Federal Reserve Bank of Chicago, Staff Memoranda.

A Series of Occasional Papers in Draft Form Prepared by Members of the Research Department for Review and Comment.

75-3

Lissajous Figures and Cyclical Fluctuations in Unemployment and Inflation

Thomas A. Gittings

Federal Reserve Bank of Chicago

STAFF MEMORANDA

Research Paper No. 75-3

LISSAJOUS FIGURES AND CYCLICAL FLUCTUATIONS IN UNEMPLOYMENT AND INFLATION

By

Thomas A. Gittings

Department of Reseach

Federal Reserve Bank of Chicago

1. INTRODUCTION

Fifty years ago Irving Fisher [4] obtained a high correlation between the rates of inflation and unemployment. Working with data for the United States economy, he observed that major fluctuations in the rate of change of the price level "foreshadowed" fluctuations in the level of employment. Employment, in turn, had a strong inverse correlation with the rate of unemployment. Using a distributed lag weighting method to project forward the inflation index increased the correlation with the level of employment.

In his classic 1958 article, A. W. Phillips [21] plotted the rate of change of money wage rates with the unemployment rate. Using historical data for the United Kingdom, he obtained fairly regular counterclockwise loops for the years 1861-1913. This type of scatter diagram had a clockwise loop for the years 1948-1957. He noted that the direction of this loop could result from a time lag in the adjustment of money wage rates. In the ensuing years there have been numerous theoretical and statistical studies of this relationship between the percentage unemployment and an inflation index.

The purpose of this paper is to examine the basic mathematical structure that is found in all models which generate dynamic Phillips curves that are loops. We begin by reviewing some of the mathematical properties of comparing two time-series that follow cyclical patterns. When the amplitudes of two simple harmonic motions are plotted together, a Lissajous figure is produced. If these two cycles have the same period, then the Lissajous figure is a closed loop. The rotation and direction of this loop is determined by the phase difference of the two cycles. A simple model of inflation and unemployment is presented to illustrate this linkage between cyclical time-series and scatter diagram loops.

2. CYCLES AND LOOPS

Assume one has two time-series which display periodic fluctuations as shown in Figure 1. When the amplitudes of these two cycles are plotted, the result is the counterclockwise Lissajous curve in Figure 2. In order to review the basic properties of such loops, consider the simplest case where the two time-series are sine waves about their respective average levels X_0 and Y_0. The equations for these time-series can be expressed as

$$X_t = X_0 + \alpha \sin(\theta t - \emptyset); \quad (2.1)$$
$$Y_t = Y_0 + \beta \sin(\theta t). \quad (2.2)$$

These two series have the same frequency but are out of phase.[1/] Without loss of generality the time-series X is said to lag the series Y, where X is the variable plotted on the abscissa of the Lissajous graph. Let x_t and y_t represent the deviations about the means of X_t and Y_t, respectively. Therefore

$$x_t = \alpha \sin(\theta t - \emptyset); \quad (2.3)$$
$$y_t = \beta \sin(\theta t). \quad (2.4)$$

Expand equation (2.3) to get

$$x_t = \alpha[\cos(\emptyset) \sin(\theta t) - \sin(\emptyset) \cos(\theta t)]. \quad (2.5)$$

By using equation (2.4) and the relationship

$$1 = \sin^2(\theta t) + \cos^2(\theta t), \quad (2.6)$$

one can derive the following equation for the graph in x-y space:

$$\alpha^2 \sin^2(\emptyset) = x^2 - 2(\alpha/\beta)\cos(\emptyset)\, xy + (\alpha/\beta)^2 y^2, \quad (2.7)$$

where the time subscripts t have been omitted. This is the equation for an ellipse which has been rotated through the angle Ψ where

$$\Psi = 1/2 \sin^{-1}\left(\frac{2\, \alpha\beta \cos(\emptyset)}{a^2 - b^2}\right); \quad (2.8)$$

$$a^2 = \alpha^2 + \beta^2 - b^2; \quad (2.9)$$

$$b^2 = 1/2(\alpha^2 + \beta^2) \pm 1/2[(\alpha^2 + \beta^2)^2 - 4\alpha^2\beta^2\sin^2(\emptyset)]^{1/2}. \quad (2.10)$$

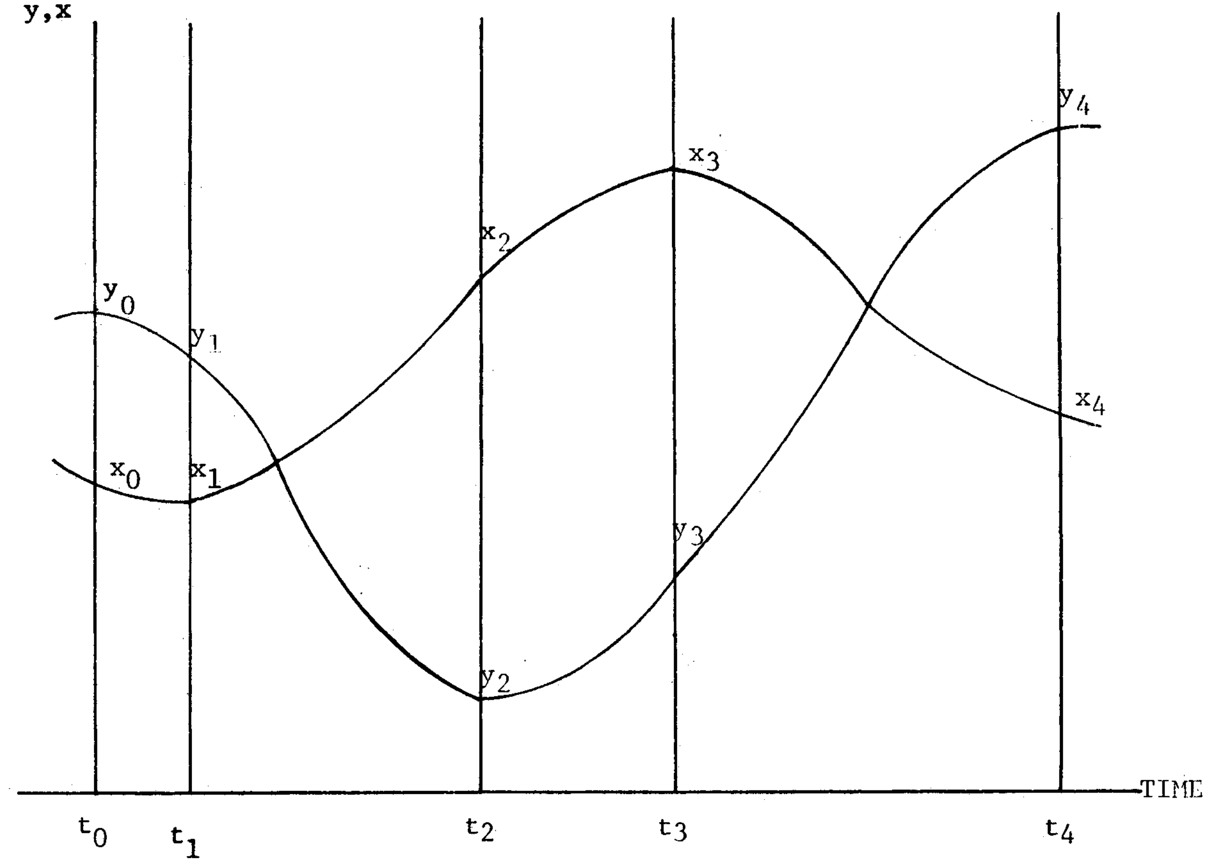

FIGURE 1. TWO "CYCLES" THAT ARE OUT OF PHASE

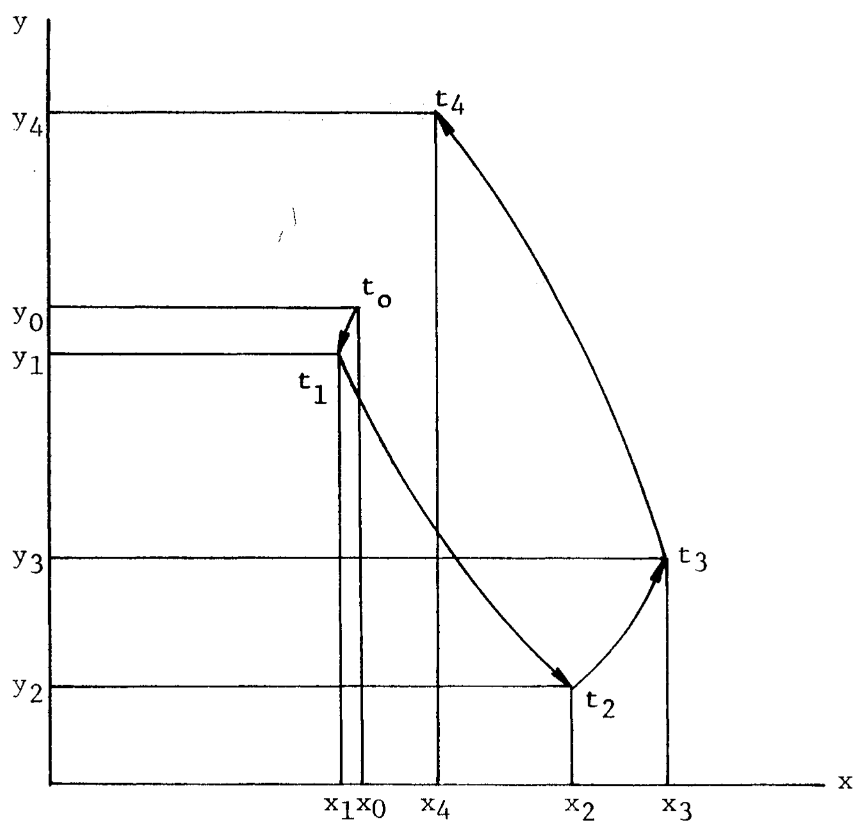

FIGURE 2. COUNTERCLOCKWISE LISSAJOUS FIGURE

In the quadratic solution equation (2.10) the appropriate value for b^2 is the root that will make $a^2 > b^2$. If the two cycles have the same amplitude, then equation (2.8) is simply $\sin(2\Psi) = \pm 1$.

The major axis of the ellipse is negatively sloped when the lag term $\emptyset$ for the X time-series is between $\pi/2$ and $3\pi/2$. For such cases the direction of the loop as it is generated in time can be determined by considering the "top" of the ellipse where

$$\frac{dy}{dt} = 0 \tag{2.11}$$

and

$$\frac{d^2y}{dt^2} < 0. \tag{2.12}$$

Since this time-series is assumed to be a simple sine wave, these local maxima occur whenever

$$\theta t = 2k\pi + \pi/2, \quad k = 0, 1, \ldots \tag{2.13}$$

The ellipse is considered to be clockwise (counterclockwise) if at this point the value of X is increasing (decreasing). If $\emptyset \varepsilon (\pi/2, \pi)$, then the loop is clockwise; if $\emptyset \varepsilon (\pi, 3\pi/2)$, then the loop is counterclockwise. The ellipse collapses into a negatively sloped straight line segment if the two series are perfectly out of phase or $\emptyset = \pi$. As the lag term deviates from π, the width of the ellipse increases until $\emptyset$ equals $\pi/2$ or $3\pi/2$.[2/]

3. SOME IMPLICATION FOR EMPIRICAL STUDIES

When estimating a relationship between two time-series that have fairly regular cycles, one has several basic options. The first option is to estimate a single-valued function with one of the time-series being the independent variable. This estimated equation can be linear or nonlinear with the variables being the historical data or indexes based on simple transformations of the data. Some of the common transformations used in statistical studies of inflation and unemployment include taking logarithms, using multipicative inverses, and varying the method for constructing the inflation index. Usually the equation is expressed as a wage adjustment function with the rate of change of money wages being the dependent variable. Such single-valued Phillip curves are incapable of generating loops and can be considered to be an estimation of the major axis of the loops that are observed.

Two methods which enable one to measure this dynamic relationship between inflation and unemployment are to estimate a multivalued equation, such as an ellipse, or to include additional independent variables that move cyclically. This latter approach is the option used in existing empirical studies. References to these studies are included in the recent articles by Berg and Dalton [1], Cargill and Meyer [2], Desai [3], McCallum [15], and Scully [23].

To review why including a second independent variable that also has periodic fluctuations can be used in a linear regression to estimate the cyclical path of a dependent variable, consider the simple case where there are three time-series that are out of phase with each other.

$$Y_t = Y_0 + \beta \sin(\theta t) \quad (3.1)$$
$$X_t = X_0 + \alpha \sin(\theta t - \emptyset) \quad (3.2)$$
$$Z_t = Z_0 + \gamma \sin(\theta t - \rho) \quad (3.3)$$

Where $\emptyset$ and ρ are not equal to $k\pi$, $k = 0, 1, \ldots$, and $\emptyset$ is not equal to ρ. Y_t can be expressed as a linear combination of X_t and Z_t.

$$Y_t = a + bX_t + cZ_t \quad (3.4)$$

where

$$a = Y_0 - bX_0 - cZ_0; \quad (3.5)$$

$$b = \frac{\beta \sin(\rho)}{\alpha \sin(\rho - \emptyset)}; \quad (3.6)$$

$$c = \frac{-\beta \sin(\emptyset)}{\gamma \sin(\rho - \emptyset)}; \quad (3.7)$$

Some of the additional independent variables that commonly are used in statistical studies and which could satisfy the previously mentioned "phase" requirements when there are sample business cycles include the rate of change of unemployment, lagged values of unemployment, previous rates of wage inflation, current and/or lagged rates of price inflation, and the corporation profit rate. A weighted average of previous rates of inflation is often interpreted to be a measure of inflationary expectations. These empirical models enable one to estimate the dynamic relationship between inflation and unemployment.

4. AN EQUILIBRIUM MODEL OF UNEMPLOYMENT AND INFLATION

Most economic theories which link the rate of unemployment to the rate of price and/or wage inflation are based on a disequilibrium model of price adjustment. The dynamic hypothesis is that the rate of change of money wages varies directly with a measure of excess demand for labor's input services. Unlike in an equilibrium tatonnement process, the speed of adjustment is assumed to be finite. When the unemployment rate varies inversely with excess demand, a negatively sloped Phillips curve is specified. Loops can be generated with this type of model by introducing a time difference between cyclical fluctuations in the relative amount of excess demand and the rate of inflation and/or the rate of unemployment. The theoretical models with such dynamic properties are formed by explaining how another cyclical variable can shift the momentary adjustment relation between unemployment and inflation. Using a wide variety of arguments, economists usually propose that the rate of change of unemployment and/or an inflation expectation measure shifts the short run Phillips curve. A sampling of representative dynamic models include those presented by Lipsey [13], Kuska [12], Rose [22], Phelps [19, 20], Lucas and

Rapping [14], Mortensen [16], and Nagatani [18]. Some more recently proposed models are in the articles by Gordon [7], H. I. Grossman [8], L. Johnson [11], J. L. Stein [24], and Vanderkamp [25].

The model I have developed is an extension of the "accelerationist" theory of unemployment in which the level of unemployment is related to the difference between expected and actual price levels (or rates of inflation). This theoretical relationship, which was developed by Phelps [19] and Friedman [5], is incorporated into an input market model with an equilibrium tatonnement process. Appealing to Occam's razor, the model is presented in its simplest form.

There are three major components of this partial equilibrium model--an input market, an expectations equation, and an exogenous time path for prices. In order to define an unemployment rate it is necessary to distinguish between labor employed and the labor force. The labor force is assumed to be constant and independent of the money wage rate, actual price level, and expected price level. Given an expected price level, the proportion of the labor force that would accept employment is distributed with respect to the money wage level. The effective labor supply equation is the corresponding cumulative distribution function of reservation wage rates. This supply curve shifts proportionally as labor adjusts its price expectations. The equation in the model is an extension of the labor supply function used by Holmes [10].

$$W = \alpha P_L^e \left((\beta/UL)^{\gamma} - \delta UL + \varepsilon\right), \tag{4.1}$$

where W is money wage rate, P_L^e is labor's expected price level, and UL is the unemployment rate of labor. If L is the amount of labor employed and LS is the total labor force, then

$$UL = 1 - L/LS \tag{4.2}$$

In this simplest form the model can be considered as a neoclassical model of "voluntary" unemployment.

The demand for labor services is taken from the traditional profit maximization conditions for perfect competition. The demand function is in terms of real wage rates and is the partial derivative of the production function with respect to labor. A money wage demand function is derived by multiplying the real wage demand function by the current price level of the market period. By using a linearly homogeneous C.E.S. (constant elasticity of substitution) production function, the value of the output (P·Q) equals wages paid workers (W·L) plus rents received by the owners of the capital inputs (R·K).

The market equilibrium condition is that the amount of labor demanded equals the amount supplied at a given wage rate. When the expected price level equals the actual price level, unemployment is at its "natural" rate. In Figure 3 the corresponding supply and demand equations are plotted where the "natural" rate of unemployment is assumed to be six percent. If the distribution of reservation wages had one mode which is near the mean wage, then the corresponding supply function would follow the dashes for higher rates of unemployment. The constant labor force is represented by the vertical line.

When the expected price level differs from the actual price level, the unemployment rate varies from its normal rate. As specified in the "accelerationist" theory, there exists a stable relationship between the rate of unemployment and the ratio of actual to expected prices. Three such Friedman-Phelps curves have been plotted in Figure 4. The steepest negatively sloped curve assumes that the amount of capital remains constant ($\overline{K}$). To generate the other curves the model was modified so that the input market for capital services is similar to the labor input market. UK is the "unemployment" or underutilization rate for capital, and KS is the available supply of capital. The curve of dashes corresponds to the case where the capital owners' price

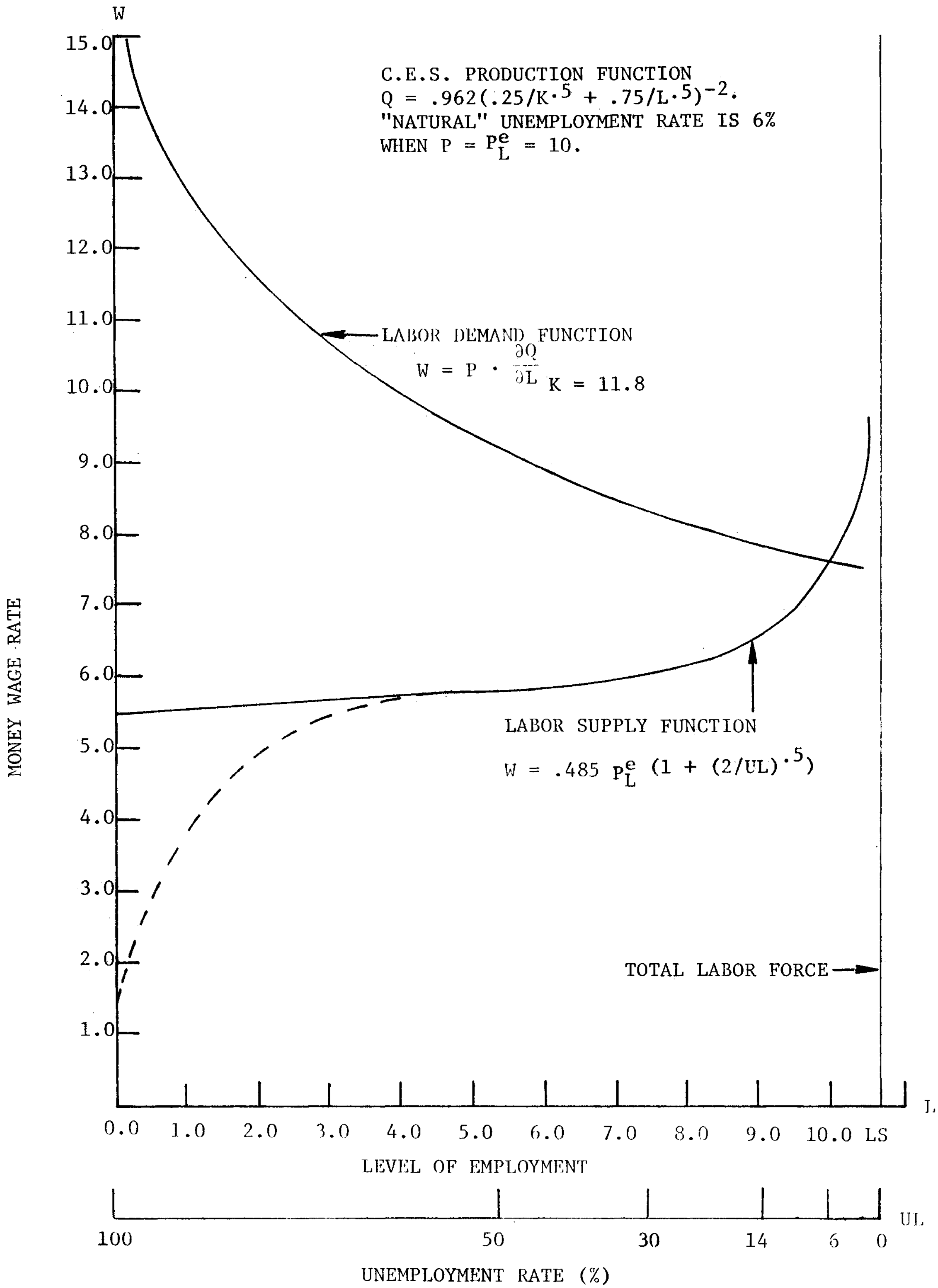

FIGURE 3. INPUT MARKET FOR LABOR

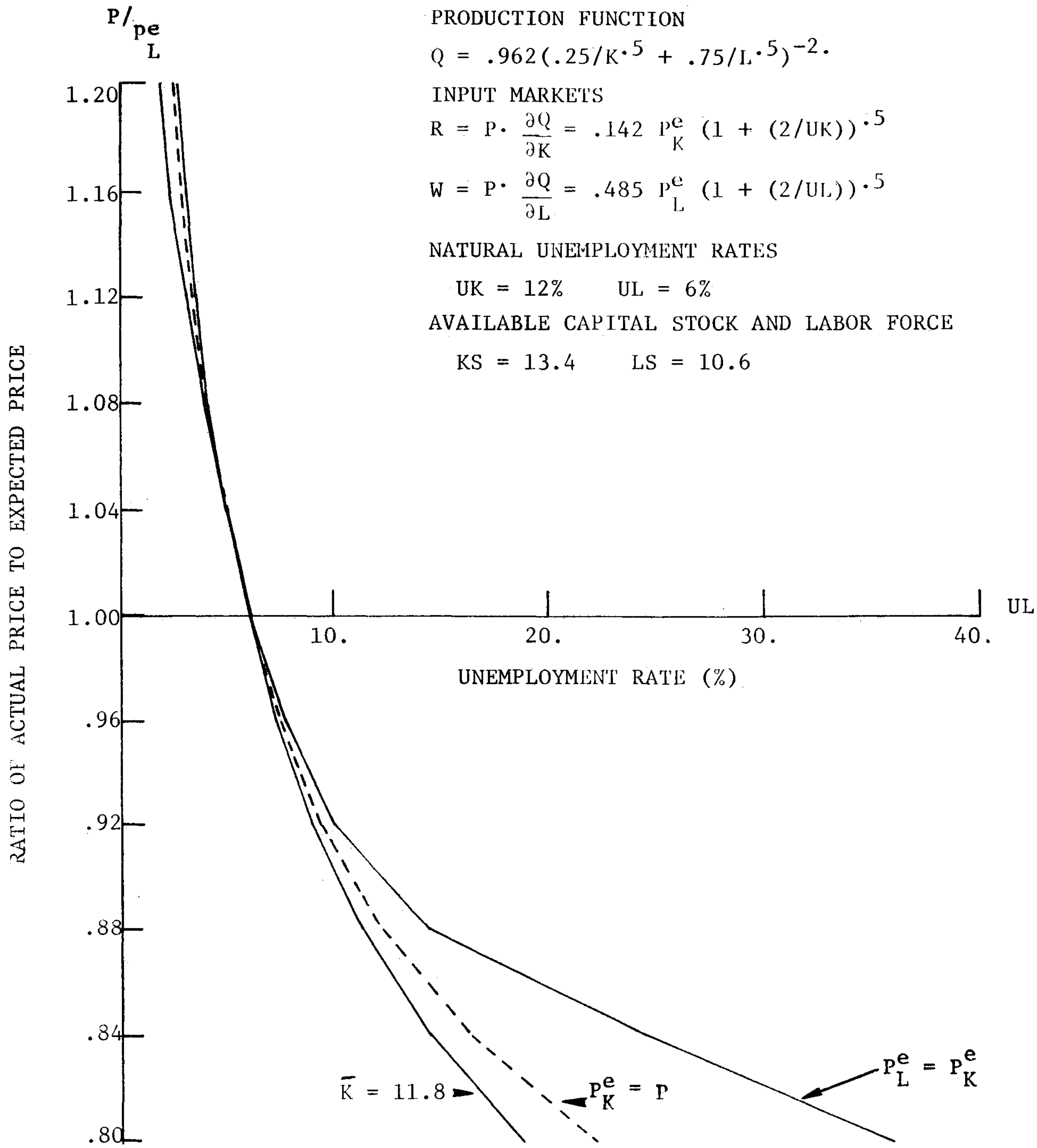

FIGURE 4. FRIEDMAN-PHELPS CURVES

expectations are equal to the actual price level ($P_K^e = P$). The third curve assumes that the two input owners have the same price expectations ($P_K^e = P_L^e$).

Dynamic Phillips curves are generated with this input market model by specifying an expectations equation and exogenous time path for prices. For example, suppose that prices follow a cyclical pattern where the price level in period t is

$$P_t = 10. + \sin(\pi t/6), \quad t = 1, 2, \ldots \tag{4.3}$$

and the rate of inflation (p) is defined as

$$\dot{p}_t = P_t/P_{t-1} - 1. \tag{4.4}$$

If both input owners have the same adaptive expectations by the level of prices with

$$P_t^e = (P_{t-1} + P_{t-1}^e)/2. \quad t = 1, 2, \ldots \tag{4.5}$$

than the dynamic Phillips curve is the counterclockwise loop plotted in Figure 5. The model is simulated over several cycles before this loop was plotted so that the damped transient response caused by the initial expected price (P_0^e) can be ignored.

If both input owner have the same adaptive expectations by the rate of change of prices with

$$P_t^e = P_{t-1}\,(1 + \dot{p}_t^e) \quad t = 1, 2, \ldots \tag{4.6}$$

and

$$\dot{p}_t^e = (\dot{p}_{t-1} + \dot{p}_{t-1}^e)/2, \tag{4.7}$$

then the dynamic patter of inflation and unemployment is the clockwise loop plotted in Figure 5. Again the model was simulated over several cycles so that the loop converges onto its equilibrium position. The corresponding loops in wage inflation and unemployment space are very similar to those plotted in Figure 5. These results are essentially unchanged when one simulates the alternative models which assume a constant utilization rate of capital or

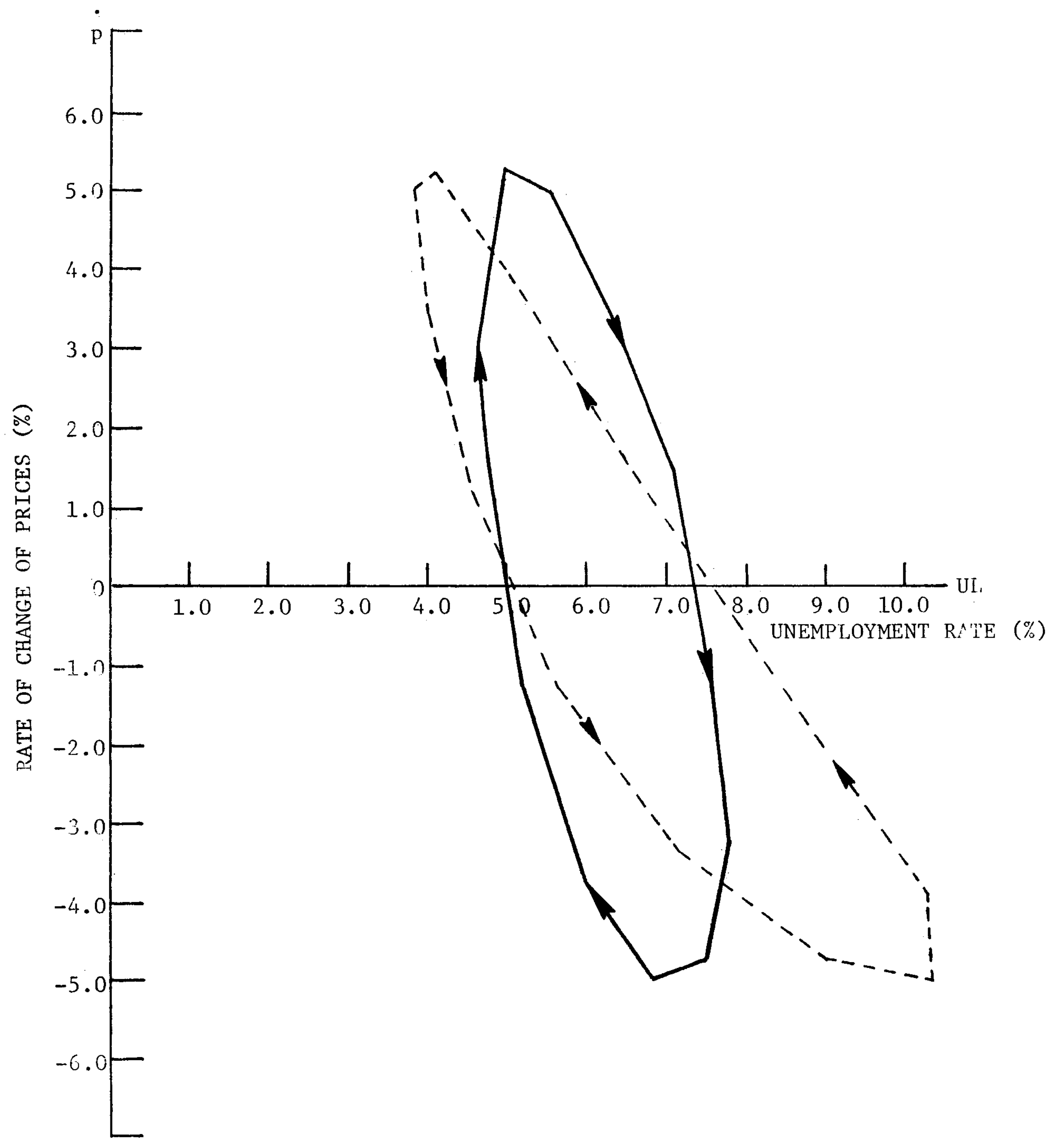

FIGURE 5. **DYNAMIC PHILLIPS CURVES** , $p_K^e = p_L^e$

perfect "myoptic" expectations of capital input owners. This easily is explained by observing the similarly shaped Friedman-Phelps curves in Figure 4.

Some of the interesting cycles that are generated by these models with adaptive expectations are plotted in Figures 6 and 7. The exogenous time-series for the price level is shown in Figure 6 along with the equilibrium cyclical fluctuations in the expected price levels.[3/] Figure 7 plots the ratios between the actual price in period t and the previous period's price and between the actual price and expected prices. Note that these cyclical fluctuations are not symmetric about their relative extreme or inflection points. For the model with adaptive expectations by the level of prices, the loop in the dynamic Phillips curve reflects both a phase difference between the inflation and unemployment cycles and the asymmetry of these cycles. Recall that the ratio between two successive periods' prices is a linear function of the percentage rate of inflation and that unemployment varies inversely with the ratio of actual and expected price levels. Since there is a one period difference between the two relative maxima of these ratio cycles in Figure 7, there is one segment of the corresponding Phillips curve where both inflation and unemployment are declining. This portion of the curve is adjacent to the "top" of the counterclockwise loop in Figure 5. Because these ratio cycles in Figure 7 have their relative minima in the same time period (which implies a relative maxima in unemployment), the "bottom" corner of the corresponding loop in Figure 5 represents the asymmetry of the inflation and unemployment cycles. Since there is not a phase difference for this portion of the two cycles, there is not a segment adjacent to the "bottom" of the loop where unemployment and inflation are both increasing. The dynamic Phillips curve for the model with adaptive expectations by the rate of change of prices has three segments where unemployment and inflation are changing in the same direction.

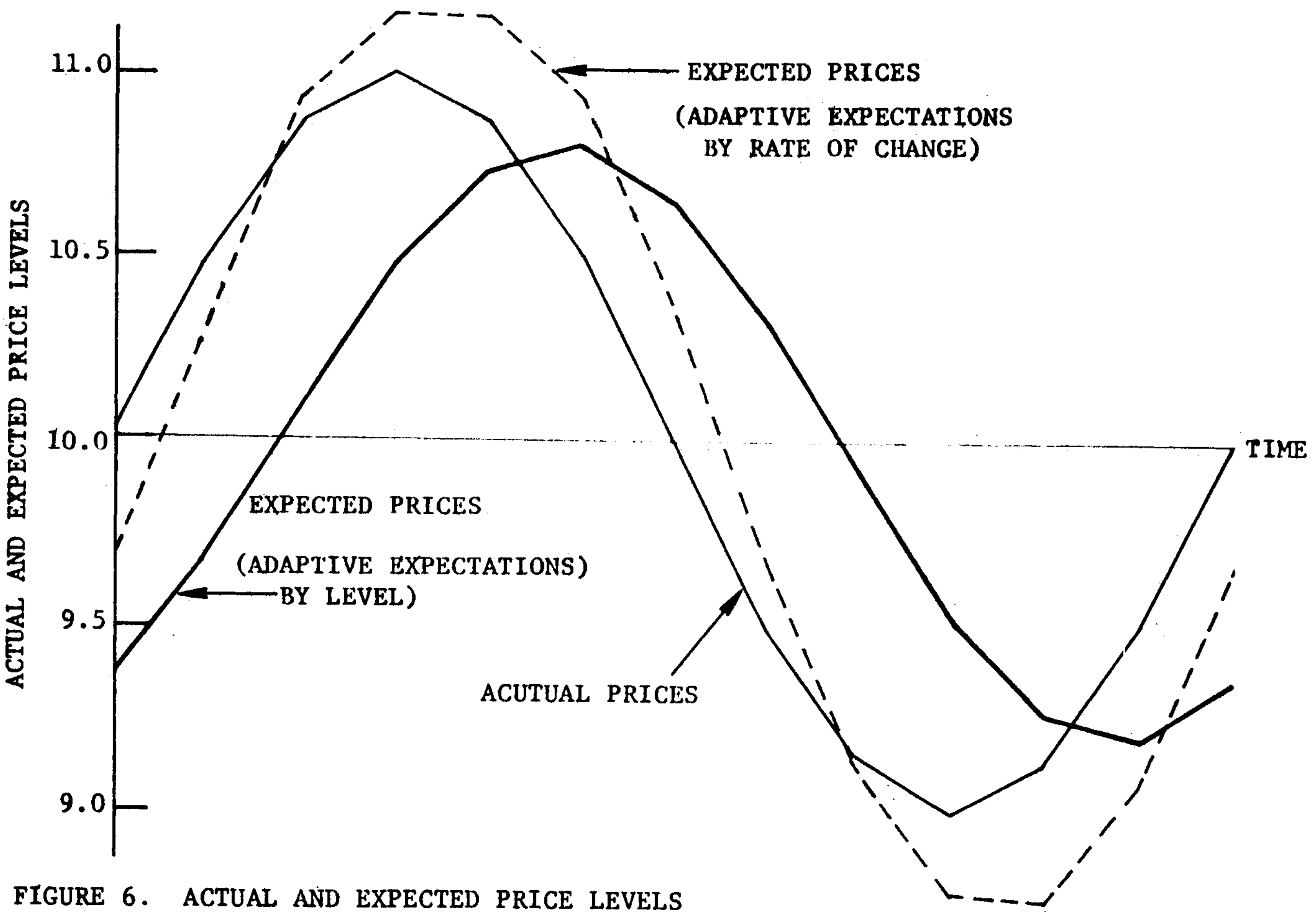

FIGURE 6. ACTUAL AND EXPECTED PRICE LEVELS

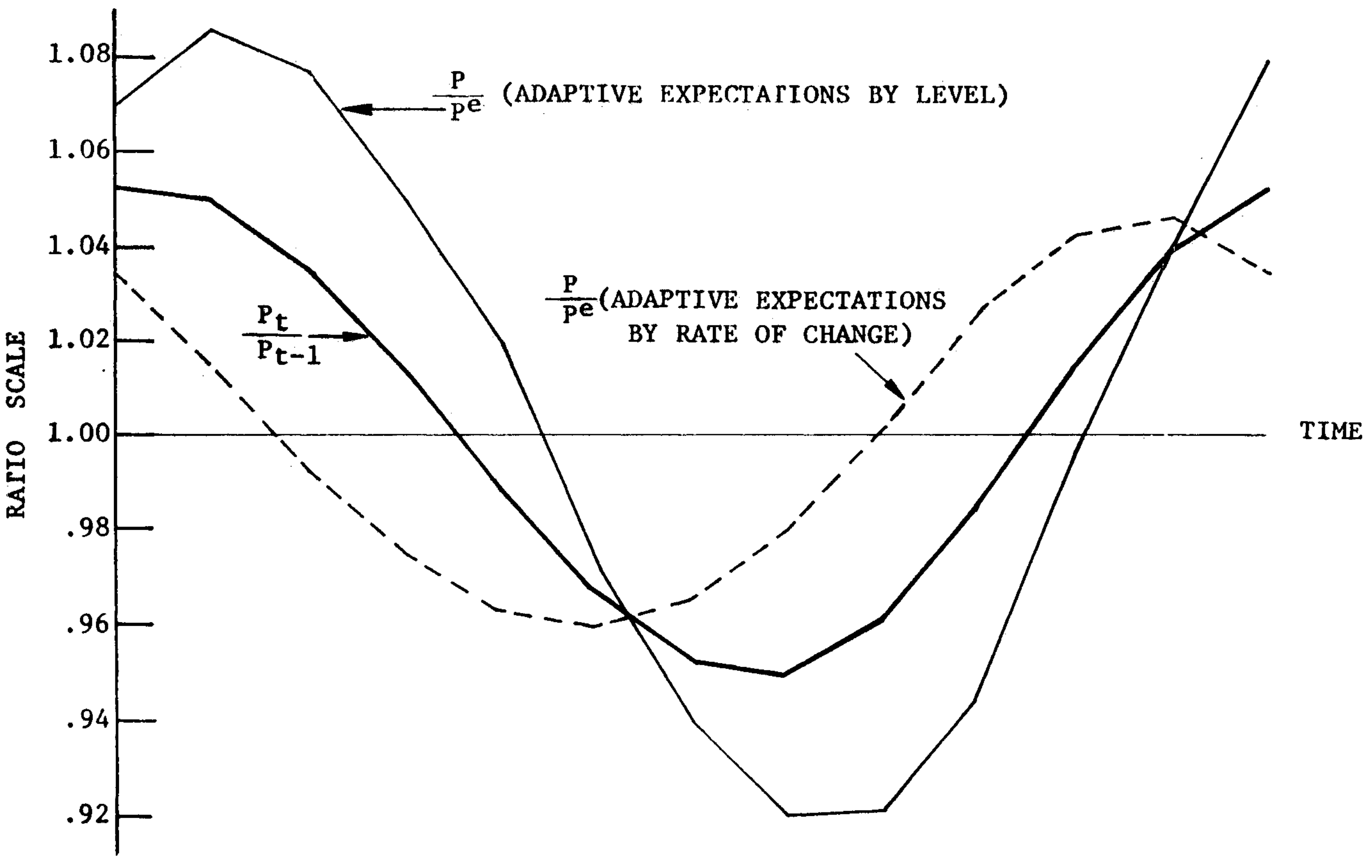

FIGURE 7. RATIOS OF ACTUAL PRICE TO PAST AND EXPECTED PRICES.

One segment reflects the one period difference between the relative maxima of the ratio cycles in Figure 7. The other two segments correspond to the two period difference between the relative minima of these ratio cycles.

These models show how the direction of a looping Phillips curve can be determined by properties of how expectations are formed. In his recent article H. I. Grossman [8] reaches a similar conclusion using a disequilibrium model with finite speeds of adjustment. His linear differential equation model produces a rotated ellipse when the rate of wage inflation follows a simple harmonic path. He determines that the loop more likely will be clockwise the faster the speed of adjustment of employment and expectations. This model includes both a time lag between unemployment and excess demand for labor services and a lag between wage inflation and excess demand. It is interesting that similar conclusions can be derived from a model with an equilibrium tatonnement process and from a disequilibrium model with a finite speed of adjustment.

5. CONCLUSIONS

This paper has reviewed some of the mathematical relationships that link cyclical time-series with loops in corresponding scatter diagrams. A simple equilibrium model of unemployment and inflation has been presented to illustrate how cyclical fluctuations in these time-series correspond to dynamic Phillips curves that are loops. The direction of these loops depends on the phase difference between the cycles and/or asymmetries in these cycles. In the simple models simulated, adaptive expectations by the level of prices generate a counterclockwise loop, whereas adaptive expectation by the rate of inflation produces a clockwise loop. This type of partial equilibrium model can be used for the input markets in theoretical macroeconomic models which also include markets for commodities, bonds, and money. [6]

FOOTNOTES

1/ If the two cycles have different angular frequencies θ_x and θ_y, then the Lissajous curve is closed if θ_x/θ_y is a rational number. Some portions of this closed figure are clockwise and some are counterclockwise. If the ratio of frequencies is not a rational number, then the curve will pass through every point lying in the rectangle bounded by $x = \pm \alpha$ and $y = \pm \beta$. [9]

2/ A phase difference which is not equal to $k\pi$, $k = 0, 1, 2, \ldots$ is a necessary and sufficient condition for a closed loop in the scatter diagram for any two cyclical series that have the same frequency and which are symmetric about their relative extrema. This loop can be "banana-shaped" when one or both of the series is not symmetric about the inflection points. When one or both of the cycles is asymmetric about the relative extrema, a phase difference is no longer a necessary condition for a loop in amplitude space.

3/ Since the path of expected prices systematically deviates from the actual price cycle, these adaptive expectations are "irrational," in the sense of Muth [17], for models with cyclical fluctuations.

REFERENCES

[1] Berg, S. V. and T. R. Dalton, "Sectoral Employment and Shifts in the Aggregate Phillips Curve," Southern Economic Journal, 41 (April 1975), 593-601.

[2] Cargill, T. F. and R. A. Meyer, "Wages, Prices and Unemployment: Distributed Lag Estimates," Journal of the American Statistical Association, 69 (March 1974), 98-107.

[3] Desai, M., "The Phillips Curve: A Revisionist Interpretation," Economica, 42 (February 1975), 1-19.

[4] Fisher, I., "A Statistical Relation Between Unemployment and Price Changes," International Labor Review, 13 (June 1926), 785-92. Reprinted in the Journal of Political Economy, 81 (March/April 1973), 496-502.

[5] Friedman, M., "The Role of Monetary Policy," American Economic Review, 58 (March 1968), 1-17.

[6] Gittings, T. A., "A Computer Program for Simulating Theoretical Macroeconomic Models," unpublished Ph.D. dissertation, University of Colorado, 1974.

[7] Gordon, D. F., "A Neo-Classical Theory of Keynesian Unemployment", Economic Inquiry, 12 (December 1974), 431-59.

[8] Grossman, H. I., "The Cyclical Pattern of Unemployment and Wage Inflation," Economica, 41 (November 1974), 403-13.

[9] Halliday, D. and R. Resnick, Physics, New York: John Wiley and Sons, 1967.

[10] Holmes, J. M., "The Keynesian Aggregate Supply Function for Labor," Journal of the American Statistical Association, 69 (December 1972), 797-802.

[11] Johnson, L., "Alternative Adjustment Mechanisms and the Long Run Phillips Relation: Job Search vs. the 'Neoclassical' Tatonnement," Special Study Paper No. 40, Division of Research and Statistics, Federal Reserve Board, January 1974.

[12] Kuska, E. A., "The Simple Analytics of the Phillips Curve," Economica, 33 (November 1966), 462-67.

[13] Lipsey, R. G., "The Relation Between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1862-1957: A Further Analysis," Economica, 28 (February 1960), 1-31.

[14] Lucas, R. E. and L. A. Rapping, "Real Wages, Employment and the Price Level," Journal of Political Economy, 77 (September/October 1969), 721-54.

[15] McCallum, B. T., "Wage Rate Changes and the Excess Demand for Labour: An Alternative Formulation," Economica, 41 (August 1974), 269-77.

[16] Mortensen, D. T., "Job Search, the Duration of Unemployment, and the Phillips Curve," American Economic Review, 60 (December 1970), 847-62.

[17] Muth, J. F., "Rational Expectations and the Theory of Price Movements," Econometrica, 29 (July 1961), 315-35.

[18] Nagatani, K., "A Monetary Growth Model with Variable Employment," Journal of Money, Credit and Banking, 1 (May 1969), 188-205.

[19] Phelps, E. S., "Phillips Curves, Expectations of Inflation and Optimal Unemployment over Time," Economica, 34 (August 1967), 254-81.

[20] Phelps, E. S., et al., Microeconomic Foundations of Employment and Inflation Theory, New York: W. W. Norton, 1970.

[21] Phillips, A. W., "The Relation between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1861-1957," Economica 25 (November 1958), 283-99.

[22] Rose, H., "On the Non-Linear Theory of the Employment Cycle," Review of Economic Studies, 34 (April 1967), 153-73.

[23] Scully, G. W., "Static vs. Dynamic Phillips Curves," Review of Economics and Statistics, 56 (August 1974), 387-90.

[24] Stein, J. L., "Unemployment, Inflation, and Monetarism," American Economic Review, 64 (December 1974) 867-87.

[25] Vanderkamp, J., "Inflation: A Simple Friedman Theory with a Phillips Twist," Journal of Monetary Economics, 1 (January 1975), 117-22.