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The recent record-setting economic expansion and the accompanying record-setting bull market in stocks are often attributed to Federal Reserve interest rate policy and increased productivity. But if interest rates behave differently when productivity changes, interest rate policy may need to change as well. This *Review* examines how productivity changes affect the entire term structure of interest rates, from short-term rates to long-term rates. The article works out a model based on a representative agent framework, which considers one person as a price taker who makes investment choices depending on prices, and these choices are used to determine the prices that will clear the markets. The person decides how much to consume and how much to invest in each of two assets—a short-term, risk-free, real bond, and a risky productive investment. A key feature of the model is that risk and return change over time in a way that is directly related to productivity. The results show that productivity plays a crucial role in how various interest rates interact, but its effect is not simple. Productivity is not a single factor that affects interest rates uniformly, and parameters of the productivity process, such as the duration of a productivity increase or the speed of adjustment, affect the term structure differently. These parameters, which might otherwise be overlooked, must be specified before the appropriate interest rate policy can be identified.

The authors present simple versions of the models used in the search-theoretic approach to monetary economics. They discuss results on the existence of monetary equilibria, the potential for multiple equilibria, and welfare. They consider models where prices are fixed as well as models where prices are determined endogenously by bilateral bargaining. After discussing the frictions necessary to construct a model with an essential role for money, they conclude the paper by reviewing many extensions and applications in the related literature.

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Productivity and the Term Structure

by Joseph G. Haubrich

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Introduction

In discussions about the recent record-setting economic expansion and the accompanying record-setting bull market in stocks, two factors often receive credit: Federal Reserve interest rate policy and increased productivity. This conjunction naturally raises the question of what interest rate policy is appropriate in the face of changing productivity. Before such a policy question can be answered, however, the logically prior question of the effects of productivity changes on interest rates must be addressed. The Federal Reserve controls one particular short-term rate: the federal funds rate. Investors, consumers, and businesses generally care about long-term rates: mortgages, car loans, corporate bonds. This *Economic Review* considers how productivity changes affect the entire term structure of interest rates. It thus may serve as a prelude to thinking about how the Federal Reserve (FOMC) should move the Federal Funds rate in response to productivity changes or movement in other interest rates.

To get a handle on the economics behind productivity's effect on the term structure, this paper works out a simplified theoretical model based on the work of Cox, Ingersoll, and Ross (1985a,b) and Sun (1992). This approach abstracts from reality: it posits a very simple production structure and it ignores money and inflation.[1] This means that it ignores several vital influences on the term structure, principally how productivity affects the part of interest rates that depends on inflation and expectations about inflation.

One big advantage of writing down a theoretical model in all its detail is that we are forced to answer a host of questions that might not naturally come up. When we talk about a productivity change (or shock) is that change permanent or temporary? Do we expect it to be repeated—or offset—in the near future? To get answers we must be precise about our assumptions.

I. The Basic Economy

Whereas the Cox, Ingersoll, and Ross (CIR) model of the economy uses continuous time, this article uses discrete time to make it more comparable with other macroeconomic work. The approach builds on the explanation of dynamic portfolio theory presented in Sargent (1987).

The basic plan is to use a representative-agent framework, that is, to consider one person as a price taker and, after finding out how that person's choices depend on prices, use the results to determine what prices will clear the markets. The person decides how much wealth to invest in each of two available assets, and how much to consume now. One asset is a risky, productive investment opportunity, something like planting wheat or building a factory. The other is a one-period, risk-free, real bond, something like a government-guaranteed CD. The agent's decision depends on the assets' risk and return. The key point of the model is that the risk and return will change over time—and change in a way directly related to productivity, as a productive factory is a profitable factory. The underlying productivity changes interact with the choices made by the representative agent to yield the prices and interest rates we wish to examine.

More formally, if A_t denotes today's wealth (that is, wealth in period t), c_t denotes consumption, s_t denotes the amount put in the productive investment, and b_t denotes the amount in the bond, the basic budget constraint for this economy is: $A_t = c_t + s_t + b_t$. The transition equation, showing how wealth tomorrow depends on decisions made today, becomes: $A_{t+1} = R_t s_t + r_t b_t$, where R_t is the (gross) return on the risky investment and r_t is the return on the safe asset. Notice that R_t can be thought of as productivity: The higher R_t is, the higher the return from investing in the factory (or the

1 For papers that tackle these more difficult issues, see den Haan (1995) and Bakshi and Chen (1996).

more the factory produces for a given level of investment s_t). It is important to notice that at time t, r_t is known with certainty but R_t, being risky, is unknown.

Next, we assume that the agent has some utility function $u(c_t)$ and some discount factor β so that total utility is $U = \sum_{t=0}^{\infty} \beta^t u(c_t)$. Since the future is uncertain, expected utility is

$$U = \sum_{t=0}^{\infty} \beta^t E_0 u(c_t). \tag{1}$$

The agent's problem is to choose values for s_t and b_t to maximize (1). The appendix carries out this calculation, which results in the *Euler equation*:

$$u'(c_t) = \beta E_t \left[R_t u'(c_{t+1}) \right]. \tag{2}$$

This equation says that the agent balances the gain from consuming a little more today (left side) against the expected gain from investing that little bit and getting more tomorrow (right side).

Next, we simplify the problem in two steps by making assumptions first about the form of the utility function u and then about the stochastic process driving R_t.

We specialize utility to take the log form:

$$u(c) = \log(c). \tag{3}$$

Fortunately, as the appendix shows, log utility results in an especially simple form for investment demand, namely $s_t = kA_t$ and $b_t = \ell A_t$, where k and ℓ are just two constants (to be determined later). An even more convenient form of (2), due to Grossman and Shiller (1981), provides a compact representation for both interest rates, R_t and r_t.

$$\begin{aligned} 1 &= \beta E_t \left[R_t \frac{1}{R_t k + r_t \ell} \right] \\ 1 &= \beta E_t \left[r_t \frac{1}{R_t k + r_t \ell} \right]. \end{aligned} \tag{4}$$

General Equilibrium

Equation (4) solves the individual's portfolio selection problem. If our goal was to provide investment guidance, we could determine how much to invest in each asset by carefully specifying the stochastic processes for R_t and r_t and solving for k and ℓ. We will use (4) in a different way. Instead of taking the interest rates as given, we will use (4) to determine them, in effect using it as a demand curve. For example, if you know the demand for apples you can predict how many people will buy when the price is \$5; but the more interesting use is putting demand and supply together to find the price. In determining interest rates, similarly, the focus is on turning (4) around and using it solve for the interest rate r_t.

Equation (4) by itself does not allow a solution in purely exogenous variables, so we must bring in other aspects of the economy. The key to doing this is the representative agent: There is only one person (both a consumer and investor) in this economy, acting as a price taker. This means that in aggregate, there is no borrowing or lending: The consumer can only borrow from himself, so everything nets out to zero. The total amount invested in the safe bond is thus zero, making $\ell = 0$. Another way of saying this is that bonds are "in zero net supply." Note that the individual may still invest in the productive assets, so we have not (yet) pinned down k.

Imposing the zero-net-supply assumption, equation (4) becomes

$$\begin{aligned} 1 &= \beta E_t \left[R_t \frac{1}{R_t k} \right] = \beta E_t \left[1/k \right] \\ 1 &= \beta E_t \left[r_t \frac{1}{R_t k} \right]. \end{aligned} \tag{5}$$

Because r_t is known at time t (you know what return a safe bond will pay) and because k is a constant, the equations in (5) can be combined to solve for k and r_t, yielding $k = \beta$ and

$$\frac{1}{r_t} = E_t \left[\frac{1}{R_t} \right]. \tag{6}$$

Equation (6) provides a crucial step: it describes the return on bonds, r_t, in terms of productivity, that is, the return on the productive process, R_t. The next step is to think more carefully about how R_t moves over time and what this implies, via (6), for the interest rate.

Lognormality, Kernels, and Interest Rates

Describing how R_t moves over time, in a way both interesting and tractable, is harder than you might think. In fact, it is best done by approaching the problem indirectly. A convenient approach is to use the lognormal distribution (that is, where $\log X$ is distributed normally), because it has a particularly nice form for expectations. If X is lognormal,

$$\log E_t(X) = E_t(\log X) + 1/2 \operatorname{var}(\log X).$$

Exactly *what* should be distributed lognormally, however? For this it pays to revisit (6).

Equation (6) can be rewritten, or perhaps we should say, re-interpreted, as a way to express the *price* (not return) of the safe bond in terms of something called a pricing kernel.[2] While it is possible to give a suitable economic description of the pricing kernel (as the intertemporal marginal rate of substitution or the probability-weighted state price), at this point it's best to think of it as a step that makes the derivation easier. Consider the interest rate r_t again. Since the bond in question is a one-period, zero-coupon bond for which the owner will get one unit of consumption tomorrow for a price of P_t today, the return is $r_t = \frac{1}{P_t}$. This makes the left side of (5) $1/(r_t) = P_t$. (Recall that r_t and R_t are gross returns, that is, of the form 1.05, rather than 5 percent.)

For the right side of (6), redefine $1/(R_t)$ to be the pricing kernel, (or, as it is sometimes called by the real fun-loving types, the stochastic discount factor), M_{t+1}. These substitutions lead to

$$P_t = E_t\left(M_{t+1}\right). \tag{7}$$

Thus, the price is just the expectation of the pricing kernel. Next, we *define* the interest rate (or yield) on the safe bond as the negative of the log of the price, so that $y_{1t} = -\log P_t$, and define m_t as $\log M_t$.[3] If m_{t+1} is distributed lognormally, then (7) becomes

$$-y_{1t} = E_t\left(m_{t+1}\right) + 1/2\text{var}\left(m_{t+1}\right). \tag{8}$$

We arranged this detour because it is easier to put an interesting and tractable structure on m_{t+1} rather than directly on R_t. At long last, we are ready to do this.

We assume that the log pricing kernel takes the following form:

$$\begin{aligned} -m_{t+1} &= x_t + x_t^{1/2}\gamma\varepsilon_{t+1}, \\ x_{t+1} &= \mu + \phi\left(x_t - \mu\right) + x_t^{1/2}\varepsilon_{t+1}. \end{aligned} \tag{9}$$

If we think about this equation in light of the original question we posed about the effect of productivity on interest rates, the new term, x_t, may be thought of as a factor that moves productivity around. While equation (9) may seem rather unintuitive at first, it has several nice properties that, as we show later, will carry over to interest rates. It has a long-run mean, μ, and it tends to revert to that mean with a speed that depends on ϕ. That is to say, the process has first-order serial correlation. The other term that perhaps looks a little strange is the $x^{1/2}$ factor on the shock, which makes the effect of the shock (and thus the variance of the process) depend on the level of x_t. If x_t is large, the shock will have large effects. This means that interest rates move around more when they are high than when they are low. If rates are at 10 percent, movements up to 11 or down to 9 will be common, but if rates are at 3 percent, it will be a rare move that reaches 4 or 2 percent.

This "square-root process" has another important aspect. As x drops toward zero, the variance (the effect of ε shocks) decreases, making it less likely that the process will fall below zero. For a large value of x, the odds are small that the shock would be big enough to send x_t negative. For a small value of x, the variance is very low, so the odds of going negative are also small. In the limit, with a continuous time process (as in Cox, Ingersoll, and Ross [1985b]) the probability is zero. To make life easier, we will adopt that approximation (also used in Sun [1992] and Campbell, Lo, and MacKinlay [1997]), although for discrete-time processes it is not strictly true.

If our goal was only to price bonds and other financial assets, we could simply have started with equation (7), but that would have omitted mention of any connection between productivity and interest rates, which is our main concern.

II. Term Structure

Putting the pricing equation (8) together with the assumptions on the productivity factor (9) finally puts enough structure on the problem to get some meaningful results. Substituting (9) into (8) yields

$$\begin{aligned} p_{1t} = -y_{1t} = E_t &\left[-\left(x_t + x_t^{1/2}\gamma\varepsilon_{t+1}\right)\right] \\ &+ (1/2)\,\text{var}\left[-\left(x_t + x_t^{1/2}\gamma\varepsilon_{t+1}\right)\right] \end{aligned}$$

or

$$y_{1t} = x_t\left(1 - \gamma^2\sigma^2/2\right). \tag{10}$$

This gives the short-term yield (under the standard approximation of logs, $y_{1t} \approx r_t - 1$). This equation has a fairly intuitive explanation. The factor affecting return to capital (here, x_t) has a big influence on the interest rate, which increases with return to capital. That's not quite the end of the story, however, because investment in capital, the productive asset, is risky. The bond is safe, and therefore risk-averse investors are willing to pay a premium to put their assets in bonds. A premium price on bonds translates into a lower interest rate. So a risk factor offsets some of the direct productivity effect. Notice the importance of the square-root process

2 For more involved descriptions of using the pricing kernel to derive interest rates, the now-standard reference is Campbell, Lo, and MacKinlay (1997) chapter 11; for a more specific application along the lines of this article, see Haubrich (1999).

3 Notice that the uncertain return from t to $t+1$ is indexed as R_t, but that the associated kernel is indexed as $t+1$. This is standard usage.

here. An increase in productivity, x_t, *also* increases the variance, and thus the uncertainty, of the productivity shocks.

Long Rates

Now let's consider what the model tells us about the longer-maturity bond. With an approach analogous to that used in section I, one can obtain an expression for the price of a two-period bond, noting that the two-period yield will be the negative of one-half the log price. Thus,

$$P_{2t} = E_t\left[M_{t+1}P_{1,t+1}\right]$$

or

$$p_{2t} = E_t\left[m_{t+1} + p_{1,t+1}\right] + 1/2\text{var}\left[m_{t+1} + p_{1,t+1}\right],$$

which after substituting in (8), becomes

$$\begin{aligned} p_{2t} = E_t\Big[&-\left(x_t + x_t^{1/2}\gamma\varepsilon_{t+1}\right) \\ &-x_{t+1}\left(1-\gamma^2\sigma^2\right)\Big] \\ &+1/2\text{var}\Big[-\left(x_t + x_t^{1/2}\gamma\varepsilon_{t+1}\right) \\ &-x_{t+1}\left(1-\gamma^2\sigma^2\right)\Big]. \end{aligned} \tag{11}$$

Again using (9) to express x_{t+1} in terms of x_t and ε_{t+1}, (11) reduces to

$$\begin{aligned} p_{2t} = -x_t\Big\{&1 + \left(1-\gamma^2\sigma^2/2\right)\phi \\ &-\left[\gamma + \left(1-\gamma^2\sigma^2/2\right)\right]^2\sigma^2/2\Big\} \\ &-\left(1-\gamma^2\sigma^2/2\right)(1-\phi)\mu. \end{aligned}$$

This makes the two-period yield

$$\begin{aligned} y_{2t} = (1/2)x_t\Big\{&1 + \left(1-\gamma^2\sigma^2/2\right)\phi \\ &-\left[\gamma + \left(1-\gamma^2\sigma^2/2\right)\right]^2\sigma^2/2\Big\} \\ &+(1/2)\left(1-\gamma^2\sigma^2/2\right)(1-\phi)\mu. \end{aligned} \tag{12}$$

A more intuitive expression for the two-period rate comes from rearranging (12) into

$$\begin{aligned} y_{2t} = (1/2)\Big\{&\left(1-\gamma^2\sigma^2/2\right)x_t \\ &+\left(1-\gamma^2\sigma^2/2\right)\left[\mu + \phi\left(x_t - \mu\right)\right] \\ &-\left[2\gamma\left(1-\gamma^2\sigma^2/2\right)\right. \\ &\left.+\left(1-\gamma^2\sigma^2/2\right)^2\right]\left(\sigma^2\right)x_t\Big\}. \end{aligned} \tag{12a}$$

The first two terms in the brackets of (12a) describe the part of the two-period bond yield that is attributed to the expectations hypothesis of the term structure. The expectations hypothesis says that two-period interest rates ought to be the average of today's one-period interest rate and the expectation of next period's one-period interest rate. The first term,

$$\left(1-\gamma^2\sigma^2/2\right)x_t,$$

is just today's one-period interest rate. The next term,

$$\left(1-\gamma^2\sigma^2/2\right)\left[\mu + \phi\left(x_t - \mu\right)\right],$$

is the expectation of next period's interest rate. The reason x_t shows up is that productivity today has information about x_{t+1}, productivity tomorrow. The best guess for the productivity factor tomorrow is that it will revert somewhat toward the mean μ (exactly how much depends on the speed of adjustment, ϕ.) Next period's short rate ($y_{1,t+1}$) depends on what x_{t+1} is, so our best guess for next period's short-term rate is our best guess for next period's productivity factor multiplied by the factor $\left(1-\gamma^2\sigma^2/2\right)$. Notice that the second term is greater or less than the first term precisely when x_t is greater or less than μ. From the expectations perspective, if the productivity factor (and thus the short rate) is below the mean, rates are expected to increase, and so the term structure slopes upward. If rates are above the mean, they are expected to fall, and the term structure slopes downward.

The expectations hypothesis is not completely true, however, and (12a) has an additional term, accounting for risk, which tends to lower the two-period yield. For example, if $x_t = \mu$, the risk term would imply that $y_{2t} \leq y_{1t}$.

Questions about the term structure reduce to questions about the difference between equation (10), the short rate, and equation (12a), the long rate. Of course, one might ask more complicated questions involving three-period yields, four-period yields, or even seventeen-period yields. Restricting attention to one- and two-period yields eliminates questions about the shape of the yield curve, such as whether or not it is humped. Still, the key intuitions about many important questions—such as how productivity affects term structure level and slope—come through with only two yields.

Spreads

A convenient way to discuss many term-structure changes is to look at the spread between long and short yields. From (10) and (12), this becomes

$$y_{2t} - y_{1t} = (1/2)\left(1 - \gamma^2\sigma^2/2\right)(1-\phi)\,\mu + (1/2)\Big\{1 + (\phi - 2)\left(1 - \gamma^2\sigma^2/2\right) - \left[\gamma + \left(1 - \gamma^2\sigma^2/2\right)\right]^2 \sigma^2/2\Big\} x_t. \tag{13}$$

Now to return (and about time) to the central question of this paper: How do productivity changes affect the term structure? As may be apparent by now, getting the answer will not be easy for two very different reasons. First, the many and complicated terms in equations (10)–(13) indicate that there are fairly complicated interactions going on, and comparative statics will result in some messy algebra. A deeper reason is that the phrase "changes in productivity" now has no unambiguous meaning. Does "change" mean an increase in the average level, μ, a high (or low) value of x_t, or perhaps a higher variance, σ^2, or mean reversion parameter, ϕ? Not every change is worth looking at, but understanding a few key changes will shed light on some central aspects of the relation between productivity and the term structure.

First, consider an increase in the mean of the productivity factor μ, holding everything else constant. This indicates that the long-run average productivity of the economy has increased; we have entered a "new era" of high growth. What does this do to the term structure? A quick look at equations (10)–(13) shows that y_{1t} is unchanged, and that the effect on y_{2t} depends on the sign of

$$\left(1 - \gamma^2\sigma^2/2\right)(1-\phi). \tag{14}$$

It will also be apparent that the sign of $\left(1 - \gamma^2\sigma^2/2\right)$ should be positive if a positive level of the productivity factor, x_t, implies a positive interest rate (yield). Furthermore, as long as the factor adjusts towards the mean but does not immediately jump to the mean, (that is, $0 < \phi < 1$), both parts of (14) will be positive, and an increase in productivity will steepen the slope of the term structure.

Intuitively, this simply says that because bonds compete with real, productive assets, when the return on those productive assets is expected to be higher in the long run, real interest rates are expected to be higher as well. If that increase doesn't show up directly in today's productivity (x_t), the part of the effect that shows up in long-term rates creates a steeper term structure.

What happens if the productivity factor itself, x_t, is higher? This corresponds to a temporary shock, an increase in productivity for a limited time. A glance at (10) shows that this increase in productivity raises short-term rates, as is to be expected. The effect on long rates and thus on the slope of the term structure is more difficult to ascertain. In fact, a direct attack along the lines of equations (13) and (14) would be unenlightening. Comparing (10) and (12a), and discussing how the productivity shocks affect expected rates and risk terms, will prove more fruitful.

An increase in x_t increases y_{1t}, as discussed above. It also may increase y_{2t}, depending on the relative sizes of the expectations effect and the risk effect. What does it mean for the slope of the term structure? Using (12a) and (10) to compare the expectations part of the two-period rate with the one-period rate shows us that

$$y_{2t} \approx (1/2)\left\{\left(1 - \gamma^2\sigma^2/2\right)x_t + \left(1 - \gamma^2\sigma^2/2\right)\left[\mu - \phi\left(x_t - \mu\right)\right]\right\}.$$

This implies that

$$y_{2t} - y_{1t} \approx (1/2)\left(1 - \gamma^2\sigma^2/2\right)\left[\phi - 1\left(x_t - (1-\phi)\,\mu\right)\right].$$

Taking the derivative with respect to x_t yields

$$\partial\left(y_{2t} - y_{1t}\right)/\partial x_t = (1/2)\left(1 - \gamma^2\sigma^2/2\right)(\phi - 1). \tag{15}$$

Since $\left(1 - \gamma^2\sigma^2/2\right) > 0$ and $\phi < 1$, an increase in the productivity factor, x_t, decreases the spread between rates, meaning that the yield curve gets *flatter*. Effectively, because of reversion to the mean, a higher x_t today has less of an impact tomorrow; if x is above the mean, x_t tends to get pulled down, and if x_t is below the mean, increasing it lessens the pull upward by the mean. The net result is that the effect of the productivity shock on interest rates today is larger than the expected effect on interest rates tomorrow.

This, of course, is only one aspect of an increase in x_t. Because the economy is risky, two-period bonds are not merely the average of current and expected rates; after all, that is why equations (12) and (12a) contain variance terms. The term that remains in (12a) after accounting for the expectations hypothesis approximation is

$$-\left[2\gamma\left(1 - \gamma^2\sigma^2/2\right) + \left(1 - \gamma^2\sigma^2/2\right)^2\right]\left(\sigma^2/2\right)x_t.$$

Clearly this is negative, since γ, $\left(1 - \gamma^2\sigma^2/2\right)$, and σ^2 are positive. An increase in x_t lowers the risk factor, decreasing two-period rates and the slope of the term structure.

Why does an increase in the productivity factor, x_t, decrease the risk factor in two-period yields? There are two parts to the answer. The first has to do with the heteroskedastic aspect of the square-root process.

An increase in x_t increases the variance of the productivity process. This means that investing in the real economy is now riskier, which leads to the second part of the answer. Because the real economy is riskier, investors will pay a premium for a safe bond that delivers them from that risk. The higher price means a lower yield. If the world really does work this way, a higher productivity shock, though good in one sense (directly higher productivity), is bad in another (higher risk).

Other Assets

Bonds are not the only financial assets around. Productivity shocks will also affect stocks, options, swaps, and other derivatives. One way to price these assets is to start with (7) and (9), specifying the return process for the asset in question. Since we've only assumed one source of uncertainty in the economy, (x_t), however, the relations between the different assets might be rather simplistic. Conceptually, at least, it is straightforward to add more shocks.

This might even be done in a way that preserves the results so far. Let the pricing kernel take the form

$$P_t = E_t\left(M_{t+1}K_{t+1}\right),$$

where K_{t+1} is independent of M_{t+1} and is a martingale (that is, $E_t\left[K_{t+1}\right] = K_t$). Then $P_t = E_t\left(M_{t+1}\right)K_t$ and $P_{2t} = E_t\left(M_{t+1}P_{1\,t+1}\right)K_t$. Thus the spread between long and short rates, which depends on the ratio P_{2t}/P_t, is independent of K_t. But the extra factor, K_t, would show up in pricing other assets such as stocks.

III. Conclusion

Bond traders, stock jobbers, and risk managers all have their own reasons for understanding the course of interest rates. The Federal Reserve's Federal Open Market Committee derives its concern from its mandate for monetary policy, and that policy involves correctly setting one interest rate among many. Setting the path for the federal funds rate is itself complicated by the complex interactions of the funds rate with T-bill, mortgage, and other interest rates.

Productivity plays a crucial role in the interactions of the various interest rates, but its effect is not always simple. An increase in the long-run mean of productivity will increase long-term interest rates and cause the term structure to get steeper. An increase in today's productivity tends to increase both short- and long-term interest rates, but long-term rates move less, causing the term structure to get flatter.

Real-world productivity shifts will rarely be so cut and dried. The central, as yet unanswered, questions—such as whether recent productivity increases are permanent or temporary—matter greatly for the term structure, as they yield diametrically opposed conclusions. Thus, economic theory provides some guidance about the appropriate questions to ask. It also raises further questions. For example, in a truly "new paradigm economy" shouldn't we expect to see changes in other parameters of the productivity process, —such as the speed of adjustment—that theory tells us are important for the term structure?

So, in one sense, a more sophisticated view has complicated the matter. Just as a wine connoisseur would not hazard a recommendation until he knew whether beef or fish were being served, advice about interest rates often requires that we specify more details about the underlying economy.

Appendix

Finding the values that maximize expected lifetime utility is perhaps easiest done using dynamic programming (see Sargent [1987] for an excellent exposition). The state variables are $\left[A_t, r_t, R_{t-1}\right]$ and the control variables are $\left[s_t, b_t\right]$. Forming Bellman's equation gives

$$V\left(A_t, r_t, R_{t-1}\right) = \max_{s_t, b_t}\Big\{u\left(A_t - s_t - b_t\right) + \beta E_t V\left(s_t R_t + b_t r_t, r_{t+1}, R_t\right)\Big\}. \tag{A.1}$$

The first-order necessary conditions for the "max" part of (A.1) are given by:

$$\frac{\partial V}{\partial s_t} = 0 \Rightarrow -u'\left(A_t - s_t - b_t\right) + \beta E_t R_t V_1\left(s_t R_t + b_t r_t, r_{t+1}, R_t\right) = 0 \tag{A.2}$$

$$\frac{\partial V}{\partial b_t} = 0 \Rightarrow -u'\left(A_t - s_t - b_t\right) + \beta E_t r_t V_1\left(s_t R_t + b_t r_t, r_{t+1}, R_t\right) = 0. \tag{A.3}$$

Next, using the Benveniste and Scheinkman (1979) results on the differentiability of the value function V to evaluate V_1 yields $V_1\left(A_t, r_t, R_{t-1}\right) = u'\left(c_t\right)$. Substituting this into (A.2) yields the *Euler equation*, $0 = -u'\left(c_t\right) + \beta E_t\left[R_t u'\left(c_{t+1}\right)\right]$ or

$$u'\left(c_t\right) = \beta E_t\left[R_t u'\left(c_{t+1}\right)\right]. \tag{A.4}$$

(A.4) is equation (2) in the text.

Given equation (A.4), the next step is to solve for the *policy functions* $s_t\left(A_t, R_{t-1}, r_t\right)$ and $b_t\left(A_t, R_{t-1}, r_t\right)$. Substituting these into the Euler equation (A.4) gives

$$u'\left[A_t - s_t\left(A_t, R_{t-1}, r_t\right) - b_t\left(A_t, R_{t-1}, r_t\right)\right] = \beta E_t\left\{R_t u'\left[A_t - s_t\left(A_t, R_{t-1}, r_t\right) - b_t\left(A_t, R_{t-1}, r_t\right)\right]\right\} \tag{A.5}$$

for R_t and substituting them into the corresponding Euler equation for bonds implies

$$u'\left[A_t - s_t\left(A_t, R_{t-1}, r_t\right) - b_t\left(A_t, R_{t-1}, r_t\right)\right] = \beta E_t\left\{r_t u'\left[A_t - s_t\left(A_t, R_{t-1}, r_t\right) - b_t\left(A_t, R_{t-1}, r_t\right)\right]\right\} \tag{A.6}$$

for the bond rate, r_t.

Using log utility implies that the Euler equation (A.5) takes the form

$$\left[A_t - s_t(\cdot) - b_t(\cdot)\right]^{-1} = \beta E_t\Big\{R_t\left[R_t s_t(\cdot) + r_t b_t(\cdot) - s_{t+1}(\cdot) + b_{t+1}(\cdot)\ \right]^{-1}\Big\}. \tag{A.7}$$

The point is now to guess a form for the policy functions s_t and b_t and to see if they work. Fortunately, log utility results in an especially simple form, namely, $s_t = kA_t$ and $b_t = \ell A_t$, where k and ℓ are just two constants (to be determined later). This transforms the Euler equation (A.7) into

$$\left[(1-k-\ell)A_t\right]^{-1} = \beta E_t\left[R_t\left(R_t k A_t + r_t \ell A_t - kA_{t+1} + \ell\ A_{t+1}\right)^{-1}\right]. \tag{A.8}$$

This simplifies to

$$\left(A_t - kA_t - \ell A_t\right)^{-1} = \beta E_t\Big\{R_t\left[R_t k A_t + r_t \ell A_t (1-k-\ell)\right]^{-1}\Big\},$$

which further reduces to equation (4) in the text.

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The Search-Theoretic Approach to Monetary Economics: A Primer

by Peter Rupert, Martin Schindler,
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Introduction

This paper presents some results in monetary theory derived using very simple game-theoretic models of the exchange process. The underlying model is a variant of *search theory*, a framework that has been used extensively in a wide variety of applications. This approach is well suited to discussing the process of exchange and money's role in the process. The approach utilized here is explicitly strategic, in the following natural sense: When I decide whether to accept in trade a certain object other than one I desire for my own consumption—for example, *money*—I must conjecture as to the probability that other agents will accept it from me in the future. This evidently ought to be modeled as a game.

In search theory, the type of game to be considered is explicitly dynamic, and exchange takes place in real time. Also, the models allow us to focus precisely on various frictions in the exchange process that might give money a role in an equilibrium, or efficient, arrangement. Among the frictions are these: Agents are not always in the same place at the same time; long-run commitments cannot be enforced; and agents are anonymous in the sense that their histories are not public information. Such frictions are crucial for a logically coherent theory of money; the approach described here helps to clarify each one's role.

This approach contrasts with attempts to model the role of money in a competitive equilibrium (Walrasian) model—a difficult task that has met with mixed success at best. In a competitive equilibrium model, the exchange process is not explicitly modeled. That is, agents start with an initial allocation A and choose a final allocation B so as to maximize utility, subject to the latter not costing more than the former, but how they get from point A to point B is not discussed. Does some unmodeled agent (maybe the auctioneer) make the necessary trades with a "pick-up and delivery service"? Or do the agents trade directly with each other? Do they trade bilaterally or multilaterally? In real time, or before production and consumption activity starts? Do they barter directly or trade indirectly using media of exchange? The standard competitive equilibrium paradigm does not address such questions. Search models, in contrast, are designed with exactly these issues in mind and therefore are logical tools for studying monetary economics, as we shall illustrate.

I. The Basic Model

To model anonymous trade, it is natural to start with a large number of agents—formally, we assume a $[0,1]$ continuum. For simplicity, we assume these agents live forever and discount the future at rate r. There is a $[0,1]$ continuum of indivisible consumption goods. To generate gains from trade, we need to assume that agents are specialized. There are many ways to do this, but an easy one is to assume that each agent i is able to produce just one type of good. The unit production cost for any agent is $c \geq 0$. For convenience, we assume that these goods cannot be stored and so must be consumed immediately after they are produced. Obviously, this means that consumption goods cannot serve as media of exchange, allowing us to highlight the role of money.

To make trade interesting in the model, we need to assume that tastes are heterogeneous. Again, there are many ways to do this, but for simplicity we assume the following: First, given any two agents i and j, write iWj to mean "i wants to consume the good that j produces"—in the sense that i derives utility $u > c$ from consuming what j produces if iWj, and he derives utility 0 from consuming what j produces otherwise. Then, for any randomly selected agents, we assume $prob(iWi) = 0$, $prob(jWi) = x$, and $prob(jWi|iWj) = y$. The first assumption, $prob(iWi) = 0$, means that no agent ever wants to consume his own output (which is why they trade). The second assumption parameterizes the extent of the basic search friction: The smaller x is, the lower the probability that a random trader has what you want. However, the third assumption is the important one, since it parameterizes Jevons' (1875) famous double coincidence of wants problem: The smaller y is, the lower the probability that a trader who has what you want also wants what you have.[1]

Besides the consumption goods already mentioned, there is another object called *money*, which consists of an exogenously fixed quantity of $M \in [0,1]$ indivisible units of a storable object (of course, money must be storable to be useful). Holding money yields utility γ: If $\gamma > 0$, money pays a dividend, like many real assets, and if $\gamma < 0$, then money has a storage cost; $\gamma = 0$ describes the case of pure fiat money. Although this last case may be the most interesting, for generality we allow $\gamma \neq 0$. Initially, one unit of money is randomly allocated to each of M agents. Although we will relax this later, for now we assume that agents holding money cannot produce (one way to motivate this is to assume that after producing, you need to consume before you can produce again). Thus, no one can ever acquire more than one unit of money, and so an agent always holds either 0 or 1 unit of money. To simplify the presentation, we do not allow agents to freely dispose of money, but this is never binding except in one case mentioned below.

We now describe the trading process. Rather than assuming a centralized (Walrasian) market, here the agents must trade bilaterally. The simplest way to model this is to assume that they meet according to a pairwise random matching process. Upon meeting, a pair decide whether to trade, then part company and reenter the matching process. Let α denote the (Poisson) arrival rate in the matching process—that is, the probability of meeting someone in a given unit of time.[2] For reasons discussed later, we assume the history of any agent's past meetings and trades is not known to anyone else.

We want to analyze agents' individual trading strategies. An agent obviously should never accept a good in trade if he does not want to consume it, since goods are not storable. Whenever possible, agents should barter for a good they do want to consume (the case of a double coincidence). What needs to be determined is whether an agent should trade goods for money and money for goods. Let π_0 denote the probability that the representative agent trades goods for money, and let π_1 denote the probability that he trades money for goods. These must satisfy the equilibrium conditions given below. We will say that money is used as a medium of exchange, or *circulates*, if and only if $\pi = \pi_0\pi_1 > 0$. Let V_0 and V_1 be the value functions (lifetime, discounted, expected utility) of agents with 0 or 1 units of money. Since we consider only stationary and symmetric equilibria, V_j does not depend on time or on the agent's name, only his money inventories.

If we think of time as proceeding in discrete periods of length τ, we can calculate the payoff of holding money as follows: The probability of meeting anyone during this period is approximately $\alpha\tau$ by the Pois-

1 Several notions of specialization in the literature are special cases of this model. For example, in Kiyotaki and Wright (1991) or Aiyagari and Wallace (1991), there are N goods and N types of agents, where type n produces good n and wants good $n+1 (\text{mod } N)$. Then $x = 1/N$, and $y = 1$ if $N = 2$, while $y = 0$ if $N > 2$. Alternatively, in Kiyotaki and Wright (1991, 1993) and much of the related literature, the events $\{iWj\}$ and $\{jWi\}$ are independent, and so $y = x$.

2 It is the bilateral rather than the random matching assumption that is important. Corbae, Temzelides, and Wright (2000) show how to redo the model, allowing agents to choose endogenously whom they meet, rather than meeting at random. Their model shares the basic insights discussed here, although it is complicated by the need to determine equilibrium meeting patterns as well as equilibrium trades.

son assumption.[3] If the person you meet can produce (meaning, in the version of the model that we are considering here, he does not have money), which occurs with probability $1-M$, and you want what he can produce, which occurs with probability x, and you both want to trade, which occurs with probability π in equilibrium, then you trade, consume, and continue without money, for a total payoff of $u+V_0$. In all other events (you meet no one, you meet someone with a good you do not want, etc.), you simply continue with your money, for a payoff of V_1. In all events, you also get $\gamma\tau$ from storing the money. Hence,

$$\begin{aligned} V_1 = & \frac{1}{1+r\tau}\Big\{\alpha\tau(1-M)x\pi(u+V_0) \\ & + [1-\alpha\tau x(1-M)\pi]V_1 + \gamma\tau + o(\tau)\Big\}, \end{aligned}$$

where $o(\tau)$ is the approximation error associated with the Poisson process and hence satisfies $o(\tau)/\tau \to 0$ as $\tau \to 0$. Rearranging, we have

$$r\tau V_1 = \alpha\tau(1-M)x\pi(u+V_0-V_1) + \gamma\tau + o(\tau).$$

Dividing by τ and taking the limit as $\tau \to 0$, we arrive at the continuous-time Bellman's equation,

$$(1) \qquad rV_1 = \alpha x(1-M)\pi(u+V_0-V_1) + \gamma.$$

An analogous argument implies that the value function for an agent without money satisfies

$$(2) \quad rV_0 = \alpha xy(1-M)(u-c) + \alpha xM\pi(V_1-V_0-c).$$

The first term in this expression represents the gain from a direct barter trade, while the second represents the gain from trading goods for money with probability π. Notice that you can only barter when there is a double coincidence of wants and the other person has no money.

II. Equilibrium

Define the net gain from trading goods for money by $\Delta_0 = V_1 - V_0 - c$, and the net gain from trading money for goods by $\Delta_1 = u + V_0 - V_1$. If we normalize $\alpha x = 1$ to reduce notation (which we can always do with no loss of generality by redefining units of time appropriately), we have:

$$(3) \qquad \Delta_1 = \frac{[M\pi + (1-M)y](u-c) + ru - \gamma}{r+\pi}$$

$$(4) \qquad \Delta_0 = \frac{(1-M)(\pi-y)(u-c) - rc + \gamma}{r+\pi}.$$

The equilibrium conditions for π_0 and π_1 are

$$(5) \qquad \pi_j \begin{cases} = 1 \\ \in [0,1] \\ = 0 \end{cases} \text{as } \Delta_j \begin{cases} > 0 \\ = 0 \\ < 0. \end{cases}$$

Notice that Δ_j depends on π, so to see if some candidate π_0 and π_1 constitute an equilibrium, one simply inserts the π_j and checks equation (5).

Consider first the case $\gamma = 0$ (fiat money). This implies $\Delta_1 > 0$ for all parameter values, so we always have $\pi_1 = 1$. Also, Δ_0 is equal in sign to $\pi_0 - \hat{\pi}$, where

$$(6) \qquad \hat{\pi} = \frac{rc + (1-M)y(u-c)}{(1-M)(u-c)}.$$

Notice that $\pi_0 = 0$ is always an equilibrium: Since $\hat{\pi} > 0$, $\pi_0 = 0$ implies $\pi_0 < \hat{\pi}$, which implies $\Delta_0 < 0$. Thus, $\pi_0 = 0$ satisfies the equilibrium condition. Naturally, there is an equilibrium in which no one accepts fiat money. However, if

$$c < \frac{(1-M)(1-y)u}{r + (1-M)(1-y)},$$

then $\hat{\pi} < 1$, which means $\pi_0 = 1$ is an equilibrium as well. So there is also an equilibrium where fiat money circulates as long as c is not too big.[4] Finally, if $\hat{\pi} < 1$, there is also a mixed-strategy equilibrium in which $\pi_0 = \hat{\pi}$. In this case, if other agents accept money with exactly the right probability $\hat{\pi}$, you are indifferent as to accepting or rejecting it, so randomizing is an equilibrium.

The above model is essentially that of Kiyotaki and Wright (1993), except that they assume $c = 0$ in addition to $\gamma = 0$. This implies that three equilibria necessarily exist, $\pi = 0$, $\pi = 1$, and $\pi = y$. In the mixed-strategy equilibrium, money is accepted with the same probability as a good (since y is the probability of a double coincidence), which makes you indifferent. When $c > 0$, money must have a strictly greater probability of being accepted than a barter trade for you to be indifferent about accepting money, because you must incur the production cost to get the money. More generally, when $\gamma \neq 0$, we have to determine π_1 endogenously; for example, if γ is large, then agents may prefer to hoard rather than spend their money.

The results for the general case are summarized as follows:

3 That is, the probability of meeting one person in a period of length τ is $\alpha\tau + o(\tau)$, and the probability of meeting more than one is $o(\tau)$, where $o(\tau)$ satisfies $o(\tau)/\tau \to 0$ as $\tau \to 0$.

4 Alternatively, for a given $c > 0$, we can say that r and M must be relatively small for a monetary equilibrium to exist (agents must be patient and money not too plentiful).

Proposition 1. *There are five types of equilibria, and they exist in the following regions of parameter space:*

1. $\pi_0 = 1$ *and* $\pi_1 = 0$ *is an equilibrium iff* $r \leq \bar{r}_2$
2. $\pi_0 = 0$ *and* $\pi_1 = 1$ *is an equilibrium iff* $r \geq \bar{r}_3$
3. $\pi_0 = 1$ *and* $\pi_1 \in (0,1)$ *is an equilibrium iff* $\bar{r}_1 < r < \bar{r}_2$
4. $\pi_0 \in (0,1)$ *and* $\pi_1 = 1$ *is an equilibrium iff* $\bar{r}_3 < r < \bar{r}_4$
5. $\pi_0 = 1$ *and* $\pi_1 = 1$ *is an equilibrium iff* $\bar{r}_1 \leq r \leq \bar{r}_4$

where the critical values of r are given by

$$\bar{r}_1 = \frac{\gamma - [M + (1-M)y](u-c)}{u}$$
$$\bar{r}_2 = \frac{\gamma - (1-M)y(u-c)}{u}$$
$$\bar{r}_3 = \frac{\gamma - (1-M)y(u-c)}{c}$$
$$\bar{r}_4 = \frac{\gamma + (1-M)(1-y)(u-c)}{c}.$$

These are the only (steady-state) equilibria.

Proof: See the appendix.

We characterized the regions where the different equilibria exist in terms of r, but we could have used some other parameter, such as c. Routine algebra implies $\bar{r}_1 < \bar{r}_2 < \bar{r}_3 < \bar{r}_4$. Also, note that our assumption of no free disposal is never binding, except possibly when $\pi_0 = 0$, and even then only if $\gamma < 0$.[5] More importantly, there are equilibria where $\pi > 0$ and $\gamma < 0$; that is, agents value money and use it as a medium of exchange despite its storage cost. We will have more to say about the economics underlying the above results in the next section, after we introduce a slight variation on the model, since it will be interesting to compare the two versions.

III. Alternative Specification

A key assumption in the above model is that agents holding money cannot produce; this is what prevents them from acquiring more than a single unit of money. The fact that agents hold either 0 or 1 unit of money is what makes the model so tractable (see below). Although the assumption that agents holding money cannot produce is common in the literature, it has some undesirable implications. For example, if two agents with money meet and there is a double coincidence of wants, they cannot trade. A related implication is that as M increases, the productive capacity of the economy necessarily decreases, which makes it difficult to interpret the effects of changes in the money. So here we present an alternative model, first discussed by Siandra (1993, 1996), where agents with money *can* produce, and we simply impose the condition that agents can store, at most, one unit of money.

The first issue to be resolved is, what happens in a double coincidence when you have money and the other person does not—do you barter or pay with cash?[6] We resolve this by allowing agents to play the following simple game: First, with probability β the agent with money is chosen and with probability $1-\beta$ the agent without money is chosen to propose either a barter or a cash transaction (in principle they could also propose not to trade at all, but we ignore this option, which will always be dominated by proposing barter). Second, the other agent responds either by accepting, which executes the proposal, or rejecting, which implies they part company (figure 1). A strategy for the agent with j units of money, $j = 1$ or 0, is denoted as ψ_j, which equals the probability that he proposes barter (and so $1-\psi_j$ is the probability he proposes cash).

Proposition 2. *In a double-coincidence meeting between an agent with and an agent without money, generically the unique subgame-perfect equilibrium in pure strategies is* $\psi_0 = \psi_1 = 1$.

Proof: See the appendix.

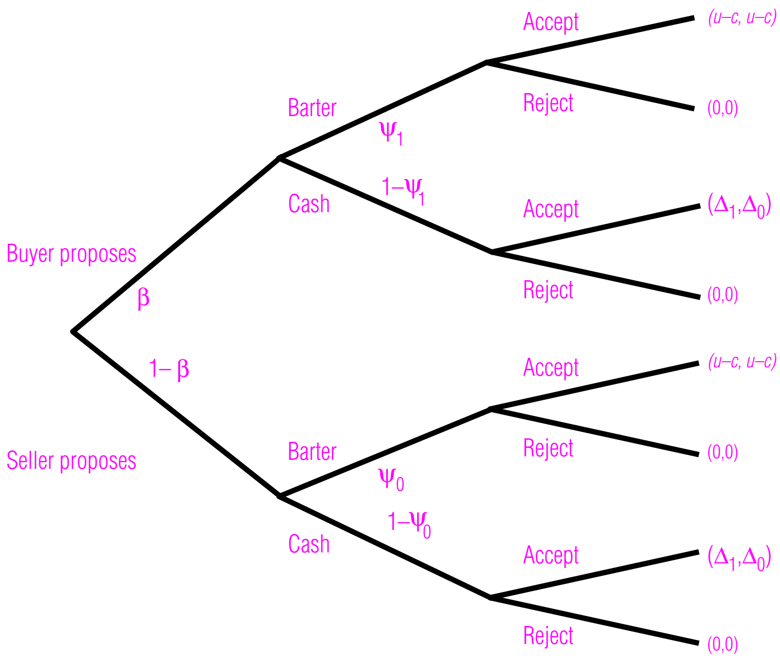
Having resolved the ambiguity that arises when both barter and cash are available, we can now derive the Bellman equations. Again, let time proceed in discrete

5 To be precise, we should say what agents do after disposing of their money. We assume here that they cannot trade, as they cannot produce. Hence, agents will dispose of money and drop out of the trading process iff $V_1 < 0$. Since it is easy to see that $V_0 \geq 0$ in any equilibrium and $V_1 \geq V_0$ in any equilibrium with $\pi_0 > 0$, the only case where disposal could potentially occur is $\pi_0 = 0$, which implies $V_1 = \gamma/r$. Hence, agents dispose of money if and only if $\pi_0 = 0$ and $\gamma < 0$.

6 This is the only ambiguous case; every other meeting has only one feasible transaction (that is, if you encounter a double coincidence and have no money, barter is the only option). The issue did not come up in the previous section, because agents with money cannot barter.

FIGURE 1

Game Tree



periods of length τ. Then

$$(7)\quad V_1 = \frac{1}{1+r\tau}\{\alpha\tau xy(u-c+V_1) + \alpha\tau(1-M)(1-y)x\pi(u+V_0) + [1-\alpha\tau xy-\alpha\tau(1-M)(1-y)x\pi]V_1 + \gamma\tau + o(\tau)$$

$$(8)\quad V_0 = \frac{1}{1+r\tau}\{\alpha\tau xy(u-c+V_0) + \alpha\tau M(1-y)x\pi(V_1-c) + [1-\alpha\tau xy-\alpha\tau M(1-y)x\pi]V_0 + o(\tau)\ ,$$

where we temporarily reintroduce αx to facilitate comparison to equations (1) and (2) in the previous model. Observe, for example, that now a money holder barters every time he encounters a double coincidence, which occurs with probability $\alpha\tau xy$. Indeed, he uses money only when he encounters a single coincidence, which occurs with probability $\alpha\tau(1-y)x$, and the other agent does not have money and they both agree to trade, which occurs with probability $(1-M)\pi$.

Rearranging, we let τ 0 and normalize $\alpha x = 1$ as before, to write the continuous-time Bellman equations for the alternative model

$$(9)\quad rV_1 = y(u-c) + (1-y)\pi(1-M)(u+V_0-V_1)+\gamma$$

$$(10)\quad rV_0 = y(u-c) + (1-y)\pi M(V_1-V_0-c).$$

Although the value functions are different across the two models, we compute Δ_j and define equilibrium exactly as in the last section. The results are as follows.

Proposition 3. *In the alternative model, where agents with money can produce, there are five potential types of equilibria and they exist in the following regions of parameter space:*

1. $\pi_0 = 1$ *and* $\pi_1 = 0$ *is an equilibrium iff* $r \le \hat{r}_2$
2. $\pi_0 = 0$ *and* $\pi_1 = 1$ *is an equilibrium iff* $r \ge \hat{r}_3$
3. $\pi_0 = 1$ *and* $\pi_1 \in (0,1)$ *is an equilibrium iff* $\hat{r}_1$ r $\hat{r}_2$
4. $\pi_0 \in (0,1)$ *and* $\pi_1 = 1$ *is an equilibrium iff* $\hat{r}_3$ r $\hat{r}_4$
5. $\pi_0 = 1$ *and* $\pi_1 = 1$ *is an equilibrium iff* $\hat{r}_1 \le r \le \hat{r}_4$,

where the critical values of r are given by

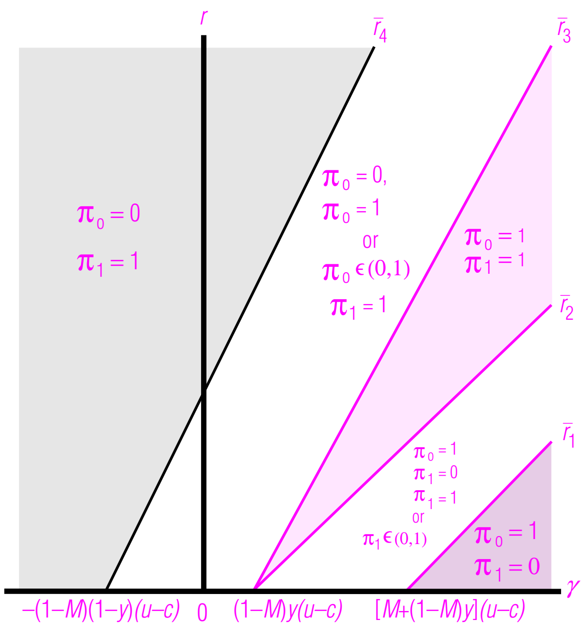
$$\hat{r}_1 = \frac{\gamma - M(1-y)(u-c)}{u}$$
$$\hat{r}_2 = \gamma/u$$
$$\hat{r}_3 = \gamma/c$$
$$\hat{r}_4 = \frac{\gamma + (1-M)(1-y)(u-c)}{c}.$$

These are the only (steady-state) equilibria.

Proof: The proof is analogous to that of Proposition 1 and so is omitted for brevity.

FIGURE 2

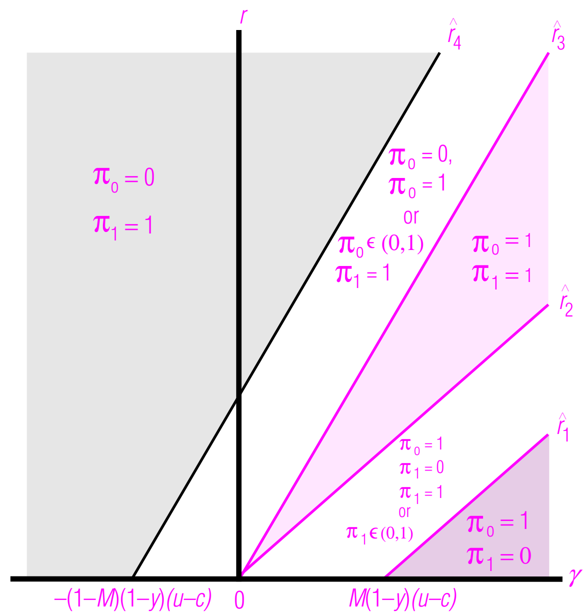
Equilibria in (γ, r)-Space When Money Holders Cannot Produce



The regions of (γ, r) space where the different equilibria exist in the two models (those where money holders cannot produce and those where they can produce) are shown in figures 2 and 3. The same five types of equilibria exist in both models; the regions where they exist are similar but quantitatively different—unless $y = 0$, of course, since the models are identical when there is no barter. In either case, when γ is very low the only equilibrium is one in which no one accepts money; if γ is very high, the only equilibrium is one in which no one spends it. Hence, money circulates if and only if its intrinsic properties are not too bad or too good. Also, both models include a region

FIGURE 3

Equilibria in (γ, r)-Space When Money Holders Can Produce



where the unique equilibrium is $\pi = 1$, as well as other regions where there are multiple equilibria: In one region, we must have $\pi_1 = 1$, but π_0 can be 0, 1, or between 0 and 1; in another region, we must have $\pi_0 = 1$, while π_1 can be 0, 1, or between 0 and 1.[7]

The models differ in that it is actually more difficult to get money to circulate when money holders can produce; the potential region where $\pi > 0$ is larger when they cannot. Intuitively, agents with money are more willing to spend it when they cannot produce, since doing so allows them to barter. If $\gamma = c = 0$, the differences between the two models are especially stark: When money holders cannot produce, there are always three equilibria, $\pi = 0$, $\pi \in (0, y)$, and $\pi = 1$; in the other model there are two, $\pi = 0$ and $\pi = 1$. The intuition is as follows: When money holders can produce, there is no cost associated with acquiring money when we set $\gamma = c = 0$, so if there is a strictly positive probability of money being accepted, you should always accept it. This is not true when money holders can-

7 It is well known that there is a strategic complementarity in the decision to accept money, π_0, but it is less well understood that the same is true of π_1. For some parameters, an agent is more willing to spend money if he believes that others will do the same. This only occurs when $\gamma > 0$.

not produce, because even when $\gamma = c = 0$, accepting money involves the opportunity cost of giving up your barter option.

The version of the model in which agents with money cannot produce is easy to motivate by saying that one must consume before producing, since this leads naturally to the result that agents in equilibrium always hold either 0 or 1 unit of money. However, the alternative version where money holders can produce also seems more natural in some respects. For example, when two agents with money meet and there is a double coincidence, they trade. Of course, this version does require assuming directly that agents can only store 0 or 1 units of money. There is not necessarily a right or wrong model, and the choice should be dictated by how well it addresses a certain question and how tractable it is in any given application.

IV. Welfare

In this section, we discuss welfare, defined by $W = MV_1 + (1-M)V_0$ (average utility). For the purpose of discussion, we also set $\gamma = 0$. Using straightforward algebra, in the two models we have

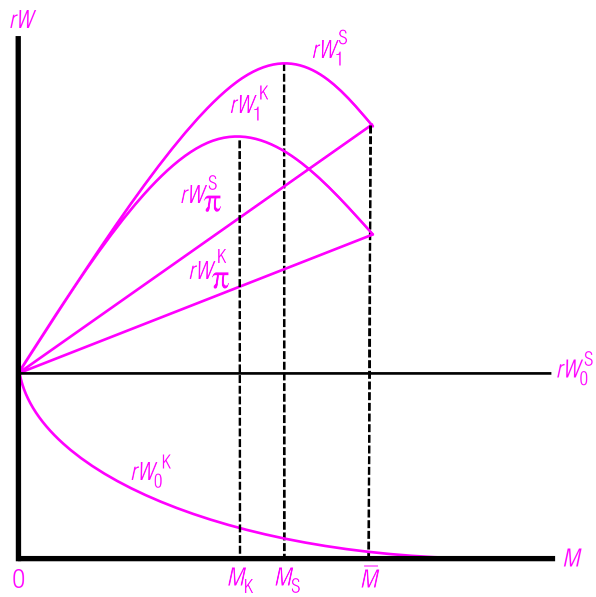
$$rW^K = (1-M)[(1-M)y + M\pi](u-c)$$
$$rW^S = [y + M(1-M)(1-y)\pi](u-c),$$

where the superscript K indicates the case where money holders cannot produce and the superscript S indicates the case where they can. Notice first that when we compare two equilibria in either model, other things being equal, W is greater in the equilibrium with the higher π. This simply says that the more acceptable money is, the more useful it is. The next thing to notice is that across the two models, given any π we have $W^S \quad W^K$ with strict inequality as long as $y > 0$ and $M > 0$; not surprisingly, agents are better off if we allow money holders to produce.

We now want to focus on equilibrium with $\pi = 1$ and consider maximizing W with respect to M. The result for the model in which money holders can produce is $M = \frac{1}{2}$. The intuition is simple. Money is useful because it facilitates trade every time an agent i with money and an agent j without money meet and iWj holds. To maximize the frequency of such meetings, we should have half the agents holding money and half of them not: $M = \frac{1}{2}$. For the other model, the welfare-maximizing policy is $M = \frac{1 \quad 2y}{2 \quad 2y}$ if $y < \frac{1}{2}$, and $M = 0$ if $y \quad \frac{1}{2}$. Hence, the optimal M is lower in this model, simply because when money holders cannot produce, increasing M "crowds out" barter. Still, for small y the welfare-maximizing M is positive because it facilitates the exchange process *even though* money "crowds out" barter.

FIGURE 4

Welfare as a Function of M



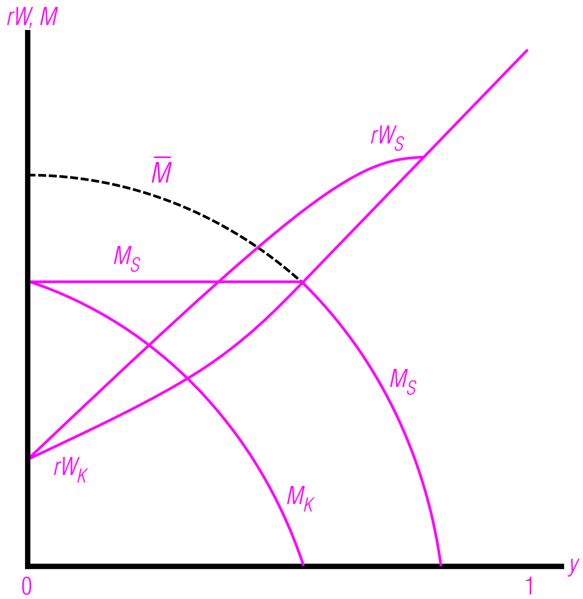
This discussion ignores the fact that $\pi = 1$ is not an equilibrium for all parameter values. If $\gamma = 0$, it is easy to see that in either version of the model, $\pi = 1$ is an equilibrium if and only if $M \leq \overline{M} = 1 - \frac{rc}{(1 \quad y)(u \quad c)}$. The true optimal policy, then, is the minimum of $\overline{M}$ and the values given above. In figure 4, we depict welfare in each type of equilibrium by the curves W_0^j, W_π^j, or W_1^j, where money is accepted with probability 0, π, or 1. The superscript $j = K$ or S refers to the two different models. The curves are drawn only for values of M such that the equilibria exist. Figure 5 shows welfare as a function of y, given that we set M to its welfare-maximizing level (which depends on y). These figures illustrate various properties, including: W is always higher when money holders can produce; and W increases with π across equilibria in either model.

V. Essentiality of Money

At this stage, it is instructive to highlight the role of various frictions in the model, to understand what makes money essential. Following Hahn (1965), we say money is essential if it allows agents to achieve allocations they cannot achieve with other mechanisms that also respect the enforcement and information con-

FIGURE 5

Welfare as a Function of y (optimal M)



straints in the environment.[8]

First, we need some sort of double coincidence of wants problem. For this it is important that not everybody can trade multilaterally. For example, consider an economy with three agents, where each agent of type $n \in \{1, 2, 3$ produces good n and wants good $n+1$ (mod 3). If all agents meet at the same place and are able to make multilateral trades, it is feasible for each agent to produce and consume in the period: Agent 2 produces for agent 1, agent 1 produces for agent 3, and agent 3 produces for agent 2. Our restriction to bilateral meetings, given specialized tastes and technology, is merely a convenient way to generate a double-coincidence problem. In the three-agent example, in every bilateral meeting one agent wants what the other produces, while the other does not.

Second, even given a double-coincidence problem, for money to be essential it is also important that agents cannot commit to long-term agreements. Consider the following *credit arrangement*: "produce for anyone you meet who wants your good." This arrangement resembles credit in the sense that agents receive consumption today in exchange for nothing but a "promise" to repay someone in kind at a future date. It is also obviously an efficient arrangement; that is, it generates the maximum possible expected utility, say $W_c = (u-c)/r$, given the normalization $\alpha x = 1$, where the subscript c on W_c stands for credit. If agents could commit to this arrangement ex ante, they would all agree to do so, and there would be no need for money. Clearly, an imperfect ability to commit to future actions is important if money is to have an essential role.

However, even in the absence of explicit commitments, cooperative agreements like the credit arrangement can sometimes be enforced by reputational considerations *if* individual actions are public information. Thus, consider the arrangement: "produce for anyone who wants your production good as long as everyone else has done so in the past; as soon as someone deviates from this, trigger to plan X," where plan X is to be determined. Of course, plan X must be self-enforcing (that is, it must be an equilibrium), and we want the outcome of plan X to be sufficiently unpleasant to keep agents from deviating from the efficient arrangement. We will assume here that plan X is to trade if and only if there is a double coincidence of wants, which generates expected utility $W_b = y(u-c)/r$, where the subscript b stands for barter.[9]

It is in individuals' self-interest not to deviate from the credit arrangement in which they are supposed to produce if and only if $-c + W_c \geq W_b$, which simplifies to $r \leq \tilde{r} = (1-y)(u-c)/c$. As always, if agents are sufficiently patient, the threat of triggering to pure barter supports the efficient outcome, and again money is not essential. Of course, this assumes that agents' trading histories can be observed publicly; otherwise, it is not possible for agents to use trigger strategies.[10] When trading histories are private information, the only sustainable outcome without money is pure barter. But if there is money, we can do better than pure barter, even when trading histories are pri-

8 For a recent treatment of this problem, see Kocherlakota (1998).

9 We do not trigger to autarky because we assume that if two agents want to barter without it being observed, they can; so the worst possible equilibrium is the one where none but double-coincidence trades occur. Nothing much hinges on this; a similar message holds if we can trigger to autarky, and it is, in fact, easier to support credit-like arrangements by triggering to autarky.

10 If the number of agents was small, then even if agents did not observe all other agents' histories but only their own, we could potentially support the efficient arrangement by the following strategy: "If ever you directly observe someone deviate (by not producing for you when you would like him to), stop producing for anyone else." This would set off a chain of agents who observe deviations and would eventually lead the economy into autarky. With a large number of agents, however, if I fail to produce for you, there is zero probability that in the future I will meet you or I will meet someone who has met you, and so the chain will never get back to me. So with a large number of agents, to support credit it does not suffice to have agents observe (only) their own histories. See Araujo (2000) for more discussion.

vate. Notice that monetary exchange generates lower welfare than the credit arrangement, although higher than pure barter.

Money does not do as well as credit because of the random-meeting technology and because money holdings are bounded, leading to some meetings where I want your good and you do not want mine, but either I have no money or you already have money. In these meetings, monetary exchange will not work, while credit could work as long as trading histories are publicly observable. Even if we relax the upper bound on money holdings, the fact that money holdings are bounded below by zero means that money cannot do as well as credit in a random-matching environment. We conclude that money has an essential role in the model for three reasons: the double-coincidence problem, the lack of commitment, and private information on trading histories. For an extended discussion of these issues, see Kocherlakota (1998, 2000) and Kocherlakota and Wallace (1998).

VI. Prices

This section provides an extension in which the assumption of indivisible goods is relaxed, although money is still indivisible and so agents will always have either 0 or 1 units of money. Following Shi (1995) and Trejos and Wright (1995), we will use bargaining theory to determine prices endogenously. For simplicity, we set $y = 0$, so there is no direct barter.[11]

Given that goods are perfectly divisible, let $u(q)$ be the utility of consuming q units of one's consumption good and $c(q)$ the disutility of producing q units of one's production good. We assume $u(0) = c(0)$, $u'(0) > c'(0) = 0$, $u'(q) > 0$, $c'(q) > 0$, $u''(q) \leq 0$, and $c''(q) \geq 0$, for $q > 0$, with at least one of the weak inequalities strict. For future reference, we define q^* by $u'(q^*) = c'(q^*)$. Also, there is a $\hat{q} > 0$ such that $u(\hat{q}) = c(\hat{q})$. When a buyer meets a seller who can produce the right good, they bargain over how much q will be exchanged for the buyer's unit of money, implying a nominal price $p = 1/q$. Otherwise, the model is exactly identical to that in the previous section.

Letting V_1 and V_0 denote the value functions and taking $q = Q$ as given, the generalizations of the dynamic programming equations can be expressed as

$$rV_1 = (1-M)[u(Q) + V_0 - V_1] \tag{11}$$

$$rV_0 = M[V_1 - V_0 - c(Q)]. \tag{12}$$

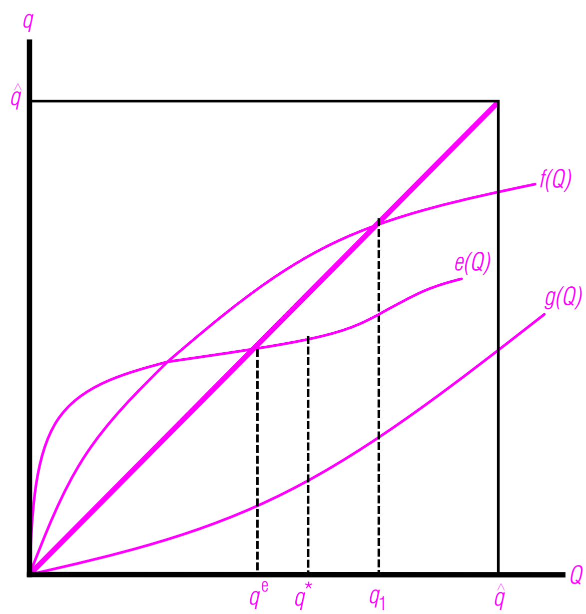
These can be easily solved for $V_1 = V_1(Q)$ and $V_0 = V_0(Q)$. Taking $V_1(Q)$ and $V_0(Q)$ as given, q will solve a bargaining problem. In equilibrium, of course, $q = Q$. The bargaining model can be formulated in several different ways. A typical approach is the generalized Nash bargaining solution,

$$q = \arg\max\, [u(q) + V_0(Q) - T_1]^{\theta} \times [V_1(Q) - c(q) - T_0]^{1-\theta}, \tag{13}$$

where θ is the bargaining power of the buyer and T_j is the threat point of the agent with j units of money. Also, the maximization is subject to $u(q) + V_0 \geq V_1$ and $V_1 - c(q) \geq V_0$. Here we will set $\theta = \frac{1}{2}$, and $T_1 = T_2 = 0$.[12]

FIGURE 6

Monetary Equilibrium in the Divisible-Goods Model



The bargaining solution in equation (13) defines a mapping $q = q(Q)$ from $[0, \hat{q}]$ into itself. That is, if other agents are giving Q units of output for one

11 This assumption makes it irrelevant whether we use the version in which money holders can produce or the one in which they cannot, since they are identical when $y = 0$. Shi (1995) and Trejos and Wright (1995) analyze the case with $y > 0$, but only for a special bargaining solution and only for the model where money holders cannot produce. Rupert, Schindler, and Wright (forthcoming) consider the general case.

12 It is also common to set the threat points equal to continuation values: $T_j = V_j$. Both can be derived from an underlying strategic model; see Osborne and Rubinstein (1990) for the bargaining theory.

unit of money, then a particular pair bargaining bilaterally will agree to $q = q(Q)$. An equilibrium is a fixed point, $q = q(Q)$. In general, we must be careful with the constraints on the bargaining problem: When $y = 0$, the constraints are never binding in equilibrium. However, if $y > 0$ the constraints may bind; therefore, it is instructive to proceed allowing for the possibility of binding constraints. The constraints can be rewritten $c(q) \leq D(Q)$ and $u(q) \geq D(Q)$, where $D(Q) = V_1(Q) - V_0(Q)$. The former constraint is satisfied if and only if $q \leq f(Q)$, and the latter is satisfied if and only if $q \geq g(Q)$, for increasing functions f and g. As figure 6 shows, both f and g go through the origin in the (Q,q) plane, and g lies below f and below the 45° line for all $Q \in [0,\hat{q}]$. Also, f crosses the 45° line at a unique $q_1 \in (0,\hat{q}]$. Hence, our search for equilibria can be constrained to the interval $[0,q_1]$.

The first-order condition for an interior solution to equation (13), taking $V_1 = V_0(Q)$ and $V_0 = V_0(Q)$ as given, is

$$[V_1(Q) - c(q)]u'(q) - [u(q) + V_0(Q)]c'(q) = 0.$$

This defines a function $q = e(Q)$, also shown in the figure. It too goes through the origin and intersects the 45° line at a unique point q^e. Hence, $q = q(Q)$ can be written as $q(Q) = \min\{e(Q), f(Q)$ for all $q \in [0,q^e]$, and $q(Q) = \max\{e(Q), g(Q)$ for all $q \in [q^e, q_1]$. For $q > q_1$, it does not really matter how we define $q(Q)$, which necessarily is below the 45° line, and we set it equal to $q(Q) = 0$. This makes it clear that for all parameter values, $q = q(Q)$ has exactly two fixed points: a nonmonetary equilibrium $q = 0$, and a unique monetary equilibrium $q = q^e > 0$.[13]

One important property of monetary equilibrium (that continues to hold even if $y > 0$) is the following. Recall that q is defined by $u'(q) = c'(q)$. Then it is easy to show that $e(q) < q$ and, therefore, $q^e < q$, as seen in figure 6. This is significant because q is the efficient outcome. More precisely, if we define welfare as before, $W = MV_1 + (1-M)V_0$, after simplification we have

$$rW = M(1-M)[u(q) - c(q)]. \tag{14}$$

Hence, W is maximized with respect to q at q. The result $q^e < q$ says that in equilibrium q^e is too low—or, equivalently, the price level is too high.[14]

The economic intuition for this result is straightforward. If a seller could turn the proceeds from his production into immediate consumption, as in a static or frictionless model, then he would produce until $u'(q) = c'(q)$. But in a monetary exchange, the proceeds from production consist of cash that can only be spent in the future when an opportunity to buy comes along. Since he discounts the future, a seller is only willing to produce less than the amount that satisfies $u'(q) = c'(q)$. Indeed, to verify that frictions are driving the result, observe that when r 0 or α ∞, we have q^e q.

Another question to ask is how q depends on M. One might expect $\partial q^e/\partial M < 0$, but it is actually possible to have $\partial q^e/\partial M > 0$ for small M (at least if r is also small). The explanation is that when M is close to zero, very little trade occurs. In this case, increasing M increases the frequency of productive meetings between buyers and sellers, which in turn increases both V_1 and V_0. The net effect on the bargaining solution can be a higher q. However, there is some threshold $\hat{M} < \frac{1}{2}$ such that $\partial q^e/\partial M < 0$ for all $M > \hat{M}$, so we can be sure that the value of money eventually begins to fall as M increases.

We can also ask how M affects welfare. It is clear that if a planner can choose both M and q to maximize W, he will choose $M = \frac{1}{2}$ and $q = q$. This is because $M = \frac{1}{2}$ maximizes the number of trades (just as in the previous section when $y = 0$), and $q = q$ maximizes the surplus that results from each trade. However, if the planner can choose only M and q is determined in equilibrium, then the value of M that maximizes W satisfies the first-order condition

$$\begin{aligned}\frac{\partial W}{\partial M} &\overset{s}{=} (1-2M)[u(q) - c(q)] \\ &\quad + M(1-M)[u'(q) - c'(q)]\frac{\partial q^e}{\partial M} = 0.\end{aligned}$$

As the second term is negative at $M = \frac{1}{2}$, the solution is $M^o < \frac{1}{2}$. This illustrates the trade-off between providing liquidity (making trade easier), and reducing the value of money (lowering the surplus from each trade). Reducing the value of money reduces welfare because, as we have already established, q is too low in equilibrium.

13 If $y > 0$, one can show that for large r there are no monetary equilibria, while for small r there are multiple monetary equilibria.

14 This is true even though bargaining is bilaterally efficient in the sense that the agreement is on the Pareto frontier in each exchange, taking as given the value of Q that prevails in other exchanges. The point is that all agents would be better off (in an ex ante sense) if they could get *everyone* to commit to increasing q. A stronger result is actually true: Not only is q^e too low according to the ex ante criterion W, it is also too low according to the ex post criteria V_0 and V_1. That is, buyers and even sellers would be better off if q were bigger. The result that q^e q^* also can be shown for models where agents can hold any amount of money; see Trejos and Wright (1995, pp. 133–4).

VII. Dynamics

We previously focused on steady states; in this section, we consider dynamic equilibria in the model with divisible goods. Although one could do things more generally,[15] we restrict attention to the case where $\theta = 1$ in the bargaining solution (equation [13]). This is equivalent to assuming that agents with money get to make take-it-or-leave-it offers. Hence, they will demand the quantity that satisfies

$$V_1 - V_0 = c(q), \tag{15}$$

since this is the most a producer would give to acquire currency. In the previous section's model, this immediately implies $V_0 = 0$ by virtue of equation (12), so we have $V_1 = c(q)$. Inserting this into equation (11) we have

$$rc(q) = (1-M)[u(q) - c(q)]. \tag{16}$$

An equilibrium is a q that solves equation (16). Although the model with take-it-or-leave-it offers is interesting in its own right, mainly because of its simplicity, we use it here to study dynamics.

We need to rederive the Bellman equations without limiting our attention to steady state. First, write the discrete time value of holding money at t as

$$V_1(t) = \frac{1}{1+r\tau}\{\alpha\tau(1-M)x\pi[u + V_0(t+\tau)] \\ +[1-\alpha\tau x(1-M)\pi]V_1(t+\tau) + \gamma\tau + o(\tau)\ ,$$

where we have reintroduced the utility of holding money γ. Rearranging, we have

$$\begin{aligned} rV_1(t) =& \alpha(1-M)x\pi[u + V_0(t+\tau) - V_1(t+\tau)] \\ &+ \frac{V_1(t+\tau) - V_1(t)}{\tau} + \gamma + \frac{o(\tau)}{\tau}. \end{aligned}$$

Taking the limit as τ 0, we arrive at

$$\begin{aligned} rV_1(t) =& \alpha x(1-M)\pi[u + V_0(t) - V_1(t)] \\ &+ \gamma + \dot{V}_1(t), \end{aligned} \tag{17}$$

where $\dot{V}_1$ indicates the time derivative. A similar derivation yields

$$rV_0(t) = M[V_1(t) - V_0(t) - c(q)] + \dot{V}_0(t). \tag{18}$$

Note that because of equation (15), the first term on the right side of equation (18) is 0.

Henceforth, we will omit the time argument t when there is no risk of confusion. Then an equilibrium is a bounded time path for each of the variables (V_0, V_1, q) satisfying (17), (18), and (15) at every point in time.

As in steady-state analysis, we want to eliminate the value functions from (15). To this end, subtract (17) and (18) to obtain

$$r(V_1 - V_0) = (1-M)[u(q) - c(q)] + \gamma + \dot{V}_1 - \dot{V}_0.$$

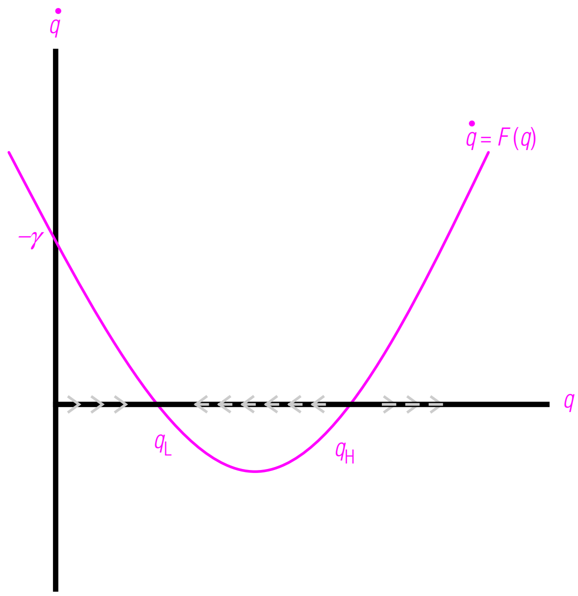
Equation (15) implies $c'(q)\dot{q} = \dot{V}_1 - \dot{V}_0$. Inserting this in the previous equation yields

$$\begin{aligned} \dot{q} =& F(q) \\ =& \frac{(r+1-M)c(q) - (1-M)u(q) - \gamma}{c'(q)}. \end{aligned} \tag{19}$$

Any bounded, non-negative path for q solving the above differential equation constitutes an equilibrium.

FIGURE 7

Dynamic Equilibria



To keep the number of cases manageable, assume $\gamma \leq 0$. Figure 7 shows equation (19) for this case. When γ 0 and large, there are no monetary equilibria; when γ 0 but not too big, there are two monetary steady states, q_L and q_H. Clearly, q_L is stable while q_H is unstable. Hence, in addition to the steady states, the set of equilibria is as follows: For all $q_0 \in (0, q_L)$, there is an equilibrium that converges monotonically up to q_L; for all $q_0 \in (q_L, q_H)$, there is an equilibrium that converges monotonically down to q_L. The former

15 See Coles and Wright (1998) and Ennis (1999).

(latter) equilibria are characterized by deflation (inflation), due simply to beliefs. If agents expect prices to change, this can be a self-fulfilling prophecy. When $\gamma = 0$, q_L coalesces with the nonmonetary equilibrium $q = 0$. Hence, in addition to the steady states, the set of equilibria includes paths starting at any $q_0 \in (0, q_H)$ and converging down to $q = 0$.

VIII. Extensions and Related Literature

In this section, we provide a short overview of some extensions to and applications of the above models in the literature. We will discuss some of the papers briefly; others we will merely mention. Our intent is to provide a bibliography rather than a review, so that the interested reader at least knows where to look.[16]

The basic search-theoretic monetary model can be generalized along several dimensions. Specialization is endogenized in more detail in Kiyotaki and Wright (1993), Burdett, Coles, Kiyotaki, and Wright (1995), Shi (1997b), and Reed (1999), for example. More general production structures are incorporated in Kiyotaki and Wright (1991) and Johri (1999). Long-term partnerships, in addition to one-time exchanges, are considered in Siandra (1996) and Corbae and Ritter (1997). Various extensions of bargaining are considered by Engineer and Shi (1998, 1999), Berentsen, Molico, and Wright (forthcoming), and Jafarey and Masters (1999). We already mentioned credit in section II, and there are several papers that attempt to have money and credit in the model at the same time: Kocherlakota and Wallace (1998) assume histories are imperfectly observed over time; Cavalcanti and Wallace(1999a,b) and Cavalcanti, Erosa, and Temzelides (1999) assume that the histories of only some agents are observed; and Jin and Temzelides (1999) assume only histories of local neighbors are observed. Following Diamond (1990), some papers have bilateral credit and money, with repayment (explicitly or implicitly) enforced by collateral. These include Hendry (1992), Shi (1996), Schindler (1998), and Yiting Li (forthcoming).

Many papers deal with commodity money, as opposed to fiat money. The basic idea is to determine endogenously which of many possible goods become media of exchange. Kiyotaki and Wright (1989) consider a version of the model where type i consumes good i and produces $i+1$, with $N = 3$ types. The goods are all storable, although at different costs. It is shown that goods with low storage costs may or may not come to serve as money, depending on parameter values as well as which equilibrium the economy is in; that is, there can be equilibria in which high-storage-cost goods are used as money. Aiyagari and Wallace (1991, 1992) generalize this to N types and consider several applications. Wright (1995) extends the model to allow agents to choose their type. Renero (1994, 1998b, 1999) considers several extensions of the framework. Among other things, he shows that equilibria in which goods with high storage costs serve as money can have good welfare properties, perhaps surprisingly (the intuition is that there is more trade in such equilibria). Other related papers include Kehoe, Kiyotaki, and Wright (1993), Cuadras-Morato and Wright (1997), and Renero (1998a).

There is also a literature on search models with private information. Williamson and Wright (1994) assume there is uncertainty concerning the quality of goods. In such an environment, a generally recognizable money has the potential role of mitigating the informational frictions and inducing agents to adopt strategies that increase the probability of acquiring high-quality output. So money may be valued even if the double-coincidence problem vanishes (that is, even if $y = 1$).

Trejos (1997) presents a simplified version of the model (essentially by setting $y = 0$), which allows him to obtain analytical solutions to the model. Kim (1996) endogenizes the extent of the private information problem. Cuadras-Morato (1994) and Yiting Li (1995b) use a version of this model to study commodity money. All the above papers assume indivisible goods. Trejos (1999) combines private information with divisible goods and bargaining. Velde, Weber, and Wright (1999) and Burdett, Trejos, and Wright (forthcoming) use commodity money models with private information to study some issues in monetary history, including Gresham's Law. Other related papers include Wallace (1997b), Williamson (1998) and Katzman, Kennan, and Wallace (1999).

Several papers attempt to model policy as follows: There is a subset of agents who are subject to the same search and information frictions as everyone else

16 A large body of work in search theory is tangentially related to the approach to monetary economics presented here. This brief review cannot discuss all such work, but we do want to mention Diamond (1982); although there is no money in that model, it is in some respects quite similar to that in section II. The version in Diamond (1984) does have money, but it is imposed through a cash-in-advance constraint; so although it in some ways resembles the framework presented here, its spirit is quite different. See also Diamond and Yellin (1985). We also mention Jones (1976) as well as the extensions by Oh (1989) and Iwai (1996), which attempt to build a model along lines similar to those presented here; see Ostroy and Starr (1990) for a review of this and related work. Other general discussions that concentrate more on models like the ones in this paper include Wallace (1996, 1997a).

but act collectively. Call these agents government agents. The idea is to see how government agents' exogenously specified trading rules affect the endogenously determined equilibrium behavior of other (private) agents. Papers in this group include Victor Li (1994, 1995a), Ritter (1995), Aiyagari, Wallace, and Wright (1996), Aiyagari and Wallace (1997), Li and Wright (1998), Green and Weber (1996), Wallace and Zhou (1997), and Berentsen (2000). For example, in Victor Li (1994, 1995a, 1997), government agents can tax money holdings when they meet private agents. A key result is that taxing money holdings may be efficient. The reason is that in his model (which also endogenizes search intensity) there is too little search by agents holding money, for standard reasons. Taxing them increases their search effort, and this can improve welfare.

The matching model seems a natural one for studying issues related to international monetary economics. For example, one can think about parameterizing differences in the efficiency of economic activity as well as degrees of openness across countries in terms of arrival rates. Among the first authors to analyze this in a model with multiple currencies and multiple countries are Matsuyama, Kiyotaki, and Matsui (1993). They find that several types of equilibria can arise, including those in which one currency circulates only locally while another emerges as an international currency; they find other equilibria in which all currencies are universally accepted. They compare these equilibria in terms of welfare. Zhou (1997) extends their model to study currency exchange. These models assume indivisible goods. Trejos and Wright (1999) endogenize prices using divisible goods and bargaining. Other examples of models with multiple currencies include Kultti (1996), Green and Weber (1996), Craig and Waller (1999), Peterson (2000), and Curtis and Waller (2000).

Some papers consider intermediation (in the form of middlemen, for example) as an alternative (or sometimes in addition) to money. An early paper to explicitly consider intermediation in a search model without money is Rubinstein and Wolinsky (1987). They generate a role for middlemen by specifying exogenously a set of agents who may have a more efficient technology for finding buyers than sellers have for finding buyers. Yiting Li (1998) is a very different model, in which private information about the quality of consumption goods combined with the existence of a costly quality verification technology give rise to a role for intermediation. In Shevchenko (2000), intermediation arises from inventory-theoretic considerations: Middlemen keep a stock of several goods on hand to increase the probability that a random buyer will find something he likes. The Shevchenko and Li papers also endogenize the number of intermediaries in the economy by means of a free entry condition. See also Camera (2000), Camera and Winkler (2000), and Hellwig (2000).

The framework's most important recent extension is perhaps the relaxation of the strong assumptions on how much money agents can hold—typically, zero or one unit of money, as we assume above. Models that consider such an extension can be an order of magnitude more complicated, but they are obviously more realistic and generate many interesting new results. They are also capable of addressing more traditional policy questions, such as the optimal rate of inflation, which are difficult to study in models where agents hold only zero or one unit of money. Such a model is contained in Molico (1999), who allows agents to hold any nonnegative amount of money and to bargain over the quantity of goods as well as the amount of money that is traded in each bilateral meeting. Because of the model's complexity, however, it can only be solved numerically. The numerical analysis generates interesting results on policy, welfare, the equilibrium distribution of prices, and other issues.

Green and Zhou (1998b) and Zhou (1999) also present a model with divisible money, where several results can be derived analytically. Unlike Molico, they assume that sellers set prices and cannot observe buyers' money holdings. Although such an environment could still have equilibria with a distribution of prices, they only look for equilibria where all sellers set the same price. Several interesting results emerge, including the existence of multiple (indeed, a continuum of) steady states, indexed by the nominal price level. Also, there can be an endogenous upper bound for money holdings: Agents with sufficient cash will not accept more. (Molico's model can also generate this.) Related references are Green and Zhou (1998a), Zhou (1998), Camera and Corbae (1999), Taber and Wallace (1999), Berentsen (1999a,b), and Rocheteau (1999). Shi (1997a) presents an analytically solvable model with perfectly divisible money, but his model is quite different in some dimensions from the rest of the literature.[17]

There are other applications and extensions that cannot all be considered in this brief review. However,

17 In Shi's model, the decision-making unit consists of a family with a large number of members (formally, a continuum), rather than a single individual. In this framework, family members share money holdings between periods, so every family starts the next period with the same amount of money by the law of large numbers. Applications and extensions of this model are contained in Shi (1998, 1999).

we want to mention some examples of papers that study evolution or learning in this framework, including Marimon, McGrattan, and Sargent (1990), Sethi (1996), Staudinger (1998) and Basçi (1999). They are interested in determining which of the equilibria are more robust; for example, can agents learn to use money? Brown (1996), Duffy and Ochs (1998, 1999), and Duffy (2001) ask the same kind of questions, but use laboratory methods with paid human subjects to test them experimentally. Although the results are by no means definitive, they are interesting in that they point to certain areas where laboratory subjects do not behave as theory predicts. However, the most recent experiments (Duffy, 2001) produce results that are encouraging from the perspective of the theory.

IX. Conclusion

In this paper, we have presented simple versions of the basic search-theoretic models of monetary exchange. Even these simple models allow a variety of questions to be addressed, and there are a wide range of extensions and applications. We hope this illustrates the usefulness of the framework for monetary economics and will encourage the reader to pursue these issues further.

Appendix

Proof of Proposition 1: For pure strategy equilibria, insert π_0 and π_1 into equations (3) and (4) and determine the region of parameter space in which the inequalities in (5) hold. Consider $\pi_0 = \pi_1 = 1$. For this to be an equilibrium, we require $\Delta_0 \geq 0$ and $\Delta_1 \geq 0$. Inserting $\pi_0 = \pi_1 = 1$ into (3) and (4), one finds $\Delta_0 \geq 0$ and $\Delta_1 \geq 0$ if and only if $r \in [\bar{r}_1, \bar{r}_4]$, as stated in the proposition. The other pure strategy cases are similar. For mixed strategies, solve $\Delta_j = 0$ for π_j and then determine the region of parameter space in which $\pi_j \in (0,1)$. Consider $\pi_0 \in (0,1)$ and $\pi_1 = 1$. For this to be an equilibrium, we require $\Delta_0 = 0$ and $\Delta_1 \geq 0$. Now $\Delta_0 = 0$ implies

$$\pi_0 = y + \frac{rc - \gamma}{(1-M)(u-c)}.$$

It is easy to see that $\pi_0 > 0$ iff $r > \bar{r}_3$ and $\pi_0 < 1$ iff $r < \bar{r}_4$, and the condition $\Delta_1 \geq 0$ is redundant. Hence, this equilibrium exists iff $r \in (\bar{r}_3, \bar{r}_4)$, as stated. The other mixed-strategy cases are similar. In this way, we obtain the complete set of equilibria.

Proof of Proposition 2: First note that rejecting a barter offer is always strictly dominated by accepting, given $u > c$. Now suppose that the seller gets to propose and that he proposes a cash transaction. There are three possibilities. First, if $\Delta_1 < 0$ the proposal will be rejected, so the seller would have been strictly better off proposing barter. Second, if $\Delta_1 > 0$ then the proposal will be accepted, but in this case $\Delta_0 < u - c$ (because $\Delta_0 + \Delta_1 = u - c$), so again the seller would have been strictly better off proposing barter. So a seller would never propose a cash trade over barter except possibly if $\Delta_1 = 0$. A symmetric argument implies that a buyer would never propose a cash trade except possibly if $\Delta_0 = 0$. This gives us two cases to consider: (i) $\Delta_0 = 0$, which implies $\Delta_1 = u - c$, which further implies $\psi_0 = 1$ (since $\Delta_1 > 0$ implies the seller strictly prefers barter), which is the only case in which we can have $\psi_1 < 1$; and (ii) $\Delta_1 = 0$, which implies $\Delta_0 = u - c$ and $\psi_1 = 1$, which is the only case in which we can have $\psi_0 < 1$.

Consider case (ii), where $\Delta_1 = 0$, $\Delta_0 = u - c$, $\psi_0 < 1$ and $\psi_1 = 1$ in equilibrium. Note $\Delta_0 = u - c$ implies $\pi_0 = 1$. Suppose $\pi_1 < 1$; then the agent without money gets $\pi_1 \Delta_0 = \pi_1(u-c)$ from proposing a cash transaction, which is strictly less than what he gets proposing barter. So $\psi_0 < 1$ requires $\pi_0 \pi_1 = 1$. Now the value functions can be written

$$\begin{aligned}
rV_1 =& yM(u-c) + (1-y)(1-M)\Delta_1 \\
&+ y(1-M)[\beta\psi_1 + (1-\beta)\psi_0](u-c) \\
&+ y(1-M)[1 - \beta\psi_1 - (1-\beta)\psi_0](1-\psi_0)\Delta_1 + \gamma \\
rV_0 =& y(1-M)(u-c) + (1-y)M\Delta_0 \\
&+ yM[\beta\psi_1 + (1-\beta)\psi_0](u-c) \\
&+ yM[1 - \beta\psi_1 - (1-\beta)\psi_0](1-\psi_0)\Delta_0.
\end{aligned}$$

Since we are in case (ii), we have $\Delta_1 = 0$, $\Delta_0 = u - c$ and $\psi_1 = 1$. Hence, subtracting V_1 and V_0 and simplifying, we have

$$\psi_0 = \frac{ru - \gamma + [(1-y)M + y(1-M)(1-\beta)](u-c)}{y(1-M)(1-\beta)(u-c)}. \tag{20}$$

This equality is violated for generic parameter values when $\psi_0 = 0$. Hence there is no equilibrium where sellers propose cash with probability 1. A symmetric argument for case (i) implies there is no equilibrium where buyers propose cash with probability 1. This means that the unique pure strategy equilibrium is for agents to propose barter with probability 1: $\psi_0 = \psi_1 = 1$.

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