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76-1

Money, Prices, and Interest Rates in Stable Monetary Growth Models

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MONEY, PRICES, AND INTEREST RATES IN STABLE MONETARY GROWTH MODELS

By

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I. INTRODUCTION*

The purpose of this paper is to analyze a series of theoretical macroeconomic models that describe the possible dynamic relationships among money, a price index, and an interest rate. Having been subject to Occam's razor, the models are presented in their simplest form and from a macroscopic point of view. All behavioral equations are assigned specific functional forms so that the models can be solved analytically or simulated numerically. This enables one to study both the qualitative and quantitative properties of the alternative models.

Each of the models is formulated as a system of ordinary differential equations. The first three models are presented as "equilibrium" models where the real quantity of money is inversely related to the rate of inflation in the "long run." Two of these three models have been analyzed extensively in the literature; the third is a subtle extension. After discussing the dynamic relationship between the rate of growth of the quantity of money and the rate of inflation, we extend the third model by including an interest rate equation and a simple unemployment relationship. Finally, a corresponding "disequilibrium" model is derived to show the mathematical equivalence between equilibrium and disequilibrium growth models. This equivalency is based on the concept that a system of differential equations is a system of differential equations.

In the spirit of mathematical growth models, we make a series of heroic assumptions. The models represent a closed economy that produces and consumes a constant amount of a single homogeneous nondurable good. There is no investment, depreciation, technological change, or inventories. The population, level of employment, and capital utilization rate remain constant. In this initial state the nominal quantity of money and price level are constant and are normalized to equal one. Money is assumed to be a type of fiat money that is costless to produce and is injected into the economy without causing any changes in the income distribution.

Several basic experiments will be conducted on some of the alternative models. One test is to have the nominal amount of money grow at a constant rate, beginning from the initial time period zero, and observe the corresponding time path for the rate of inflation that is generated by the model. A second test assumes that the nominal money supply remains constant but that the economy begins from a position off its steady state path. This initial displacement could reflect an exogenous shock or an expected rate of inflation not equal to zero. A third test is to assign a simple feedback rule for the rate of growth of money and simulate the model following some initial disturbance. With the exception of the first model, we shall use parameter values which ensure stability in the alternative models.

We begin by reviewing two models formulated by Cagan in his study on hyperinflation [2]. A period of hyperinflation is reasonably compatible with the assumptions of our models. Real output remains fairly constant and the primary cost of holding cash balances in the rate of change in prices. The difference in return on money and on alternative forms of reserves, such as bonds, equities, or consumer durables, is dominated by the rate of inflation. During such a period of hyperinflation, real cash balances decrease as the rate of inflation exceeds the rate of growth of the quantity of money. This potentially destabilizing force is summarized succinctly in the theoretical work of Marshall [9, p. 48] where he states:

> The total value of an inconvertible paper currency therefore cannot be increased by increasing its quantity; an increase in its quantity, which seems likely to be repeated, will lower the value of each unit more than in proportion to the increase.

In Monetary Reform Keynes [8] stresses that this type of inflationary taxation is self-defeating because of the rapid increase in the velocity of circulation due to the public's loss of confidence in the currency. Friedman [5] argues that this inverse relationship between the rate of inflation and real balances is a fundamental long-run property of the theoretical framework for monetary analysis. Given such a consensus, let us now look at the mathematical implications.

II. MODEL I

The first model is a modified Cambridge cash-balance equation,

(1) $M/P = \exp(-\alpha\pi)$,

where M is the nominal quantity of money, P is the price index, M/P is real cash balances, π is the instantaneous rate of price inflation, exp is the natural base of logarithms, and α is a positive constant. The rate of inflation is defined as

$$\pi \equiv DP/P = D\ln P,$$

where the operator D is interpreted as the right-hand time derivative and lnP is the natural logarithm of the price index.

Taking the logarithms of equation (1) yields

$$\pi = (\ln P - \ln M)/\alpha.$$

This model is unstable since the derivative of the rate of inflation with respect to the logarithm of prices is positive.

$$\frac{\partial D\ln P}{\partial \ln P} = \frac{1}{\alpha} > 0.$$

When the nominal quantity of money is constant and the initial price level is above its equilibrium level, the model states that there will be self-generating inflation. In other words, when there is not enough money one has self-generating inflation, and when there is too much money one has self-generating deflation. This is the antithesis of the Marshall, Keynes, Friedman, Cagan, et. al. thesis.

In this heavenly model where there are perfect foresight and no frictions,[1/] the stabilizing monetary policy is to increase nominal money balances at a rate greater than the rate of inflation. Let μ be the rate of growth of the quantity of money and η be the logarithm of real cash balances:

$$\mu \equiv DM/M = D\ln M,$$

$$\eta \equiv \ln(M/P) = \ln M - \ln P.$$

Assume that μ is positively related to π so that

$$\mu = \theta\pi.$$

Since

$$D\eta = \mu - \pi = (\theta-1)\pi$$

and

$$\eta = -\alpha\pi,$$

therefore

$$D\eta = (1-\theta)\eta/\alpha.$$

This model is stable if and only if θ is greater than one so that

$$\frac{\partial D\eta}{\partial\eta} = (1-\theta)/\alpha < 0.$$

If heaven is stable, St. Peter must conduct such an aggressive monetary policy.

MODEL II

An alternative model[2/] used by Cagan is the following:

(2) $M/P = \exp(-\beta\pi^*)$,

(3) $D\pi^* = \gamma(\pi-\pi^*)$,

where π^* is the expected rate of inflation. β and γ are positive constants. This model is stable if the reaction index ($\beta\gamma$) is less than unity. To see why, take the logarithm of equation (2) to get

(4) $\eta = -\beta\pi^*$.

Differentiating with respect to time yields

(5) $D\eta = \mu - \pi = \beta D\pi^*$.

Use equations (2)-(5) to obtain the following relationships:

(6) $\pi^* = ((\beta\gamma-1)\pi+\mu)/(\beta\gamma)$,

(7) $\eta = ((1-\beta\gamma)\pi-\mu)/\gamma$,

$D\eta = \gamma(\eta+\beta\mu)/(\beta\gamma-1)$.

When $\beta\gamma$ is less than one, the model is stable and real cash balances are positively related to the rate of inflation and negatively related to the rate of change of nominal money balances. In the long run, which corresponds to the particular solution of a differential equation model, the rate of inflation equals the rate of growth of the quantity of money and is inversely related to real cash balances. Notice in equation (6) that the expected rate of inflation is negatively related to the actual rate of inflation and serves as a counterbalance.

Figure 1 plots the dynamic responses of these two models when the quantity of money is increasing at a constant rate $\bar{\mu}$. Notice that the rate of inflation asymptotically approaches its long-run level from above, so that real cash balances instantaneously fall whenever the nominal quantity of money is increased. This may be a reasonable description of a theoretical economy where new fiat money is introduced by having the monetary authority bid directly for goods and services. When money is introduced via transfers or a bond market, one would expect that real money balances initially increase, reach a maximum, and then

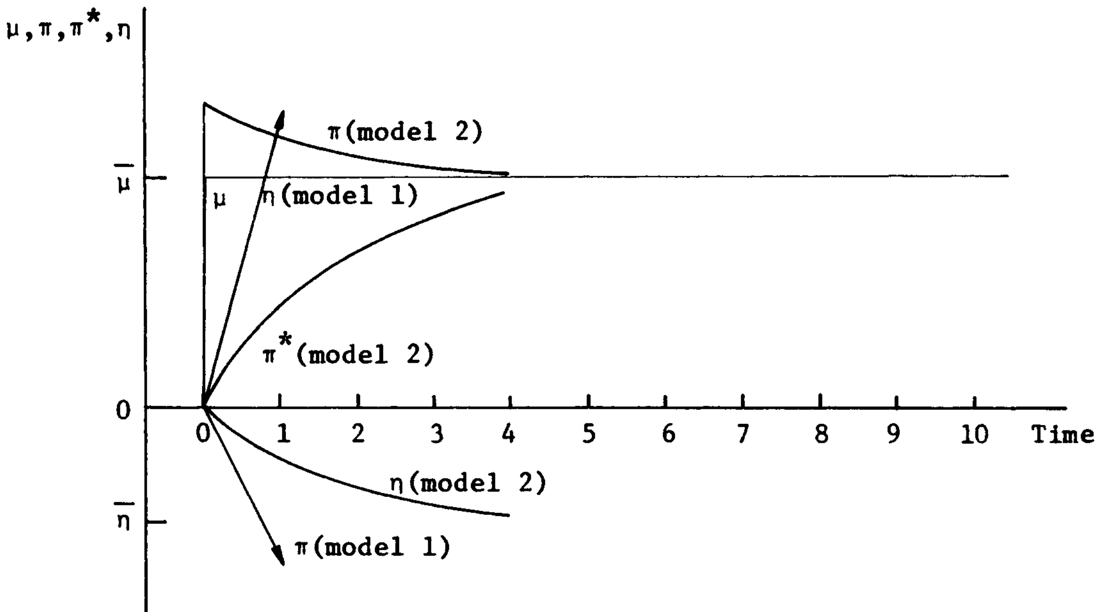

FIGURE 1. Cagan's First-order Differential Equation Models

Model 1.	$\eta = -\alpha\pi$	$\alpha = .5$
Model 2.	$\eta = -\beta\pi^*$	$\beta = .5$
	$D\pi^* = \gamma(\pi-\pi^*)$	$\gamma = .5$

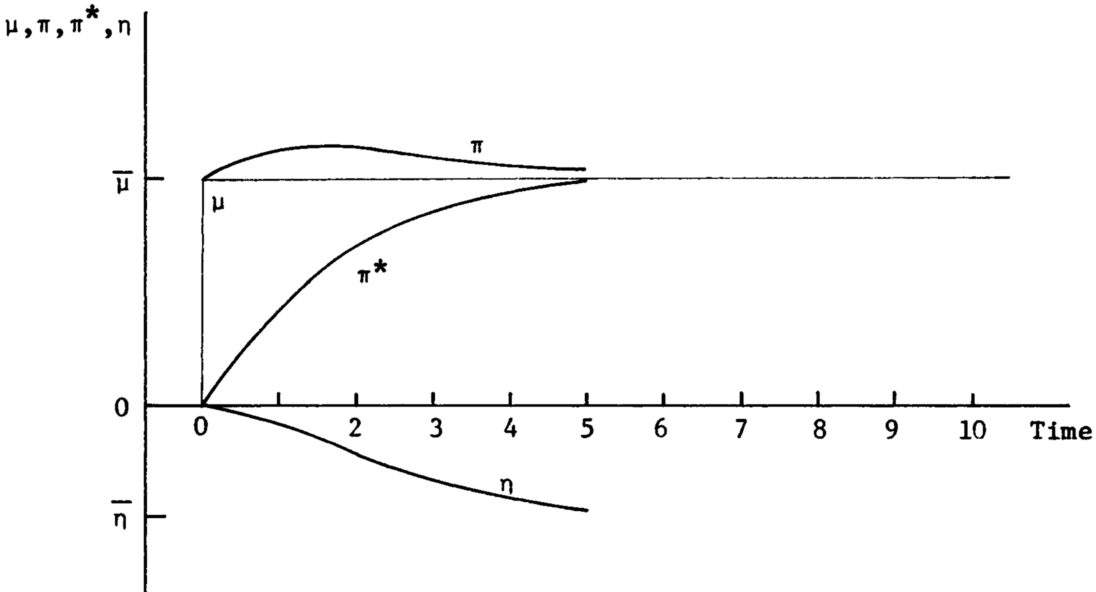

FIGURE 2. Cagan's Second-order Differential Equation Model

$\eta^d = -b\pi^*$	$b = .5$
$D\pi^* = c(\pi-\pi^*)$	$c = .5$
$D\eta = a(\eta^d-\eta)$	$a = 1.$

decline to its new and lower long-run position. In other words, the rate of inflation is initially below its new long-run value. In order for real cash balances to fall eventually, the rate of inflation must at some time rise above its long-run value and then settle onto this new equilibrium position. The minimum mathematical requirement for a model to generate this cyclical process is that it be a second-order differential equation model. The previous two models considered are first-order differential equations linear in the logarithm of real cash balances(η). Let us now analyze some linear second-order models.

III. SECOND-ORDER MODELS

Assume that the dynamic relationship between money and prices can be reduced to the following linear second-order nonhomogeneous differential equation:

(8) $D^2\eta + \rho D\eta + \sigma\eta = -\tau\mu + \phi D\mu$,

where D^2 denotes the second-order time derivative; ρ, σ, τ, and ϕ are constants. This model is stable if and only if ρ and σ are positive since ρ is the determinant of the corresponding Jacobian matrix of partial derivatives and σ is the negative of the trace of the Jacobian. The thesis that real cash balances are inversely related to the rate of inflation in the long run implies that τ is positive. The form of the solution depends upon whether the roots of the characteristic equation are real and distinct, real and equal, or complex.

An interesting exercise is to derive the corresponding cash balance equation in logarithmic form. By definition the following equations exist:

$$D\eta = \mu - \pi,$$

$$D^2\eta = D\mu - D\pi.$$

Substituting these equations into equation (8) and rearranging yields the following equation for the logarithm of real balances:

(9) $\eta = (\rho\pi + D\pi - (\rho+\tau)\mu)/\sigma + (\phi-1)D\mu/\sigma.$

When the model is stable, the quantity of real balances is positively related to the rate of inflation and the rate of change of the rate of inflation and is negatively related to the rate of growth of money. Since economists usually do not specify an explicit demand for real balances with this form, it is instructive to analyze some models that implicitly assume such an equation.

IV. MODEL III

The third model assumes that the logarithm of real cash balances is proportional to a weighted difference between the actual rate of inflation and the expected rate of inflation. The equations of the model[3/] are as follows:

(10) $\eta = (\pi-\beta\pi^*)/\alpha,$

$D\pi^* = \gamma(\pi-\pi^*).$

α and γ are assumed to be positive, and β is assumed to be greater than one so that in the long run real balances and the rate of anticipated inflation are inversely related.

Since the rate of growth of real balances equals the rate of growth of the quantity of money minus the rate of inflation, this model can be written as the following system of linear differential equations:

(11) $D\eta = -\alpha\eta - \beta\pi^* + \mu,$

$D\pi^* = \alpha\gamma\eta + (\beta-1)\gamma\pi^*.$

By taking the time derivative of equation (11) and using some simple substitutions to eliminate the expected rate of inflation term, this model can be expressed as a linear second-order differential equation:

$$D^2\eta + (\alpha+(1-\beta)\gamma)D\eta + \alpha\gamma\eta = (1-\beta)\gamma\mu + D\mu.$$

The stability condition therefore is that

$$\alpha + (1-\beta)\gamma > 0.$$

Equation (9) gives the functional form which relates real cash balances to the relevant observable variables.

It is interesting to compare this model with one Cagan proposed that includes a lag in forming expectations and a lag in adjusting actual levels of real cash balances to their desired levels. He uses two unobservable variables: an expected rate of inflation and the logarithm of "desired" real balances or η^d. The three basic equations of his model are as follows:

$$\eta^d = -b\pi^*,$$

$$D\pi^* = c(\pi-\pi^*),$$

$$D\eta = a(\eta^d-\eta),$$

where a, b, and c are positive constants. In a manner similar to our previous analysis, this model can be reduced to a system of two simultaneous differential equations. After substituting to remove the desired real cash balances and the actual rate of inflation terms, the model can be written as follows:

(12) $D\eta = -a\eta - ab\pi^*,$

$$D\pi^* = ac\eta + (ab-1)c\pi^* + c\mu.$$

The corresponding second-order differential equation is

$$D^2\eta + (a+c - abc)D\eta + ac\eta = -abc\mu.$$

This model is locally stable if and only if

$$a + c - abc > 0.$$

By using equation (12) and the definition of $D\eta$, one can obtain the following equation, which relates the logarithm of real balances to the expected rate of inflation:

$$\eta = (\pi-\mu)/a - b\pi^*.$$

The easiest way to get a feel for the difference between this model of

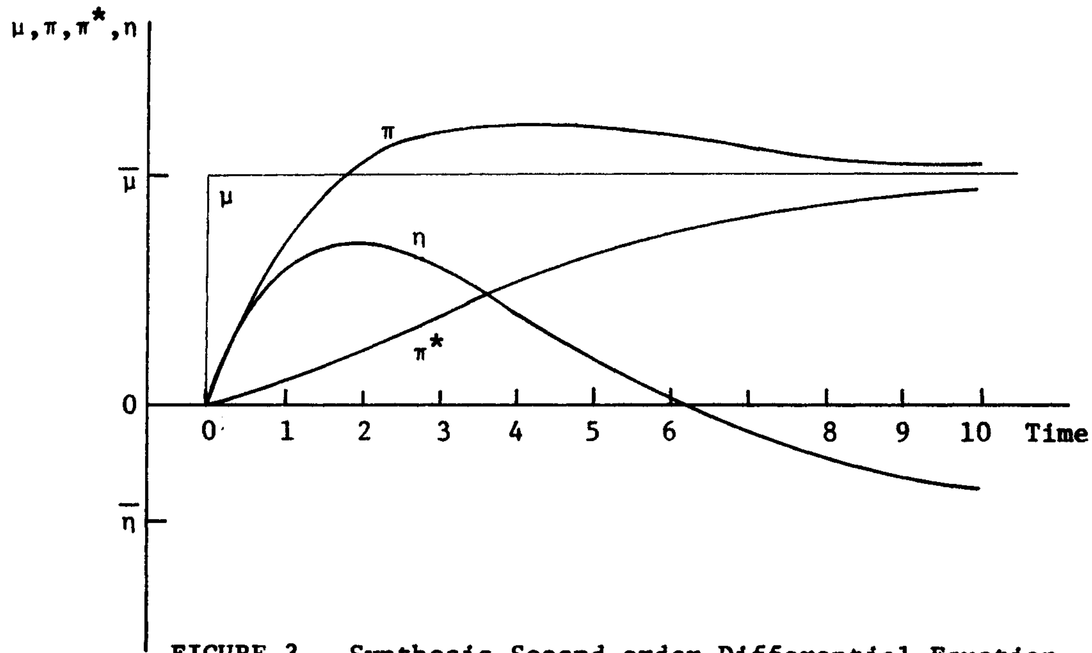

FIGURE 3. Synthesis Second-order Differential Equation Model (Roots Real and Equal)

$\eta = (\pi - \beta\pi^*)/\alpha$ $\alpha = 1.$ $\beta = 1.5$

$D\pi^* = \gamma(\pi - \pi^*)$ $\gamma = 10 - 4\sqrt{6}$

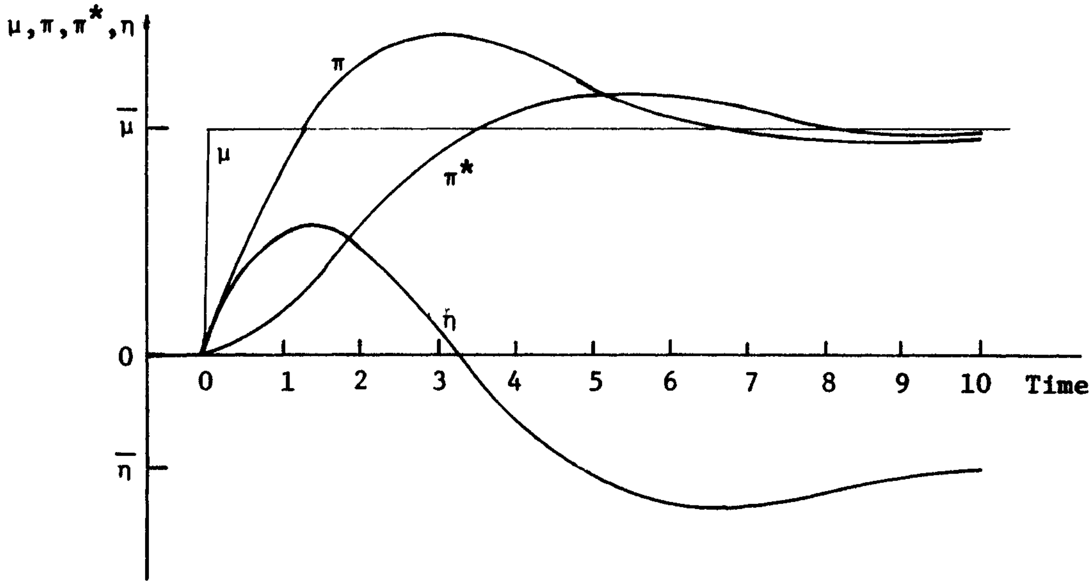

FIGURE 4. Synthesis Second-order Differential Equation Model (Roots Complex Conjugates)

$\eta = (\pi - \beta\pi^*)/\alpha$ $\alpha = 1.$ $\beta = 1.5$

$D\pi^* = \gamma(\pi - \pi^*)$ $\gamma = .5$

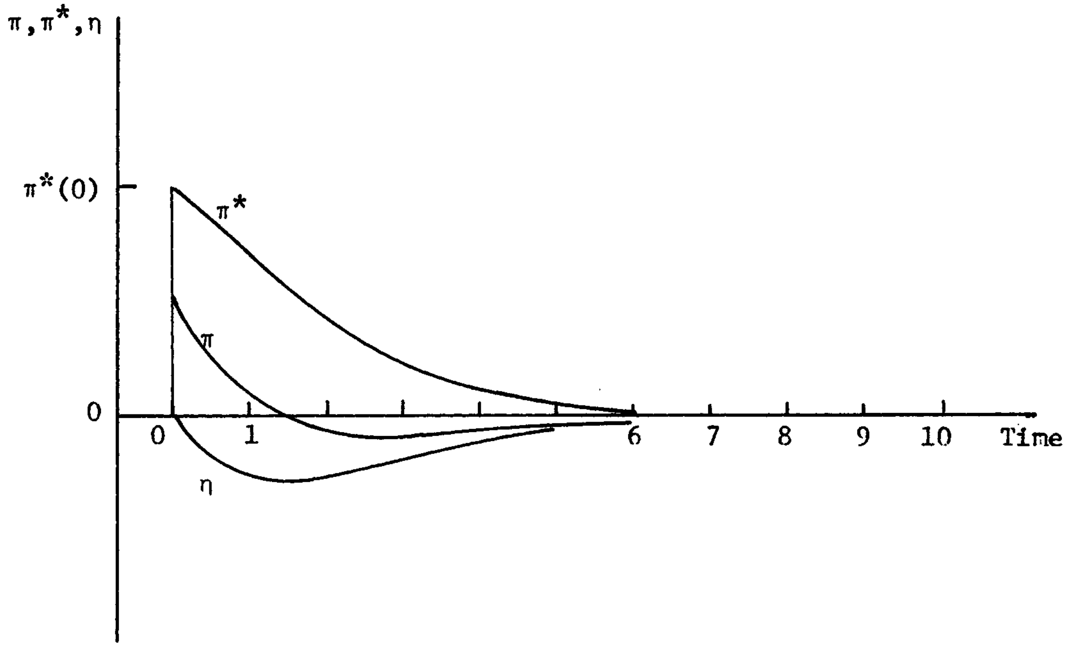

FIGURE 5. Cagan's Second-order Differential Equation Model

$\eta^d = -b\pi^*$ $\quad b = .5$

$D\pi^* = c(\pi-\pi^*)$ $\quad c = .5$

$D\eta = a(\eta^d-\eta)$ $\quad a = 1.$

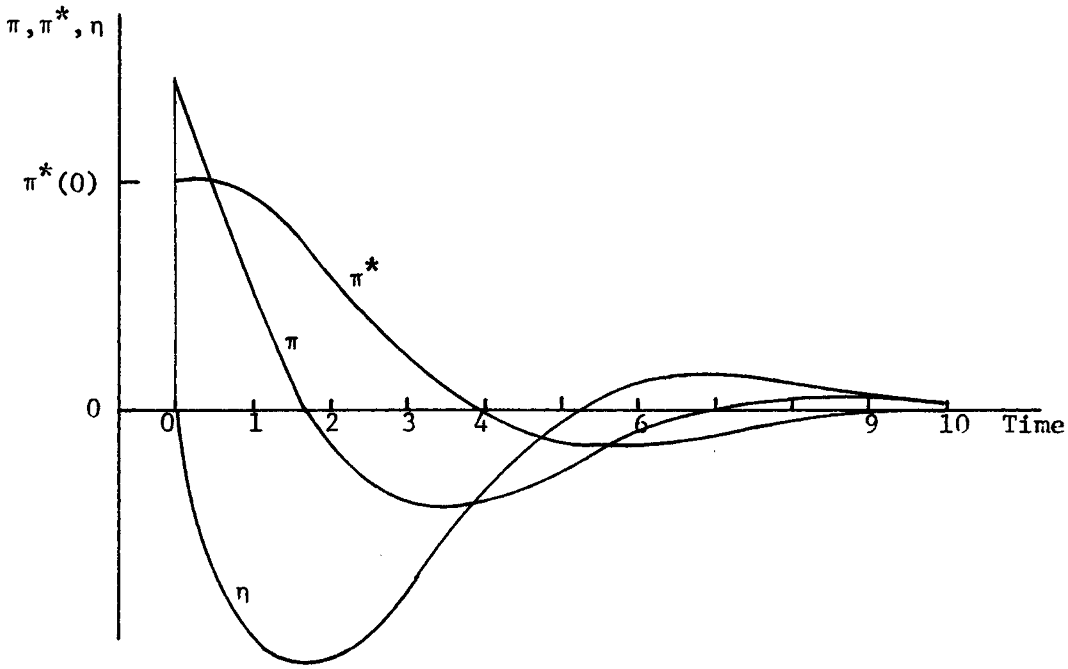

FIGURE 6. Synthesis Second-order Differential Equation Model

$\eta = (\pi-\beta\pi^*)/\alpha$ $\quad \alpha = 1.$ $\quad \beta = 1.5$

$D\pi^* = \gamma(\pi-\pi^*)$ $\quad \gamma = .5$

Cagan and of this paper is to compare their response paths following some disturbance. Figures 2-4 plot the time paths of the actual and expected rates of inflation and real cash balances after the nominal quantity of money begins to increase at some constant rate. In these simulations the initial expected rate of inflation is assumed to be equal to zero. Figures 5-6 plot the time paths generated when the quantity of money remains constant and the initial expected rate of inflation is equal to some positive value. Since the present model can have real or complex roots, two sets of parameters are simulated and plotted in the first experiment. Cagan's model only can have real roots when the constants are assumed to be positive real numbers.

V. INTEREST RATES

Up to this point we have concentrated on the dynamic relationship between the quantity of money and a price index. The model will not be extended via a combination of traditional arguments to include a macroscopic analysis of an interest rate. A nominal market rate of interest (R) is assumed to be equal to the sum of a natural rate of interest (R*) and a reaction index (r):

$$R = R^* + r,$$

where the natural rate of interest is constant and reflects an average rate of time preference. The reaction index is assumed to be a function of the expected rate of inflation and real cash balances. It is a proxy variable used to synthesize the arguments of Keynes [7] and Fisher [3].4/

According to Keynes' liquidity-preference function, real balances and a composite interest rate are inversely related. Since we have normalized the initial quantity of real balances to be equal to one, a simple liquidity-preference function can be expressed as follows:

$$r^* = -\zeta\eta,$$

where r^* is a Keynesian reaction index. If there is a sustained period of a constant rate of inflation, one would expect this reaction index to equal the rate of inflation. In terms of our third model this implies that

$$\zeta = \alpha/(\beta-1).$$

Fisher's reaction index in a continuous time model is the expected rate of inflation π^*. The synthesis reaction index is assumed to be a weighted average of these two alternative indices:

$$r = \delta\pi^* + (1-\delta)r^*,$$

where δ is a positive constant.

Figure 7 plots the actual and expected rate of inflation, the Keynesian reaction index, and the synthesis index in a simulation where the nominal quantity of money begins to increase at some constant rate. The difference between the actual rate of inflation and the synthesis index is plotted to show the time path of the "real" rate of interest. There is an initial and short-lived decrease in the nominal market rate of interest. The real rate of interest decreases for a longer duration before it eventually increases, overshoots, and then returns to its long-run position.

In this simple experiment it is easy to see how unexpected inflation can redistribute income. When the real rate of interest is negative, lenders are subsidizing borrowers and when it is positive, borrowers are subsidizing lenders. If such a monetary policy were conducted, the persons with better foresight would initially borrow and later lend money so as to maximize their effective subsidy. The fiscal analogue of this monetary policy is a tax on those persons with "poorer" foresight and a transfer payment to those persons who have better foresight or "inside" information.

VI. AN ALTERNATIVE INTEREST RATE

An implicit assumption of the previous method of introducing an interest rate variable is that the fundamental relationship between money and a price

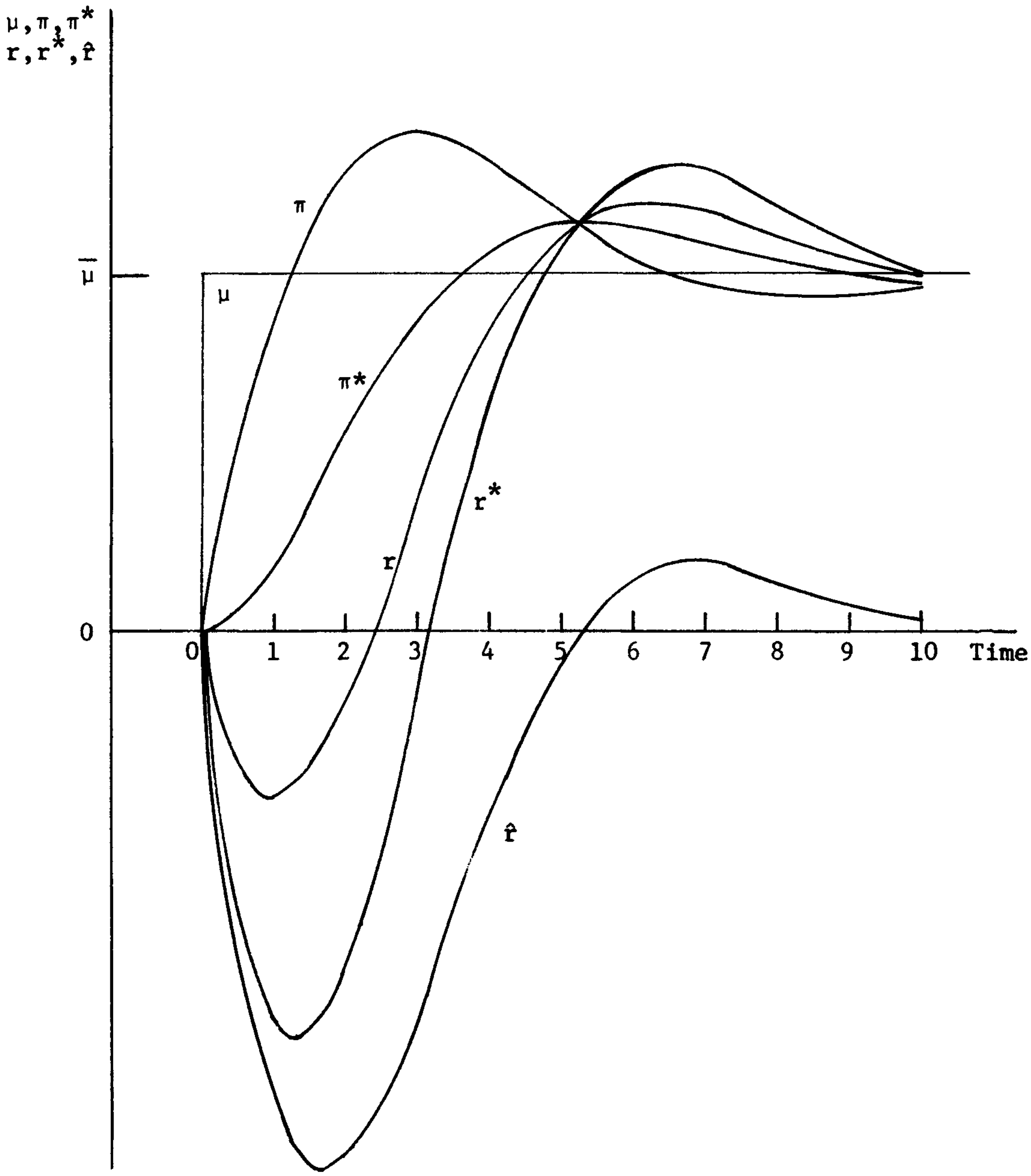

FIGURE 7. Synthesis Second-order Differential Equation Model

$\eta = (\pi - \beta\pi^*)/\alpha$ $\alpha = 1.$

$D\pi^* = \gamma(\pi - \pi^*)$ $\beta = 1.5$

$r^* = \alpha\eta/(1-\beta)$ $\gamma = \delta = .5$

$r = \delta\pi^* + (1-\delta)r^*$

$\hat{r} = r - \pi$

index is unaltered when borrowing and lending is allowed. The coefficients of such an extended model may change, but the dynamic function between money and prices is assumed to remain a linear second-order differential equation. An alternative approach is to allow this fundamental relationship to change after interest rates are introduced. This change may be of the form of introducing a nonlinear relationship and/or a different order differential equation. An example of the latter possibility is the following model:

$$\eta = (\pi-\beta\pi^{*}-\chi r)/\alpha,$$

$$D\pi^{*} = \gamma(\pi-\pi^{*}),$$

$$\tilde{r} = \pi^{*} - \psi D\eta,$$

$$Dr = \omega(\tilde{r}-r),$$

where ψ and ω are positive constants. $\tilde{r}$ is a reaction index of an actual or a hypothetical interest rate that is instantaneously decreased by an increase in the rate of growth of money. If money is introduced via an open-market operation, this interest rate may be a proxy for the federal funds rate. r is the reaction index for an interest rate of a longer-term bond. This model can be rearranged into the following system of three linear differential equations:

$$D\ln P = -\alpha\ln P + \beta\pi^{*} + \chi r + \alpha\ln M,$$

$$D\pi^{*} = -\alpha\gamma\ln P + \gamma(\beta-1)\pi^{*} + \gamma\chi r + \alpha\gamma\ln M,$$

$$Dr = -\alpha\psi\omega\ln P + \omega(1+\beta\psi)\pi^{*} + \omega(\chi\psi-1)r + \psi\omega(\alpha\ln M-D\ln M).$$

Representative simulations of this type of model are presented in the sections on variable output and disequilibrium models.5/

VII. AUTOREGRESSIVE FORM

When the rate of growth of money is constant, it is possible to derive the corresponding second-order autoregressive equation where the current value of the logarithm of real balances is expressed as a function of real balances in two previous and equally spaced time periods. Suppose that our initial model has real and distinct roots so that the solution equation is the following form:

$$\eta(t) = c_1 \exp(m_1 t) + c_2 \exp(m_2 t) + (1-\beta)\bar{\mu}/\alpha.$$

c_1 and c_2 are constants of integration and depend upon the initial conditions. m_1 and m_2 are the roots of the characteristic equation

$$m^2 + (\alpha+(1-\beta)\gamma)m + \alpha\gamma = 0.$$

Therefore

$$m_1 = p + q,$$

$$m_2 = p - q,$$

where

$$p = -(\alpha+(1-\beta)\gamma)/2,$$

$$q = \sqrt{p^2 - \alpha\gamma}.$$

The values of η in the time periods $t-\theta$ and $t-2\theta$ are given by the following equations:

$$\eta(t-\theta) = c_1 \exp(m_1(t-\theta)) + c_2 \exp(m_2(t-\theta)) + (1-\beta)\bar{\mu}/\alpha,$$

$$\eta(t-2\theta) = c_1 \exp(m_1(t-2\theta)) + c_2 \exp(m_2(t-2\theta)) + (1-\beta)\bar{\mu}/\alpha.$$

The autoregressive equation is

$$\eta(t) = \phi_0 + \phi_1 \eta(t-\theta) + \phi_2 \eta(t-2\theta),$$

where

$$(13) \quad 1 = \phi_1 \exp(-m_1\theta) + \phi_2 \exp(-2m_1\theta),$$

$$(14) \quad 1 = \phi_1 \exp(-m_2\theta) + \phi_2 \exp(-2m_2\theta),$$

$$\phi_0 = (1-\phi_1-\phi_2)(1-\beta)\bar{\mu}/\alpha.$$

Solving equations (13) and (14) and substituting in the functions for the roots yields the following equations for the two autoregressive parameters:

$$\phi_1 = 2 \cosh(q\theta)\exp(-p\theta),$$

$$\phi_2 = -\exp(2p\theta),$$

where cosh is the hyperbolic cosine. When the roots are real and equal, the first autoregressive term becomes

$$\phi_1 = 2\exp(-p\theta),$$

and when the roots are complex,

$$\phi_1 = 2\cos(q\theta)\exp(-p\theta).$$

Notice that the intercept term is proportional to the constant rate of growth of nominal balances.

A similar exercise is to determine the corresponding equation which relates the current value of real balances to the current rate of interest and a lagged real balances term. Again we assume that nominal balances are increasing at a constant rate and that the model is initially off its long-run position. In this case the second-order differential equation with interest rates included does not generate such simple equations for the first lag coefficient and the interest rate coefficient. Consider the case where the model has real and equal roots. When initial real balances equal one, the initial and expected rates of inflation equal zero, and nominal balances are beginning to increase at a constant rate $\bar{\mu}$, the corresponding equation can be derived (see Appendix A):

$$\eta(t) = \phi_0 + \phi_1\eta(t-\theta) + \phi_2 r(t).$$

A numerical example of these coefficients yields the following equations:

$$\phi_2 = -2\theta/(1-(\sqrt{6}-4)\theta),$$

$$\phi_1 = \exp((2 - \sqrt{6})\theta)/(1-(\sqrt{6} - 4)\theta),$$

$$\phi_0 = -((1-\phi_1)/2+\phi_2)\bar{\mu},$$

when $\alpha = 1.0$, $\beta = 1.5$, $\gamma = 10-4\sqrt{6}$, and $\delta = .75$.

With the appropriate amount of cranking, these coefficients can be determined for the cases where there are real and distinct roots or complex roots and where the coefficients and initial conditions are different. What happens to these relationships when stochastic elements are introduced and the rate of change of the quantity of money is allowed to vary could be an interesting area for further study.

VIII. VARIABLE OUTPUT

One of our initial assumptions was that the level of real output remains constant following a change in nominal balances. This assumption can be relaxed by introducing the accelerationist theory of unemployment as developed by Friedman [4] and Phelps [10]. The rate of unemployment is assumed to be inversely related to the difference between the actual and expected rate of inflation. Real output is assumed to be inversely related to the rate of unemployment. Therefore, real output is above (below) its long-run level whenever the actual rate of inflation is above (below) the expected rate of inflation. Furthermore this relationship is assumed to be nonlinear.[6/] If the expected rate of inflation is zero, then a given rate of inflation will increase real output by a smaller amount that an equal rate of deflation will decrease it.

One possible equation[7/] that characterizes this relationship is the following:

$$\ln q \equiv \ln(Q/V_o) = \iota(\kappa^{-1}-(\kappa+\pi-\pi^*)^{-1}),$$

where Q is the level of real output, V_o is the income velocity when the price index is constant, q is the ratio of Q divided by V_o. ι and κ are positive constants. The cash-balance equation is now assumed to have the following form:

$$\eta \equiv \ln(M/P) = \ln(Q/V_o) + (\pi-\beta\pi^*-\chi r)/\alpha,$$

or

$$\eta = \iota(\kappa^{-1}-(\kappa+\pi-\pi^*)^{-1}) + (\pi-\beta\pi^*-\chi r)/\alpha.$$

This model has a nonlinear differential equation for the rate of inflation which can be expressed only in implicit form. When this equation is combined with differential equations for the expected rate of inflation and the interest rate, the solution can be approximated numerically, using a combination of Newton's and Runge-Kutta's algorithms.[8/]

Figures 8 and 9 plot a simulation run when nominal balances begin to increase at a constant rate in the third-order differential equation model with variable output. By comparing figures 7 and 8, one observes that the primary effect of allowing output to vary in such a prescribed manner is to lengthen the period of adjustment. In figure 9 the synthesis interest rate equation of our second-order differential equation model is plotted for purposes of comparison. Notice the timing of the first turning points for the alternative interest rates, output, and the rate of inflation. The first impact of increasing nominal balances is to reduce interest rates for a relatively short period of time. Next, real output reaches a peak when the difference between the actual and expected rates of inflation is at a maximum. Finally, the rate of inflation attains its maximum level and then begins to converge downwards onto its long-run value.

IX. DISEQUILIBRIUM APPROACH

A parallel way of looking at the dynamics of a monetary growth model is the disequilibrium apprach, where the rate of inflation is a function of the excess demand for goods and the expected rate of inflation. This excess demand is the difference between "planned" investment and "planned" savings. Actual investment usually is assumed to be a weighted average of planned investment and savings. These hypothetical variables are assumed to be functions of real balances, an interest rate and/or the expected rate of inflation when the real output and quantities of capital

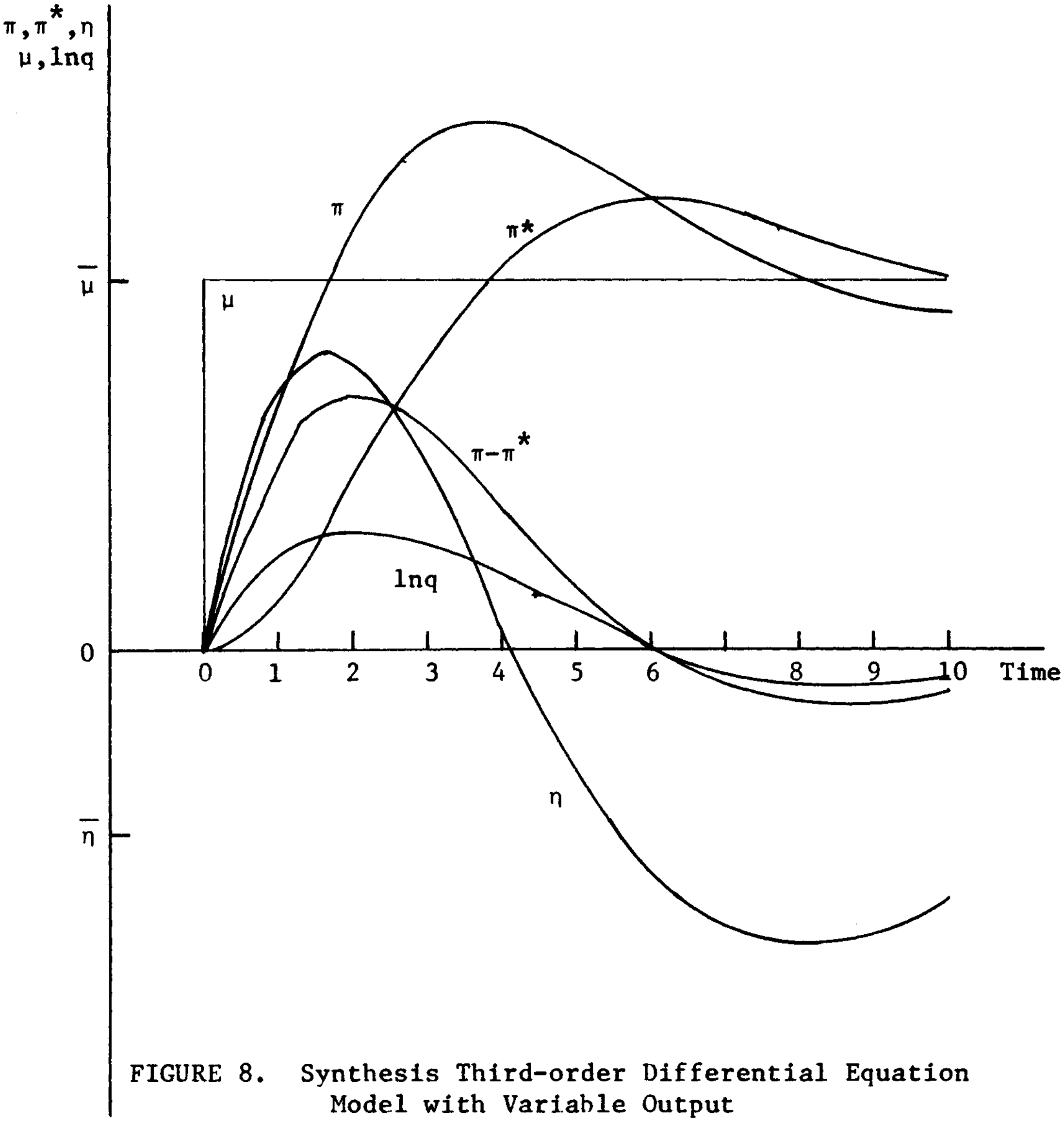

FIGURE 8. Synthesis Third-order Differential Equation Model with Variable Output

$\eta = \ln q + (\pi - \beta\pi^* - \chi r)/\alpha$	$\alpha = 1.0$
$\ln q = \iota(\kappa^{-1} - (\kappa + \pi - \pi^*)^{-1})$	$\beta = 1.5$
$D\pi^* = \gamma(\pi - \pi^*)$	$\gamma = \delta = \psi = \omega = .5$
$\tilde{r} = \pi^* - \psi D\eta$	$\iota = .015$
$Dr = \omega(\tilde{r} - r)$	$\kappa = .15$
	$\chi = 0.$
	$\bar{\mu} = .1$

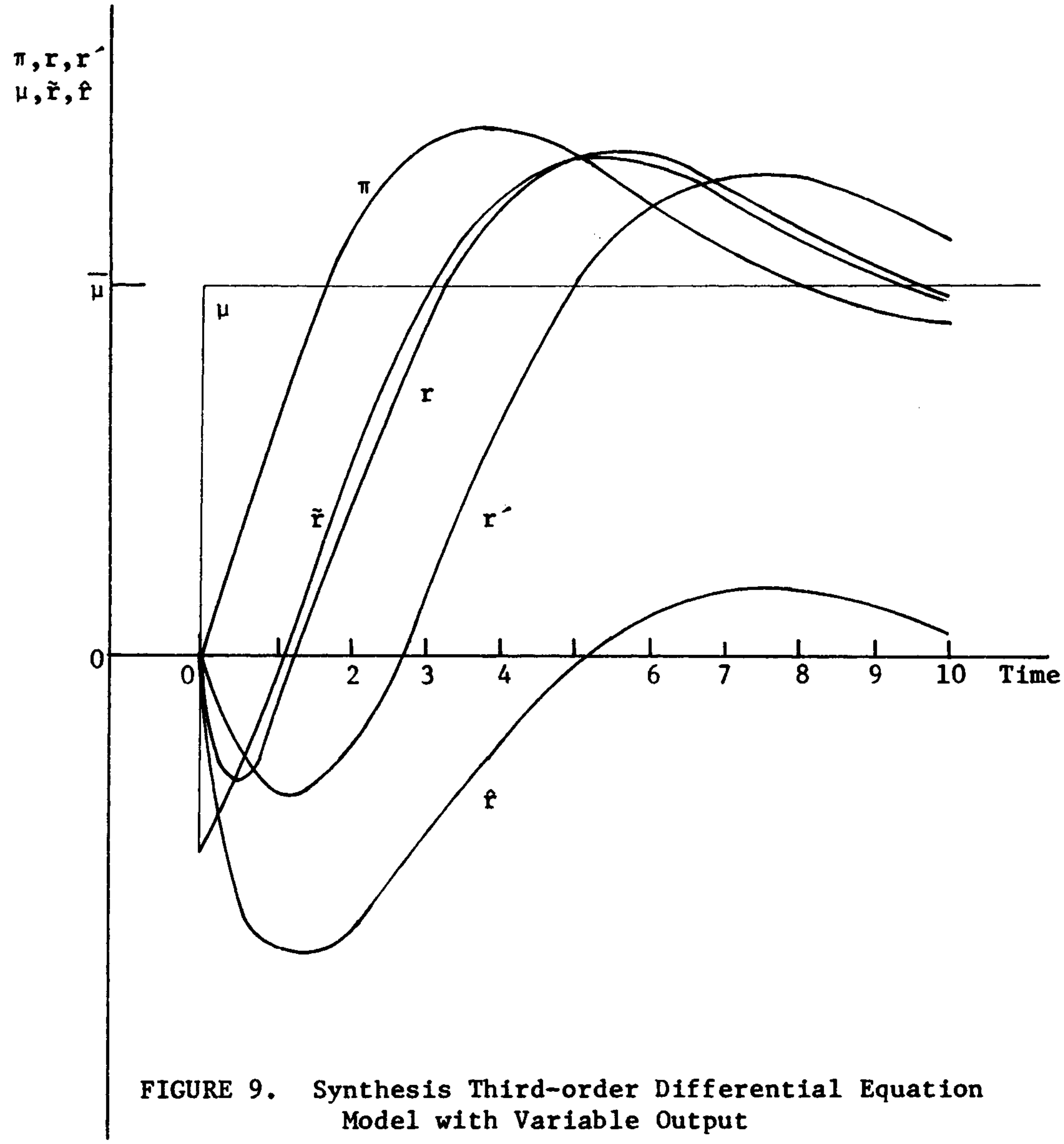

FIGURE 9. Synthesis Third-order Differential Equation Model with Variable Output

$\eta = \ln q + (\pi - \beta\pi^* - \chi r)/\alpha$	$\alpha = 1.0$
$\ln q = \iota(\kappa^{-1} - (\kappa + \pi - \pi^*)^{-1})$	$\beta = 1.5$
$D\pi^* = \gamma(\pi - \pi^*)$	$\gamma = \delta = \psi = \omega = .5$
$\tilde{r} = \pi^* - \psi D\eta$	$\iota = .015$
$Dr = \omega(\tilde{r} - r)$	$\kappa = .15$
$\hat{r} = r - \pi$	$\chi = 0.$
$r^* = \alpha\eta/(1-\beta)$	$\bar{\mu} = .1$
$r' = \delta\pi^* + (1-\delta)r^*$	

and labor are constant. References to these growth models can be found in the works by J. Stein, [11], Burmeister and Dobell [1] , Urrutia [12], and L. Johnson [6].

To see how this method is equivalent to the previous approach, one need only derive the corresponding disequilibrium model. Suppose the cash-balances equation model is the following:

(15) $\eta \equiv \ln(M/P) = (\pi-\beta\pi^{*}-\chi r)/\alpha.$

The market rate of interest is assumed to equal a natural rate plus a reaction index:

(16) $R = R^{*} + r.$

These equations can be combined into the following differential equation for the logarithm of the price index:

$$\pi \equiv D\ln P = \alpha\ln M - \alpha\ln P + \beta\pi^{*} + \chi R - \chi R^{*}$$

or

$$\pi = \alpha\eta + \beta\pi^{*} + \chi r.$$

A representative disequilibrium form for this differential equation is

$$\pi = \lambda(I^{*}(\cdot)-S^{*}(\cdot)) + \pi^{*},$$

where λ is a positive parameter. $I^{*}(\cdot)$ and $S^{*}(\cdot)$ are the unobservable functions for planned investment and savings, respectively. The actual level of investment is assumed to equal zero and is a weighted average of the planned levels of investment and savings:

$$I = aI^{*}(\cdot) + (1-a)S^{*}(\cdot) = 0,$$

where a is a nonnegative constant that is less than one. Simple substitutions yield the following equations:

(17) $S^{*} = -a(\alpha\eta + (\beta-1)\pi^{*} + \chi R - \chi R^{*})/\lambda,$

(18) $I^{*} = (1-a)(\alpha\eta + (\beta-1)\pi^{*} + \chi R - \chi R^{*})/\lambda.$

The usual assumption that the partial derivative of planned savings with respect to the interest rate is positive and the partial derivative of planned investment with respect to the interest rate is negative implies that χ is less than zero. This assumption means that the partial derivative of real balances with respect to the interest rate is positive. In order to get an idea of how the sign of χ affects the dynamic properties of such a model, several simulations were run using the third-order model with a constant level of output. One simulation has a negative value for χ; the other simulation uses a positive value. The coefficient of the price expectations variable is adjusted so that the simulations have the same long-run, or particular, solutions. The dynamic paths of the rate of inflation and the reaction index for the rate of interest are plotted in Figure 10.

While these simulations show the sign of the interest rate coefficient is not necessarily of particular importance in determining the dynamic response of a model, most disequilibrium models specify that planned savings is positively related to the interest rate, while real balances and planned investment are negatively related to the interest rate. In order for all these conditions to be satisfied, it is necessary to impose an explicit relationship between real balances and the rate of interest. This means that the partial derivative of planned savings and investment with respect to the interest rate cannot treat real balances and the interest rate as independent variables. For example, the partial derivative of planned savings in equation becomes

$$\frac{\partial S^*}{\partial R} = -a(\alpha\frac{\partial \eta}{\partial R} + \chi)/\lambda.$$

The explicit relationship between real balances and the interest rate can be used to express planned savings and investment as functions of only

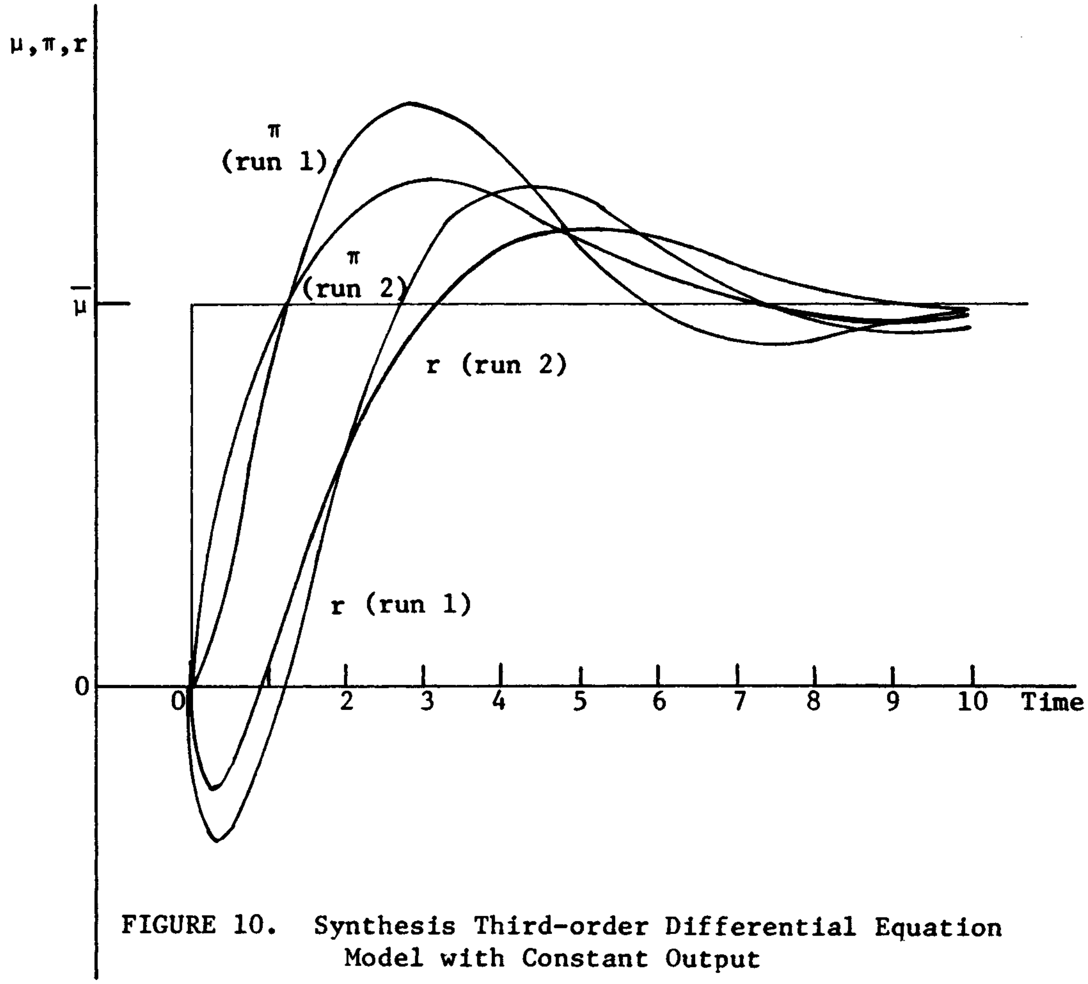

FIGURE 10. Synthesis Third-order Differential Equation Model with Constant Output

$$\eta = (\pi-\beta\pi^{*}-\chi r)/\alpha \qquad \alpha = 1.0$$

$$D\pi^{*} = \gamma(\pi-\pi^{*}) \qquad \gamma = \omega = .5$$

$$\tilde{r} = \pi^{*} - \psi D\eta$$

$$Dr = \omega(\tilde{r}-r)$$

Run 1. $\beta = 1.0$ $\chi = .5$

Run 2. $\beta = 2.0$ $\chi = -.5$

the actual and expected rates of inflation. Substituting equations (15) and (16) into equations (17) and (18) gives the following equations for planned savings and investment:

$$S^* = -a(\pi-\pi^*)/\lambda,$$

$$I^* = (1-a)(\pi-\pi^*)/\lambda,$$

This makes the disequilibrium form for the differential equation of the price index a tautology.

Let us now consider the more general disequilibrium model where we begin by specifying the following functions for planned savings and investment:

$$S^* = f(M/P,\ R,\ \pi^*),$$

$$I^* = g(M/P,\ R,\ \pi^*),$$

where the level of output is assumed to remain constant. Next, one makes the initial assumption that real balances and the interest rate are independent variables and that the partial derivatives have the following signs:

$$\frac{\partial S^*}{\partial(M/P)} < 0, \qquad \frac{\partial S^*}{\partial R} > 0.$$

$$\frac{\partial I^*}{\partial(M/P)} > 0, \qquad \frac{\partial I^*}{\partial R} < 0.$$

By the implicit-function rule these assumptions imply that

$$\frac{\partial(M/P)}{\partial R} > 0.$$

In order to avoid this conclusion, we impose the restriction that real balances and the interest rate are dependent and inversely related. A representative restriction is the following explicit function for the rate of interest:

$$R = h(M/P),$$

where the derivative is assumed to be negative, or

$$\frac{dR}{d(M/P)} < 0.$$

The differential equation for the rate of inflation can now be expressed as

$$\pi = \lambda(g(M/P,\ h(M/P),\ \pi^*) - f(M/P,\ h(M/P),\ \pi^*)) + \pi^*$$

or

$$\pi = F(M/P, \pi^*)$$

A semi-logarithmic approximation of this general function is

$$\eta \equiv \ln(M/P) = (\pi - \beta\pi^*)/\alpha,$$

which is the cash-balance equation (10) of the third model. The semi-logarithmic approximation of the interest rate equation is the Keynesian version of the synthesis second-order differential equation model.

X. CONCLUSION

This paper has analyzed a series of dynamic macroeconomic models designed to reflect an inverse relationship between the rate of inflation and real balances in the long run. These models were presented as systems of ordinary differential equations. The particular solutions of differential equation models correspond to the economic concept of the long run. Each of the alternative models has been simulated using parameter values that ensure stability. This enables one to study the subtle differences in the quantitative responses of dynamic models that have the same qualitative properties. The synthesis models proposed are shown to be compatible with a broad spectrum of economic models. Some possible areas for future studies would be to relax some of the initial and very restrictive assumptions, to develop a microeconomic framework for the synthesis models with an interest rate included, and to analyze the stochastic behavior of the models using different monetary policies.

FOOTNOTES

*I am grateful for comments and suggestions offered by my colleagues at the Federal Reserve Bank of Chicago and by participants at a talk given to the Special Studies Section of the Board of Governors. Bob Laurent, Vince Snowberger, and my brother Mike provided challenging discussions that influenced my research. The motivation for this paper stems from some of the difficult and fundamental questions raised by my former teacher Nicholas Schrock. All biases and errors are, of course, mine.

1/ An alternative model of such a perfectly perfect world is

$$\eta = -\alpha\mu.$$

For a discussion of this type of model, see Stein [11].

2/ A variation of this model is

$$\eta = -\beta\mu^*,$$

$$D\mu^* = \gamma(\mu-\mu^*)$$

where β and γ are assumed to be positive. μ^* is the expected rate of change of nominal balances and is used as a proxy for the expected rate of inflation. In a manner similar to that used in analyzing Cagan's model, real balances can be expressed as a function of the actual rate of inflation and the rate of change of nominal balances.

$$\eta = (\pi-(1+\beta\gamma)\mu)/\gamma.$$

This model is stable when γ is positive.

3/ An extension of this model is the following cash-balances equation:

$$\eta = (\pi-\beta\pi^*-\beta^*\mu)/\alpha,$$

where β^* is a positive constant. This model implies that an increase in the rate of growth of nominal balances instantaneously causes an increase in the rate of inflation.

4/ It is often difficult, if not impossible, to represent the logical patterns in the arguments of Keynes and Fisher by elementary functions. The equations used should be viewed as stylized constructs of broader theories.

5/ When testing feedback rules where the rate of growth of money is a function of the rates of inflation and/or the interest rates, a fourth differential equation must be added to this type of model. The values of the parameters in the feedback rule should be selected so as to maintain the stability of the model.

Another way of extending this model is to specify a production function, where the level of real output is related to the capital stock, and an investment function that is the differential equation for capital. Care should be exercised when formulating the investment function since it will include the implicit assumption about the neutrality or nonneutrality of money. A "neutral" model is considered to be one in which the long-run capital stock is independent of the rate of growth of nominal balances.

6/ A linear relationship between real output and the difference between the actual and expected rates of inflation may be simulated by just changing the parameters of the second-order model. If

$$\ln q \equiv \ln(Q/V_o) = \iota(\pi-\pi^*)$$

and

$$\eta \equiv \ln(M/P) = \ln q + (\pi-\beta\pi^*)/\alpha,$$

then

$$\eta = ((1+\iota\alpha)\pi - (\beta+\iota\alpha)\pi^*)/\alpha,$$

or

$$\eta = (\pi-\beta^*\pi^*)/\alpha^*$$

where

$$\alpha^* = \alpha/(1+\iota\alpha),$$

$$\beta^* = (\beta+\iota\alpha)/(1+\iota\alpha).$$

7/ This equation was derived from a translated rectangular hyperbola. Assume that the nonlinear relationship between lnq and the difference between the actual and expected rates of inflation is the following:

$$(\kappa+\pi-\pi^*)(\ln q - \bar{Q}) = -\iota,$$

where $-\kappa$ and $\bar{Q}$ are the asymptotes. The model is normalized so that lnq equals zero when π equals π^*. Therefore,

$$\bar{Q} = \iota/\kappa,$$

and

$$\ln q = \iota(\kappa^{-1} - (\kappa+\pi-\pi^*)^{-1})$$

8/ The implicit differential equation is evaluated using Newton's method for solving a nonlinear equation. The system of simultaneous differential equations is numerically integrated using Gill's modification of a fourth-order Runge-Kutta's algorithm. Variable step integration was used with an absolute error criterion of exp(-8). The subroutines used were XCNLSB and XCRKGM in the CDC Library of Mathematical Subprograms (publication number 86614900).

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APPENDIX A

Structural equations of the model:

(1) $\eta = (\pi-\beta\pi^*)/\alpha$,
(2) $D\pi^* = \gamma(\pi-\pi^*)$,
(3) $r^* = \alpha\eta/(1-\beta)$,
(4) $r = \delta\pi^* + (1-\delta)r^*$,

where

$\eta \equiv \ln(M/P)$,
$\pi \equiv D\ln P$,
$\mu \equiv D\ln M$,
$R = R^* + r$.

α, β, γ, δ, R^*, and μ are assumed to be constant.

Initial conditions:

$$\eta(0) = \pi^*(0) = r^*(0) = r(0) = 0.$$

By the general rules of differentiation:

$$D\eta = \mu - \pi,$$

$$D^2\eta = D\mu - D\pi = -D\pi.$$

Model as a system of differential equations:

$$D\eta = -\alpha\eta - \beta\pi^* + \mu,$$
$$D\pi^* = \alpha\gamma\eta + (\beta-1)\gamma\pi^*.$$

Two corresponding second-order differential equations:

$$D^2\eta + aD\eta + b\eta = (1-\beta)\gamma\mu,$$

$$D^2\pi^* + aD\pi^* + b\pi^* = \alpha\gamma\mu,$$

where

$$a = \alpha + (1-\beta)\gamma,$$

$$b = \alpha\gamma.$$

Stability condition when α, β, and γ are positive:

$$a > 0.$$

CASE I: ROOTS REAL AND EQUAL

When roots are real and equal,

$$a^2 = 4b,$$

$$p = -a/2,$$

where p is the common root.

Solutions of the nonhomogeneous second-order differential equations:

(5) $\eta(t) = A_1 \exp(pt) + A_2 t \exp(pt) + (1-\beta)\mu/\alpha$,

(6) $\pi^*(t) = B_1 \exp(pt) + B_2 t \exp(pt) + \mu$.

Constants of integration given the initial conditions:

$A_1 = (\beta-1)\mu/\alpha$,

$A_2 = \mu - pA_1$,

$B_1 = -\mu$,

$B_2 = p\mu$.

From equations (1), (5), and (6):

$\pi = \alpha\eta + \beta\pi^*$

(7) $\pi(t) = C_1 \exp(pt) + C_2 t \exp(pt) + \mu$,

where

$C_1 = -\mu$,

$C_2 = (\alpha+p)\mu$.

From equations (1), (3), and (4)

(8) $r = \varepsilon\pi + (1-\varepsilon)\pi^*$,

where

$\varepsilon = (1-\delta)/(1-\beta)$.

Solution for interest rate's reaction index from equations (6), (7), and (8):

$r(t) = D_1 \exp(pt) + D_2 t \exp(pt) + \mu$,

where

$D_1 = -\mu$,

$D_2 = (\alpha\varepsilon+p)\mu$.

Lagged value of η:

(9) $\eta(t-\theta) = E_1 \exp(pt) + E_2 t \exp(pt) + (1-\beta)\mu/\alpha$,

where

$E_1 = (A_1-\theta A_2)\exp(-p\theta)$,

$E_2 = A_2 \exp(-p\theta)$.

$\eta(t)$ as a function of $r(t)$ and $\eta(t-\theta)$:

$\eta(t) = \phi_o + \phi_1\eta(t-\theta) + \phi_2 r(t)$, where

(10) $A_1 = \phi_1 E_1 + \phi_2 D_1$,

(11) $A_2 = \phi_1 E_2 + \phi_2 D_2$,

$\phi_o = ((1-\beta)(1-\phi_1)/\alpha-\phi_2)\mu$.

Solve equations (10) and (11) using Cramer's rule:

Let $\Delta = E_1 D_2 - D_1 E_2$

$= ((X-\theta(1-pX))(\alpha\varepsilon+p) + (1-pX))\mu^2 \exp(-p\theta)$,

where

$X = (\beta-1)/\alpha$,

$\phi 1 = (A_1 D_2 - D_1 A_2)/\Delta$

$= ((\beta-1)\varepsilon+1)\mu^2/\Delta$,

$\phi_2 = (E_1 A_2 - A_1 E_2)/\Delta$

$= -\theta(1-pX)^2 \mu^2 \exp(-p\theta)/\Delta$.

When α=1, β=1.5, δ=.75 and the roots are real and equal:

$\gamma = 10-4\sqrt{6}$,

$\varepsilon = -.5$,

$p = 2-\sqrt{6}$,

$X = .5$,

$\Delta = -.75((\sqrt{6}-4)\theta-1)\mu^2 \exp(-p\theta)$,

$\phi_2 = -1.5\theta\mu^2 \exp(-p\theta)/\Delta$,

$= 2\theta/((\sqrt{6}-4)\theta-1)$,

$\phi_1 = .75\ \mu^2/\Delta$

$= -\exp(p\theta)/((\sqrt{6}-4)\theta-1)$

$= -\phi_2 \exp(2-\sqrt{6}\theta)/2\theta$,

$\phi_o = -((1-\phi_1)/2+\phi_2)\mu$.