

[12] See Whittle [1963].

[13] The recursive projection formula is

$$E[y|x,z] = Ey|x + E[(y - Ey|x)|(z - Ez|x)]$$

where y, x, z are three random variables (x and z can be interpreted as vectors of random variables).

[14] In the normal equations (22), (29), and (32), I have simplified things by assuming that the contemporaneous covariance matrix $\sum = Eu_t u_t'$ is diagonal. It is straightforward to modify these equations to deal with the case in which $\sum$ is not diagonal.

[15] It is straightforward to verify that the systematic part of the vector autoregression for $(n_t, w_t - p_t)$ would not be invariant with respect to the parameters governing a feedback rule for the marginal income tax rate. In the presence of a personal income tax, on the right-hand side of the labor supply schedule should be

$$\sum_{j=0}^{\infty} h_j E_t (w_{t+j} - p_{t+j} + T_{t+j}) \quad ,$$

where T_t is the log of one minus the marginal tax rate at time t. The reader is invited to investigate the effects of feedback policies of the form $T_t = \theta n_{t-1}$ on the stochastic process for n_t. Where u_{1t} and u_{2t} are serially uncorrelated, it is straightforward to use the method leading to Tables 3 and 4 to show that the parameter θ influences the systematic part of the vector autoregression for $(n_t, w_t - p_t)$.

[16] However, the model will continue to have the property that one deterministic money supply rule is as good as any other.

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